INTEGRABILITY OF NON-NEGATIVE TRIGONOMETRIC SERIES. II.

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There is an extensive literature on connections between the integrability of $x^{\alpha}f(x)$ and the convergence of $\sum c_n n^{-\alpha-1}$ when c_n are (in some sense) sine or cosine coefficients of $f(x)$ and either the coefficients or the function satisfies a condition of positivity or monotonicity. I considered some cases when $\alpha>0$ in [1]. I should now like to bring out the connection between these results and some older ones; I give a few additional results, and correct one of the theorems of [1].

Theorems 3 and 4 of [1] imply the following result (here $\gamma=1+\alpha$, where α is the index appearing in [1]).

THEOREM A. *If* $1<\gamma<2$, $g(x)\geqq 0$ *on* $(0,\pi)$, $xg(x)\in L$, *and* b_n *are the generalized sine coefficients of* g, *then* $\sum n^{-\gamma}|b_n|$ *converges if and only if* $x^{\gamma-1}g(x)\in L$.

In this form, Theorem A is seen to be a natural extension of the following theorem of B. Sz.-Nagy [3], which is also a special case of a Theorem of Edmonds [2].

THEOREM B_1. *If* $0<\gamma\leqq 1$, $g(x)$ *is decreasing and bounded below in* $(0,\pi)$, $xg(x)\in L$ *and* b_n *are the generalized sine coefficients of* g, *then* $\sum n^{-\gamma}|b_n|$ *converges if and only if* $x^{\gamma-1}g(x)\in L$.

Thus, as is usual with theorems of this kind, an increase in the range of γ corresponds to a weakening of the hypothesis (from decreasing g to positive g).

The proof in [1] also establishes a generalization of Theorem A.

THEOREM A′. *If* $1<\gamma<2$, $G(x)$ *increases on* $(0,\pi)$, $\int_0^{\pi} x\,dG(x)<\infty$

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and $b_n = (2/\pi)\int_0^\pi \sin nx\,dG(x)$, *then* $\sum n^{-\gamma}|b_n|$ *converges if and only if* $\int_0^\pi x^{\gamma-1}dG(x) < \infty$.

Theorem B_1 $(0<\gamma<1)$ can be deduced from Theorem A′ by integration by parts.

Theorem 6 of [1], restated in our present notation, gives an analogue of Theorem A for $\gamma=2$:

If $g(x)\geqq 0$ *on* $(0,\pi)$, $xg(x)\in L$ *and* b_n *are the generalized Fourier sine coefficients of* g *then* $\sum n^{-2}|b_n|$ *converges if and only if* $xg(x)\log x\in L$.

We might reasonably ask instead for a condition on the coefficients b_n that would be equivalent to $xg(x)\in L$, provided that we do not assume $xg(x)\in L$ to begin with. If we retain the condition $g(x)\geqq 0$, the generalized sine coefficients are not defined unless $xg(x)\in L$ and so at first sight there seems to be no possibility of a theorem of the kind we are asking for. However, if we did not have $g(x)\geqq 0$ we could have $\int_{0+}^\pi xg(x)dx=0$ and then

$$\int_{0+}^\pi g(x)\sin nx\,dx = -\int_0^\pi (nx-\sin nx)g(x)\,dx$$

even when $xg(x)\notin L$. This suggests that it would be reasonable to take

$$b_n = -2\pi^{-1}\int_0^\pi (nx-\sin nx)\,g(x)dx \tag{1}$$

as generalized sine coefficients provided that $x^3g(x)\in L$.

We have the following theorem.

THEOREM 1. *If* $g(x)\geqq 0$ *on* $(0,\pi)$, $x^3g(x)\in L$ *and* b_n *are defined by* (1) *then* $x_{,}(x)\in L$ *if and only if* $n^{-1}\sum_{k=1}^n k^{-1}b_k = o(1)$.

There are analogous results for cosine coefficients. Corresponding to Theorem B_1, Sz.-Nagy proved

THEOREM B_2. *If* $0<\gamma\leqq 1$, $f(x)$ *is decreasing and bounded below on* $(0,\pi)$, $f(x)\in L$ *and* b_n *are the cosine coefficients of* f *then* $\sum n^{-\gamma}|a_n|$ *converges if and only if* $x^{\gamma-1}f(x)\in L$ $(0<\gamma<1)$, $f(x)\log x\in L$ $(\gamma=1)$.

If we want to extend this to larger values of γ we have to generalize the cosine coefficients a_n, since when $f(x) \geqq 0$ and $f(x) \in L$ we already have $x^{\gamma-1} f(x) \in L$, $\gamma \geqq 1$. With the same sort of motivation as for generalized sine coefficients, we take

$$a_n = -2\pi^{-1} \int_0^\pi (1 - \cos nx) f(x)\, dx \tag{2}$$

provided that $x^2 f(x) \in L$.

Then we have the following corrected version of Theorems 7 and 8 of [1].

THEOREM 2. *If* $1 < \gamma < 3$, $x^2 f(x) \in L$, $f(x) \geqq 0$ *on* $(0, \pi)$, *and* a_n *are defined by* (2), *then* $\sum n^{-\gamma} |a_n|$ *converges if and only if* $f(x) x^{\gamma-1} \in L$.

There is a more general version with Stieltjes integrals, analogous to Theorem A′.

The analogue of Theorem 1 for cosine coefficients corresponds to the exceptional case $\gamma = 1$ of Theorem B_2. I state it in Stieltjes form.

THEOREM 3. *If* $F(x)$ *decreases on* $(0, \pi)$, $\int x^2 |dF(x)| < \infty$, *and*

$$a_n = -2\pi^{-1} \int_0^\pi (1 - \cos nx) dF(x)$$

(*so that* $a_n \geqq 0$) *then* $F(x)$ *is bounded if and only if* $n^{-1} \sum_{k=1}^{n} a_k = o(1)$.

We shall deduce Theorem 1 from Theorem 3. We can also deduce the following result, which can be considered as a replacement for the missing case $\gamma = 0$ of Theorem B_1.

THEOREM 4. *If* $g(x)$ *decreases on* $(0, \pi)$, $\int_0^\pi x^2 |dg(x)| < \infty$, $g(\pi-) = 0$, *and* $b_n = (2/\pi) \int_0^\pi g(x) \sin nx\, dx$ (*necessarily nonnegative*), *then* $g(0+) < \infty$ (*i.e.*, g *is bounded*) *if and only if* $n^{-1} \sum_{k=1}^{n} k b_k = o(1)$.

PROOF OF THEOREM 2. This theorem is what is actually established by the argument of [1], Theorems 7 and 8. We reproduce the argument briefly.

Suppose $\sum n^{-\gamma} a_n$ converges. Then (since everything is nonnegative).

$$\int_0^\pi \left\{ \sum_{n=1}^\infty n^{-\gamma}(1-\cos nx) \right\} f(x)\,dx = \sum_{n=1}^\infty n^{-\gamma} \int_0^\pi f(x)(1-\cos nx)dx$$

$$= -\frac{1}{2}\pi \sum_{n=1}^\infty a_n n^{-\gamma}.$$

But

$$\sum_{n=1}^\infty n^{-\gamma}(1-\cos nx) \geqq Ax^2 \sum_{n=1}^{1/x} n^{-\gamma+2} \geqq Ax^{\gamma-1}$$

and so $f(x)x^{\gamma-1} \in L$.

Suppose $f(x)x^{\gamma-1} \in L$. Then

$$\frac{1}{2}\pi \sum_{n=1}^\infty n^{-\gamma}|a_n| \leqq \sum_{n=1}^\infty n^{-\gamma} \int_0^\pi (1-\cos nx)|f(x)|dx$$

$$= \int_0^\pi \left\{ \sum_{n=1}^\infty n^{-\gamma}(1-\cos nx) \right\} |f(x)|\,dx.$$

But

$$\sum_{n=1}^\infty n^{-\gamma}(1-\cos nx) \leqq \sum_1^{1/x} + \sum_{1/x}^\infty \leqq \frac{1}{2} x^2 \sum_1^{1/x} n^{2-\gamma} + \sum_{1/x}^\infty n^{-\gamma} = O(x^{\gamma-1}),$$

and since $f(x)x^{\gamma-1} \in L$ it follows that $\sum n^{-\gamma}|a_n|$ converges.

PROOF OF THEOREM 3. If $F(x)$ is bounded it is of bounded variation and so $a_n = o(1)$. It is then trivial that $n^{-1}\sum_{k=1}^n a_k = o(1)$

If this average is bounded, we have, with $2m+1 \leqq n$,

$$\frac{1}{2}\pi n^{-1} \sum_{k=1}^n a_k = n^{-1} \sum_{k=1}^n \int_0^\pi (1-\cos kx)|dF(x)|$$

$$= \int_0^\pi \left\{ \frac{1}{n} \sum_{k=1}^n (1-\cos kx) \right\} |dF(x)|$$

$$\geqq \int_0^\pi \left\{ \frac{1}{2m+1} \sum_{k=1}^m (1-\cos(2k+1)x) \right\} |dF(x)|$$

$$\geqq \frac{1}{3} \int_0^\pi \left\{ 1 - \frac{\sin 2mx}{2m \sin x} \right\} |dF(x)|.$$

Letting $m \to \infty$, we have $\int_0^\pi |dF(x)| < \infty$ by Fatou's lemma

PROOF OF THEOREM 1. Let

$$-\frac{1}{2}\pi b_n=\int_0^\pi (nx-\sin nx)g(x)\,dx\,,$$

with $g(x)\geqq 0$. First suppose $xg(x)\in L$ so that $b_n=o(n)$ (since the generalized sine coefficients of g are $o(n)$). Then clearly

$$n^{-1}\sum_{k=1}^{n}k^{-1}b_k=O(1)\,. \tag{3}$$

Now suppose that (3) holds. Put $G(x)=\int_x^\pi g(t)dt$. Since $x^3g(x)\in L$, we have

$$x^3G(x)\leqq\int_x^\delta t^3g(t)\,dt+x^3\int_\delta^\pi g(t)\,dt\,.$$

By taking δ small and then x small, we see that $x^3G(x)\to 0$ as $x\to 0$. Then

$$\begin{aligned}\frac{1}{2}\pi b_n=-\int_0^\pi(nx-\sin nx)g(x)dx&=\int_0^\pi(nx-\sin nx)\,dG(x)\\&=-n\int_0^\pi G(x)(1-\cos nx)\,dx\,.\end{aligned}$$

We can now apply Theorem 3 since

$$\begin{aligned}\int_0^\pi x^2G(x)\,dx&=\int_0^\pi x^2\,dx\int_x^\pi g(t)\,dt\\&=\frac{1}{3}\int_0^\pi t^3g(t)\,dt<\infty\,.\end{aligned}$$

Theorem 3 then says that $G(x)\in L$, i.e.

$$\int_0^\pi dx\int_x^\pi g(t)\,dt=\int_0^\pi tg(t)\,dt<\infty\,.$$

PROOF OF THEOREM 4. Since $\int x^2dg(x)$ is finite it follows that $g(x)=o(x^{-2})$ as $x\to 0+$ (cf. the preceding proof). Then we have

$$\frac{1}{2}\pi b_n=\int_0^\pi g(x)\sin nx\,dx=-n^{-1}\int_0^\pi(1-\cos nx)\,dg(x).$$

By Theorem 3, $g(x)$ is bounded if and only if $n^{-1}\sum_{k=1}^{n} k b_k = o(1)$.

REMARK. (Added October 30, 1964) Suppose $0<\gamma\leqq 1$, $g(x)\geqq 0$ with $xg(x)\in L$, and let b_n be the generalized sine coefficients of g. Comparing Theorems A and B_1, we might naturally ask what conditions are necessary and sufficient (a) for $\sum n^{-\gamma}|b_n|$ to converge, (b) for $x^{\gamma-1}g(x)\in L$. The answer to (a) is not known, but an answer to (b) is an easy consequence of Theorem A. We have $x^{\gamma-1}g(x)\in L$ if and only if $x^{\gamma+\delta-1}g(x)C_\delta(x)\in L$, where $0<\delta<1$, $\gamma+\delta>1$, and $C_\delta(x)=\sum n^{\delta-1}\sin nx$. By Theorem A, this in turn is equivalent to $\sum n^{-\gamma-\delta}|c_n|<\infty$, where c_n are the sine coefficients of $g(x)C_\delta(x)$, provided that $xg(x)C_\delta(x)\in L$, i.e. $x^{1-\delta}g(x)\in L$. The coefficients c_n are explicitly expressible in terms of the b_n by means of convolutions. There is a similar result for cosine series.

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