A NOTE ON THE GENERALIZED HOMOLOGY THEORY

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G. W. Whitehead [6] has shown that, for any spectrum $\boldsymbol{E}, \tilde{\mathfrak{H}}(\boldsymbol{E})$ and $\tilde{\mathfrak{H}}^*(\boldsymbol{E})$ are generalized homology and cohomology theories on the category $\mathfrak{P}_0$ whose objects are finite CW-complexes with base vertex.

In this note, we show that, for any spectrum $\boldsymbol{E}$, $\tilde{\mathfrak{H}}(\boldsymbol{E})$ and $\tilde{\mathfrak{H}}^*(\boldsymbol{E})$ are defined on the category $\mathfrak{W}_0$.

1. Let $\mathfrak{W}_0$ be the category of spaces with base point having the homotopy type of a CW-complex, and a map of $\mathfrak{W}_0$ is a continuous, base point preserving map. In this note, we shall use the terms "space" and "map" to refer to objects and maps of $\mathfrak{W}_0$. Let $\mathfrak{W}_0^n$ be the category of n-ads [6].

Let T be the unit interval with base point 0, $\dot{T} = S^0$ be the subspace $\{0, 1\}$ of T, and $S = S^1 = T/\dot{T}$. The *cone* over X is the space $TX = T \wedge X$, and the *suspension* of X is the space $SX = S \wedge X$, where the space $X_1 \wedge \cdots \wedge X_n$ is the n-fold reduced join of the spaces X_i [6].

Let $[X, Y]$ be the set of homotopy classes of maps of X into Y, if $f: X \to Y$, let $[f]$ be the homotopy class of f. Then $[\ ,\ ]$ is a functor on $\mathfrak{W}_0 \times \mathfrak{W}_0$ to the category of sets with base points. If $f: X' \to X$, $g: Y \to Y'$, let

$$f^{\#} = [f, 1]: [X, Y] \longrightarrow [X', Y],$$
$$g_{\#} = [1, g]: [X, Y] \longrightarrow [X, Y'].$$

LEMMA 1.1. *Let X, Y be CW-complexes and $f: X \to Y$ be a continuous one-to-one onto map. Then the map f is a homeomorphism, if and only if, for any open cell τ of Y, there exist finite open cells $\sigma_1, \cdots, \sigma_n$ of X such that $\tau \subset f(\sigma_1 \cup \cdots \cup \sigma_n)$.*

PROOF. If the map f is a homeomorphism, then for any open cell τ of Y, $f^{-1}(\bar{\tau})$ is a compact set in X, and hence $f^{-1}(\bar{\tau})$ is contained in a finite union of open cells $\sigma_1, \cdots, \sigma_n$ of X [4]. Thus τ is contained in $f(\sigma_1 \cup \cdots \cup \sigma_n)$. Conversely, suppose that for any open cell τ of $Y, f^{-1}(\tau)$ is contained in a finite union of open cells $\sigma_1, \cdots, \sigma_n$ of X. Then $\bar{\tau} \subset f(\bar{\sigma}_1 \cup \cdots \cup \bar{\sigma}_n)$. Since f is a homeomorphism on a compact set, $f | \bar{\sigma}_1 \cup \cdots \cup \bar{\sigma}_n$ is a homeomorphism and hence $f^{-1} | \bar{\tau}$ is continuous. Therefore f^{-1} is continuous.

2. A *spectrum* $\boldsymbol{E}$ is a sequence $\{E_n | n \in Z\}$ of spaces together with a sequence of maps

$$\varepsilon_n : SE_n \longrightarrow E_{n+1},$$

where Z is the set of integers.

For any $X \in \mathfrak{W}_0$ and $n, k \in Z$, we have homomorphisms

$$\varepsilon_{*k} : [S^{n+k}, E_k \wedge X] \longrightarrow [S^{n+k+1}, E_{k+1} \wedge X], \quad n+k > 0,$$

and

$$\varepsilon^*_{n+k} : [S^k \wedge X, E_{n+k}] \longrightarrow [S^{k+1} \wedge X, E_{n+k+1}], \quad k > 0,$$

defined by

$$[S^{n+k}, E_k \wedge X] \xrightarrow{S_*} [S^{n+k+1}, S \wedge (E_k \wedge X)] \xrightarrow{\alpha_\#^{-1}} [S^{n+k+1}, S \wedge E_k \wedge X]$$
$$\xrightarrow{\beta_\#} [S^{n+k+1}, SE_k \wedge X] \xrightarrow{(\varepsilon_k \wedge 1)_\#} [S^{n+k+1}, E_{k+1} \wedge X],$$

and

$$[S^k \wedge X, E_{n+k}] \xrightarrow{S_*} [S \wedge (S^k \wedge X), SE_{n+k}] \xrightarrow{\alpha^\#} [S \wedge S^k \wedge X, SE_{n+k}]$$
$$\xrightarrow{\beta^{\#-1}} [S^{k+1} \wedge X, SE_{n+k}] \xrightarrow{\varepsilon_{n+k\#}} [S^{k+1} \wedge X, E_{n+k+1}],$$

respectively, where S_* is the suspension homomorphism, and

$$\alpha : X_1 \wedge X_2 \wedge X_3 \longrightarrow X_1 \wedge (X_2 \wedge X_3)$$

defined by $\alpha(x_1 \wedge x_2 \wedge x_3) = x_1 \wedge (x_2 \wedge x_3)$, and

$$\beta : X_1 \wedge X_2 \wedge X_3 \longrightarrow (X_1 \wedge X_2) \wedge X_3$$

defined by $\beta(x_1 \wedge x_2 \wedge x_3) = (x_1 \wedge x_2) \wedge x_3$ are homotopy equivalences [6, (2.4)]. Then $\{[S^{n+k}, E_k \wedge X], \varepsilon_{*k}\}$, $\{S^k \wedge X, E_{n+k}], \varepsilon^*_{n+k}\}$ form direct systems. For any map $f : X \to Y$, the following diagrams are commutative

$$\begin{array}{ccc}
[S^{n+k}, E_k \wedge X] & \xrightarrow{\quad (1 \wedge f)_\# \quad} & [S^{n+k}, E_k \wedge Y] \\
\downarrow \varepsilon_{*k} & & \downarrow \varepsilon_{*k} \\
[S^{n+k+1}, E_{k+1} \wedge X] & \xrightarrow{\quad (1 \wedge f)_\# \quad} & [S^{n+k+1}, E_{k+1} \wedge Y],
\end{array}$$

$$\begin{array}{ccc}
[S^k \wedge Y, E_{n+k}] & \xrightarrow{\quad (1 \wedge f)^\# \quad} & [S^k \wedge X, E_{n+k}] \\
\downarrow \varepsilon^*_{n+k} & & \downarrow \varepsilon^*_{n+k} \\
[S^{k+1} \wedge Y, E_{n+k+1}] & \xrightarrow{\quad (1 \wedge f)^\# \quad} & [S^{k+1} \wedge X, E_{n+k+1}].
\end{array}$$

Therefore, we can define the maps

$$f_{*n} : \lim_k [S^{n+k}, E_k \wedge X] \longrightarrow \lim_k [S^{n+k}, E_k \wedge Y],$$

$$f^{*n} : \lim_k [S^k \wedge Y,\ E_{n+k}] \longrightarrow \lim_k [S^k \wedge X,\ E_{n+k}].$$

Now, let

$$\tilde{H}_n(X; \boldsymbol{E}) = \lim_k [S^{n+k},\ E_k \wedge X],\ \tilde{H}_n(f\,; \boldsymbol{E}) = f_{*n}$$

and

$$\tilde{H}^n(X; \boldsymbol{E}) = \lim_k [S^k \wedge X,\ E_{n+k}],\ \tilde{H}^n(f; \boldsymbol{E}) = f^{*n},$$

then $\tilde{H}_n(\ ; \boldsymbol{E})$, $\tilde{H}^n(\ ; \boldsymbol{E})$ are respectively covariant and contravariant functors on $\mathfrak{W}_0$ to the category $\mathfrak{A}$ of abelian groups.

3. The suspension operation is a covariant functor $S : \mathfrak{W}_0 \to \mathfrak{W}_0$.

A *generalized homology theory* $\tilde{\mathfrak{H}}$ on $\mathfrak{W}_0$ is a sequence of covariant functors

$$\tilde{H}_n : \mathfrak{W}_0 \longrightarrow \mathfrak{A},$$

together with a sequence of natural transformations

$$\sigma_n : \tilde{H}_n \longrightarrow \tilde{H}_{n+1} \circ S$$

satisfying the following conditions:

(1) If $f_0, f_1 \in \mathfrak{W}_0$ are homotopic maps, then

$$\tilde{H}_n(f_0) = \tilde{H}_n(f_1).$$

(2) If $X \in \mathfrak{W}_0$, then

$$\sigma_n(X) : \tilde{H}_n(X) \approx \tilde{H}_{n+1}(SX).$$

(3) If $(X, A) \in \mathfrak{W}_0^2$, $i : A \subset X$, and if $p : X \to X/A$ is the identification map, then the sequence

$$\tilde{H}_n(A) \xrightarrow{\tilde{H}_n(i)} H_n(X) \xrightarrow{\tilde{H}_n(p)} \tilde{H}_n(X/A)$$

is exact.

A *generalized cohomology theory* $\tilde{\mathfrak{H}}^*$ *on* $\mathfrak{W}_0$ is a sequence of contravariant functors

$$\tilde{H}^n : \mathfrak{W}_0 \longrightarrow \mathfrak{A},$$

together with a sequence of natural transformations

$$\sigma^n : \tilde{H}^{n+1} \circ S \longrightarrow \tilde{H}^n$$

satisfying the following conditions:

(1*) If $f_0, f_1 \in \mathfrak{W}_0$ are homotopic maps, then

$$\tilde{H}^n(f_0) = \tilde{H}^n(f_1).$$

(2*) If $X \in \mathfrak{W}_0$, then

$$\sigma^n(X) : \tilde{H}^{n+1}(SX) \approx \tilde{H}^n(X).$$

(3*) If $(X, A) \in \mathfrak{W}_0^2$, $i : A \subset X$, and if $p : X \to X/A$ is the identification map, then the sequence

$$\tilde{H}^n(X/A) \xrightarrow{\tilde{H}^n(p)} \tilde{H}^n(X) \xrightarrow{\tilde{H}^n(i)} \tilde{H}^n(A)$$

is exact.

For any spectrum $\boldsymbol{E}$ and $X \in \mathfrak{W}_0$, we have natural transformations

$$\sigma_n(X; \boldsymbol{E}) : \tilde{H}_n(X; \boldsymbol{E}) \longrightarrow \tilde{H}_{n+1}(SX; \boldsymbol{E}),$$
$$\sigma^n(X; \boldsymbol{E}) : \tilde{H}^{n+1}(SX; \boldsymbol{E}) \longrightarrow \tilde{H}^n(X; \boldsymbol{E})$$

which are respectively induced by the following compositions

$$[S^{n+k}, E_k \wedge X] \xrightarrow{S_*} [S^{n+k+1}, S \wedge (E_k \wedge X)] \xrightarrow{\alpha_\#^{-1}} [S^{n+k+1}, S \wedge E_k \wedge X]$$
$$\xrightarrow{(-)^k \eta_\#} [S^{n+k+1}, E_k \wedge S \wedge X] \xrightarrow{\beta_\#} [S^{n+k+1}, E_k \wedge SX],$$

and

$$[S^k \wedge SX, E_{n+k+1}] \xrightarrow{\alpha^\#} [S^k \wedge S \wedge X, E_{n+k+1}]$$
$$\xrightarrow{(-1)^k \eta^\#} [S \wedge S^k \wedge X, E_{n+k+1}] \xrightarrow{\beta^{\#-1}} [S^{k+1} \wedge X, E_{n+k+1}],$$

where $\eta : X_1 \wedge X_2 \wedge X_3 \to X_2 \wedge X_1 \wedge X_3$ is a homeomorphism defined by $\eta(x_1 \wedge x_2 \wedge x_3) = x_2 \wedge x_1 \wedge x_3$.

Then the pairs of sequences $\tilde{\mathfrak{H}}(\boldsymbol{E}) = \{\tilde{H}_n(\ ; \boldsymbol{E}), \sigma_n(\ ; \boldsymbol{E})\}$, $\tilde{\mathfrak{H}}^*(\boldsymbol{E}) = \{\tilde{H}^n(\ ; \boldsymbol{E}), \sigma^n(\ ; \boldsymbol{E})\}$ satisfy the axioms (1), (2) and (1*), (2*) for the generalized homology and cohomology theories on $\mathfrak{W}_0$.

Let (X, A) be in $\mathfrak{W}_0^2$, $i : A \subset X$, and $p : X \to X/A$ the identification map.

In $\mathfrak{T}_0$, the map $E_k \wedge X / E_k \wedge A \to E_k \wedge (X/A)$ induced by the map $(1 \wedge p)$: $E_k \wedge X \to E_k \wedge (X/A)$ is not necessarily a homeomorphism, but it is a homotopy equivalence by Lemma 1. 1 (see [6, (2. 4)]), so we have the same result as [6, (5. 4)]. Therefore $\tilde{\mathfrak{H}}(\boldsymbol{E})$ satisfies the axiom (3) for the generalized homology theory.

On the other hand, the sequence

$$[S^k \wedge (X/A), E_{n+k}] \xrightarrow{(1 \wedge p)^\#} [S^k \wedge X, E_{n+k}] \xrightarrow{(1 \wedge i)^\#} [S^k \wedge A, E_{n+k}]$$

is exact for any n,k, because the sequence

$$S^k \wedge A \xrightarrow{1 \wedge i} S^k \wedge X \xrightarrow{j} (S^k \wedge X) \cup T(S^k \wedge A)$$

is exact by Puppe [3, Satz 1] and the following diagram is commutative

$$\begin{tikzcd}
S^k \wedge X \arrow[rr, "1 \wedge p"] \arrow[dd, "j"] \arrow[rrd] & & S^k \wedge (X/A) \\
 & & S^k \wedge X/S^k \wedge A \arrow[u, "g"] \arrow[d, "h"] \\
(S^k \wedge X) \cup T(S^k \wedge A) \arrow[rr] & & [(S^k \wedge X) \cup T(S^k \wedge A)]/T(S^k \wedge A)
\end{tikzcd}$$

where the maps g,h are homotopy equivalences by Lemma 1. 1 and the map q is also a homotopy equivalence by Puppe [3, Hilfssatz 4]. Therefore the sequence

$$\tilde{H}^n(X/A; \boldsymbol{E}) \xrightarrow{p^{*n}} \tilde{H}^n(X; \boldsymbol{E}) \xrightarrow{i^{*n}} \tilde{H}^n(A; \boldsymbol{E})$$

is exact. Thus $\mathfrak{H}^*(\boldsymbol{E})$ satisfies the axiom (3*) for the generalized cohomology theory.

Hence we have the following results.

THEOREM 3. 1. *For any spectrum* $\boldsymbol{E}$, $\tilde{\mathfrak{H}}(\boldsymbol{E})$ *is a generalized homology theory on* $\mathfrak{W}_0$.

THEOREM 3. 2. *For any spectrum* $\boldsymbol{E}$, $\tilde{\mathfrak{H}}^*(\boldsymbol{E})$ *is a generalized cohomology theory on* $\mathfrak{W}_0$.

4. The spectrum $\boldsymbol{E}$ is said to be *convergent* if and only if there is an integer N such that E_{N+i} is i-connected for all $i \geqq 0$ [6, p. 242].

The spectrum $\boldsymbol{E}$ is called to be of *type* (a, b) if and only if $\pi_{k+p}(E_p) = 0$ for $k<a$ or $k>b$ and for p sufficiently large, where $-\infty \leqq a \leqq b \leqq +\infty$.

The spectrum $\boldsymbol{E}$ is called to be 1*-connected* if and only if $\pi_i(E_p) = 0$ $(i = 0, 1)$ for p sufficiently large.

Then we have the following

(4. 1) If $c \leqq a \leqq b \leqq \mathrm{d}$, then the spectrum $\boldsymbol{E}$ of type (a, b) is of type (c, d).

(4. 2) If the spectrum $\boldsymbol{E}$ is convergent, then $\boldsymbol{E}$ is of type $(-N+1, +\infty)$ for some integer N.

(4. 3) If the spectrum $\boldsymbol{E}$ is of type $(a, +\infty)$, and $a \neq -\infty$, then $\boldsymbol{E}$ is 1-connected.

(4. 4) For any abelian group π, an Eilenberg-MacLane spectrum $\mathbf{K}(\pi) = \{K(\pi,n)\}$ is of type $(0, 0)$.

5. Throughout this section we shall assume that X, Y and $f: X \to Y$ are given in $\mathfrak{W}_0$ and m is a given positive integer.

THEOREM 5.1. *Suppose that the induced homomorphism*

$$f_*: H_r(X) \to H_r(Y)$$

is a monomorphism if $r < m$ *and an epimorphism if* $r \leqq m$ *(with integral coefficients). If* $\boldsymbol{E}$ *is a spectrum of type* $(a, +\infty)$, *then the mapping*

$$f_*: \tilde{H}_p(X; \boldsymbol{E}) \longrightarrow \tilde{H}_p(Y; \boldsymbol{E})$$

is a monomorphism if $p < m + a$ *and an epimorphism if* $p \leqq m + a$.

THEOREM 5.2. *Suppose that the induced homomorphism*

$$f^*: H^r(Y; G) \longrightarrow H^r(X; G)$$

is a monomorphism if $r \leqq m$ *and an epimorphism if* $r < m$ *for any coefficient group* G. *If* $\boldsymbol{E}$ *is a spectrum of type* $(-\infty, b)$ *and* 1*-connected, then the mapping*

$$f^*: \tilde{H}^p(Y; \boldsymbol{E}) \longrightarrow \tilde{H}^p(X; \boldsymbol{E})$$

is a monomorphism if $p \leqq m - b$ *and an epimorphism if* $p < m - b$.

PROOF OF THEOREM 5.1. The map $f: X \to Y$ and the inclusion map $i: X \subset Z_f$ are homotopy equivalent [3], where Z_f is the mapping cylinder of $f: X \to Y$, so we can suppose that $f: X \to Y$ is an inclusion map. For any arcwise connected space $W \in \mathfrak{W}_0$, SW is 1-connected. By assumption on $f: X \subset Y$, $Y \cup TX$ is arcwise connected and $H_r(Y \cup TX) = 0$ for $1 \leqq r \leqq m$. Therefore $S(Y \cup TX)$ is $(m+1)$-connected. Since E_i is $(a + i - 1)$-connected for i sufficiently large, $E_i \wedge S(Y \cup TX)$ is $(m + a + i + 1)$-connected, because $X_1 \wedge X_2$ is $(p + q - 1)$-connected if X_1 is $(p-1)$-connected and X_2 is $(q-1)$-connected [6].

Therefore $[S^{p+i}, E_i \wedge S(Y \cup TX)] = 0$ if $p \leqq m + a + 1$, for i sufficiently large. So that $\tilde{H}_p(S(Y \cup TX); \boldsymbol{E}) = 0$ if $p \leqq m + a + 1$. It follows that $\tilde{H}_p(Y \cup TX; \boldsymbol{E}) = 0$ if $p \leqq m + a$ by axiom (2).

On the other hand, by axiom (3) and Puppe [3, Satz 5], the sequence

$$\tilde{H}_p(X; \boldsymbol{E}) \xrightarrow{f_*} \tilde{H}_p(Y; \boldsymbol{E}) \longrightarrow \tilde{H}_p(Y \cup TX; \boldsymbol{E}) \longrightarrow \tilde{H}_p(SX; \boldsymbol{E}) \xrightarrow{(Sf)_*} \tilde{H}_p(SY; \boldsymbol{E})$$

is exact. By axiom (2) and $\tilde{H}_p(Y \cup TX; \boldsymbol{E}) = 0$ for $p \leqq m + a$, we have the consequence.

To prove Theorem 5.2, we shall use the following result.

LEMMA 5.3. *Let* W *be an arcwise connected space and* r*-simple for all* $r \geqq 1$. *Let* A, B *and* $g: A \to B$ *in* $\mathfrak{W}_0$. *Let the induced homomorphism* $g^*: H^r(B; G) \to H^r(A; G)$ *be a monomorphism if* $r \leqq n$ *and an epimorphism if* $r < n$

for any coefficient group G. Then the mapping

$$g^{\#}:[B,W]\to[A,W]$$

is a monomorphism if $\pi_r(W)=0$ *for* $r>n$ *and an epimorphism if* $\pi_r(W)=0$ *for* $r\geqq n$, *where n is a given positive integer* [2].

PROOF OF THEOREM 5.2. Since $\boldsymbol{E}$ is 1-connected, for any p, E_{p+i} is 1-connected for i sufficiently large. Since $\boldsymbol{E}$ is of type $(-\infty,b)$, if $r>b+p+i$, then $\pi_r(E_{p+i})=0$ for i sufficiently large. Therefore, if $p\leqq m-b$, then $\pi_r(E_{p+i})=0$ for $r>m+i$ and if $p<m-b$, then $\pi_r(E_{p+i})=0$ for $r\geqq m+i$.

On the other hand, by the assumption on $f:X\to Y$, the induced homomorphism $f^*:H^r(S^i\wedge Y;G)\to H^r(S^i\wedge X;G)$ is a monomorphism if $r\leqq m+i$ and an epimorphism if $r<m+i$.

By Lemma 5.3, the mapping

$$(1\wedge f)^{\#}:[S^i\wedge Y,\ E_{p+i}]\to[S^i\wedge X,E_{p+i}]$$

is a monomorphism if $p\leqq m-b$, and an epimorphism if $p<m-b$, for i sufficiently large. Thus the proof is complete.

REMARK. Let $\mathfrak{W}$ be the category of spaces having the homotopy type of a CW-complex. Then, there is a one-to-one correspondence between generalized (co-) homologies on $\mathfrak{W}_0$ and $\mathfrak{W}$ [6].

For any spectrum $\boldsymbol{E}$, let $\mathfrak{H}(\boldsymbol{E})=\{H_n(\ ;\boldsymbol{E}),\ \partial_n(\ ;\boldsymbol{E})\}$ and $\mathfrak{H}^*(\boldsymbol{E})=\{H^n(\ ;\boldsymbol{E}),\ \delta^n(\ ;\boldsymbol{E})\}$ be the generalized homology and cohomology theories on $\mathfrak{W}$ corresponding to $\mathfrak{H}(\boldsymbol{E})$ and $\mathfrak{H}^*(\boldsymbol{E})$ on $\mathfrak{W}_0$. Then we have some results analogous to Theorems 5.1, and 5.2.

6. In this section, we apply the results of §5 to the exact sequence of G.W.Whitehead [5]. This application is suggested by K.Tsuchida and H.Andô.

Let $B\in\mathfrak{W}$ be a 1-connected space, $b_0\in B, F$ the space of loops in B based at b_0, E_1 the space of paths in B which end at b_0, $e\in E_1$ the constant path and $P_0:E_1\to B$ the map which assigns to each path its starting point. Then P_0 induces a homomorphism $H_r(E_1,F)\to H_r(B,b_0)$. Now E_1 is contractible, so that $H_r(E_1,F)\approx H_{r-1}(F,e)$, and therefore we have a natural homomorphism $\sigma:H_{r-1}(F,e)\to H_r(B,b_0)$, which is called the *homology suspension*. Similarly, we have the *cohomology suspension* $\sigma^*:H^{r+1}(B,b_0)\to H^r(F,e)$ [5]. By Milnor [1], (E_1,F,e), $(B\times B,\ B\vee B,\ b_0\times b_0)\in\mathfrak{W}_0^2\subset\mathfrak{W}^3$ and so on.

COROLLARY 6.1. *Let* $B\in\mathfrak{W}$ *be n-connected* $(n\geqq 1)$. *Then, there is an exact sequence*

$$H_{3n+a}(F,e;\boldsymbol{E})\xrightarrow{\sigma}H_{3n+a+1}(B,b_0;\boldsymbol{E})\to\cdots\to H_r(F,e;\boldsymbol{E})$$

$$\xrightarrow{\sigma}H_{r+1}(B,b_0;\boldsymbol{E})\to G_{r+1}\to H_{r-1}(F,e;\boldsymbol{E})\xrightarrow{\sigma}H_r(B,b_0;\boldsymbol{E})\to\cdots$$

such that $G_{r+1} \approx H_{r+1}(B \times B,\ B \vee B; \boldsymbol{E}) \approx H_{r-1}(F \times F,\ F \vee F; \boldsymbol{E})$, *for any spectrum* $\boldsymbol{E}$ *of type* $(a, +\infty)$.

COROLLARY 6.2. *Let* $B \in \mathfrak{W}$ *be n-connected* $(n \geqq 1)$. *Then, there is an exact sequence*

$$\cdots \longrightarrow H^r(B, b_0; \boldsymbol{E}) \xrightarrow{\sigma^*} H^{r-1}(F, e; \boldsymbol{E}) \longrightarrow G^{r+1} \longrightarrow H^{r+1}(B, b_0; \boldsymbol{E})$$
$$\xrightarrow{\sigma^*} H^r(F, e; \boldsymbol{E}) \longrightarrow \cdots \longrightarrow H^{3n-b+1}(B, b_0; \boldsymbol{E}) \xrightarrow{\sigma^*} H^{3n-b}(F, e; \boldsymbol{E})$$

such that $G^{r+1} \approx H^{r+1}(B \times B,\ B \vee B; \boldsymbol{E}) \approx H^{r-1}(F \times F, F \vee F;\ \boldsymbol{E})$, *for any* 1-*connected spectrum* $\boldsymbol{E}$ *of type* $(-\infty, b)$.

Corresponding to the diagram ([5], Fig. 2), we have the following commutative diagram

$$\begin{array}{ccccc}
H^r(B, b_0; \boldsymbol{E}) & \xrightarrow{\quad\sigma^*\quad} & H^{r-1}(F, e; \boldsymbol{E}) & \xrightarrow{\quad\lambda^*\quad} & H^{r-1}(F \times F, F \vee F; \boldsymbol{E}) \\
\downarrow p_4^* & & \downarrow \alpha^* & & \uparrow \beta^* \\
H^r(E, e; \boldsymbol{E}) & \xrightarrow{\quad i_1^*\quad} & H^r(X, e; \boldsymbol{E}) & \xrightarrow{\quad\delta_1^*\quad} & H^{r+1}(E, X; \boldsymbol{E})
\end{array}$$

where α^* and p_4^* are isomorphisms onto for all r, β^* is isomorphism onto for $r \leqq 3n - b$, and the lower horizontal line is exact.

Following G.W.Whitehead ([5], §6), an element $u \in H^{r-1}(F, e; \boldsymbol{E})$ is said to be primitive if and only if $\lambda^*(u) = 0$ and the elements of $\sigma^* H^r(B, b_0; \boldsymbol{E})$ are said to be transgressive. From the above diagram

COROLLARY 6.3. *Every transgressive element of* $H^{r-1}(F, e; \boldsymbol{E})$ *is primitive. Conversely, if* $r \leqq 3n - b$, *every primitive element of* $H^{r-1}(F, e; \boldsymbol{E})$ *is transgressive.*

7. Now we consider the Ω-spectra (see [6], p. 241). For any Ω-spectrum $\boldsymbol{E} = \{E_n\}$ and $X \in \mathfrak{W}_0$, we have a natural isomorphism $[X, E_n] \to \tilde{H}^n(X; \boldsymbol{E})$.

Let n, q be given integers and $\boldsymbol{E}, \boldsymbol{E}'$ given Ω-spectra, a *cohomology operation* θ *of type* $(n, q; \boldsymbol{E}, \boldsymbol{E}')$ is a map

$$\theta_X : \tilde{H}^n(X; \boldsymbol{E}) \longrightarrow \tilde{H}^q(X; \boldsymbol{E}')$$

defined for every $X \in \mathfrak{W}_0$ such that if $f : X \to Y$ in $\mathfrak{W}_0$ then

$$f^* \circ \theta_Y = \theta_X \circ f^*.$$

In general the map θ_X is not a homomorphism, the cohomology operation θ is said to be *additive* if and only if $\theta_X : \tilde{H}^n(X; \boldsymbol{E}) \to \tilde{H}^q(X; \boldsymbol{E}')$ is a homomorphism for every $X \in \mathfrak{W}_0$.

Let $\mathfrak{O}(n,\ q;\ \boldsymbol{E},\ \boldsymbol{E}')$ be set of cohomology operations of type $(n,\ q;$

$\boldsymbol{E}, \boldsymbol{E}'$). Then we can easily prove the following

LEMMA 7.1. *Let* $\iota \in \tilde{H}^n(E_n; \boldsymbol{E}) \approx [E_n, E_n]$ *be the fundamental class corresponding to the homotopy class of the identity map of* E_n. *Then the map* $\theta \to \theta_{E_n}(\iota) \in \tilde{H}^q(E_n; \boldsymbol{E}') \approx [E_n, E'_q]$ is a $1-1$ *correspondence between* $\mathfrak{Q}(n, q; \boldsymbol{E}; \boldsymbol{E}')$ *and* $[E_n, E'_q]$.

By an analogous method to ([5], Lemma 7.1), we have the following

LEMMA 7.2. *Let* θ *be a cohomology operation of type* $(n, q; \boldsymbol{E}, \boldsymbol{E}')$. *Then* θ *is additive if and only if* $\theta_{E_n}(\iota) \in \tilde{H}^q(E_n; \boldsymbol{E}') = \tilde{H}^q(E_n, e; \boldsymbol{E}')$ *is primitive.*

COROLLARY 7.3. *Let* θ *be a cohomology operation of type* $(n, q; \boldsymbol{E}, \boldsymbol{E}')$. *If the corresponding element* $\theta_{E_n}(\iota)$ *is transgressive, then* θ *is additive. Conversely, if* $q \leqq 3(n+a) - b - 1$, *and* θ *is additive, then* $\theta_{E_n}(\iota)$ *is transgressive, where* $\boldsymbol{E}$ *is of type* $(a, +\infty)$, $\boldsymbol{E}'$ *of type* $(-\infty, b)$ *and* 1-*connected.*

This follows from Corollary 6.3 and Lemma 7.2.

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