PRINCIPAL COFIBRATIONS

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Introduction. In this paper we shall define the notion of a principal cofibration which generalizes the cofibration $X \to C_f \to \Sigma Y$ induced by a map $f: X \to Y$ where C_f means the mapping cone of f and ΣY the reduced suspension of Y. The notion of a principal cofibration is a dual of a principal fibration in the sense of Peterson-Thomas [6].

One of the problem considered here is the following; under what conditions is a cofibration equivalent to a principal? This is answered by Theorem 2.7 in §2.

In §3 we dualize the results in [5], which are in the special case of induced cofibration. In §4 we mention an application to the Lusternik-Schnirelmann category and obtain a generalization of Berstein-Hilton's results.

1. Preliminaries. In this paper we assume that all spaces have base point denoted by $*$ and all maps (homotopies) preserve (keep fixed) base point.

A map $q: B \to E$ is called a cofibration if it has the homotopy lowering property for all spaces, i.e. if, for each space P and for all maps $f_0: E \to P$ and homotopies $g_t: B \to P$ with $g_0 = f_0 q$, there exists a homotopy $f_t: E \to P$ with $g_t = f_t q$. If q is an inclusion map, this is the homotopy extension property. The quotient space $F = E/q(B)$ is called the cofibre of q. Frequently the cofibration $q: B \to E$ with cofibre F will be denoted by the sequence $B \xrightarrow{q} E \xrightarrow{p} F$, where $p: E \to F$ is the projection.

Given a map $f: A \to B$, let C_f be the mapping cone of f, the space obtained from $CA \cup B$ by identifying $(a, 1) \in CA$ with $f(a)$, where CA denotes the reduced cone over A.

The set of all homotopy classes of maps $A \to B$ will be denoted by $\pi(A, B)$, which contains the distinguished element o, i.e. the homotopy class of the constant map $*: A \to B$. The homotopy class of a map $f: A \to B$ is denoted by $[f]$.

For maps $f: A \to C$ and $g: B \to C$, we define a map $f \bigtriangledown g: A \vee B \to C$ by

$$(f \bigtriangledown g)(a, *) = f(a) \qquad a \in A$$

$$(f \bigtriangledown g)(*, b) = g(b) \qquad b \in B$$

where $A \vee B$ is the subspace $A \times * \cup * \times B$ of $A \times B$. Then for a map $h: C \to D$ we have

$$(h \circ f) \nabla (h \circ g) = h \circ (f \nabla g). \tag{1}$$

For maps $f: A \to C$ and $g: B \to D$, we define a map $f \vee g: A \vee B \to C \vee D$ by $f \vee g = f \times g | A \vee B$.

A space X is an *H'-space* if there exists a map $\mu: X \to X \vee X$ such that the compositions $X \xrightarrow{\mu} X \vee X \xrightarrow{1 \vee *} X \vee X \xrightarrow{\Delta'} X$ and $X \xrightarrow{\mu} X \vee X \xrightarrow{* \vee 1} X \vee X \xrightarrow{\Delta'} X$ are homotopic to the identity. Here Δ' means the folding map, i.e. $\Delta'(x, *) = \Delta'(*, x) = x$ and 1 means the identity map.

A space X is an *H'-space* in the strong sense if there exists a map $\mu: X \to X \vee X$ and a map $\nu: X \to X$ such that

(i) the composition $X \xrightarrow{\mu} X \vee X \xrightarrow{1 \vee *} X \vee X \xrightarrow{\Delta'} X$ is homotopic to the identity.

(ii) the composition $X \xrightarrow{\mu} X \vee X \xrightarrow{1 \vee \nu} X \vee X \xrightarrow{\Delta'} X$ is null homotopic.

(iii) the compositions $X \xrightarrow{\mu} X \vee X \xrightarrow{1 \vee \mu} X \vee X \vee X$ and $X \xrightarrow{\mu} X \vee X \xrightarrow{\mu \vee 1} X \vee X \vee X$ are homotopic.

2. The principal cofibration.

DEFINITION 2.1. The cofibration $B \xrightarrow{q} E \xrightarrow{p} F$ is a *principal* if the following conditions are satisfied:

I). F is an H'-space with co-multiplication μ.

II). There exists a map $\phi: E \to F \vee E$ and a map $h: F \to E_*$, where E_* denotes the space obtained from $E \vee E$ by identifying $(q(b), *)$ with $(*, q(b))$ for each $b \in B$, subject to the conditions;

1) the diagram

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "p"'] & F \vee E \arrow[d, "1 \vee p"] \\ F \arrow[r, "\mu"] & F \vee F \end{tikzcd}$$

is commutative.

2) the diagram

$$\begin{tikzcd} B \arrow[r, "i_2"] \arrow[d, "q"'] & F \vee B \arrow[d, "1 \vee q"] \\ E \arrow[r, "\phi"] & F \vee E \end{tikzcd}$$

is commutative. Here i_2 denotes the injection into the second factor.
3) the diagram

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[dr, "k_2"'] & F \vee E \arrow[d, "h \nabla k_1"] \\ & E_* \end{tikzcd}$$

is homotopy-commutative. Here $k_i : E \to E_*$ is the composition of the injection $j_i : E \to E \vee E$ followed the identification map $\pi : E \vee E \to E_*$, $(i=1,2)$.

EXAMPLE 1. The well known cofibration $A \xrightarrow{i} CA \xrightarrow{p} \Sigma A$ is a principal. In fact, if we represents a point in reduced cone CA by (a,t) and a point in reduced suspension ΣA by $<a,t>$, then $\phi : CA \to \Sigma A \vee CA$ and $h : \Sigma A \to (CA)_*$ are defined as follows:

$$\phi(a,t) = \begin{cases} (<a,2t>,*) & 0 \leqq t \leqq 1/2 \\ (*,(a,2t-1)) & 1/2 \leqq t \leqq 1 \end{cases}$$

$$h<a,t> = \begin{cases} (*,(a,2t)) & 0 \leqq t \leqq 1/2 \\ ((a,2-2t),*) & 1/2 \leqq t \leqq 1 . \end{cases}$$

To show that Example 1 satisfies the conditions of Definition 2.1 is similar to the proof of Example 2, so we only prove Example 2.

EXAMPLE 2. Let $f : A \to B$ be a map. Then the cofibration $B \to C_f \to \Sigma A$ induced by $A \to CA \to \Sigma A$ via f is principal. We call such cofibration an induced cofibration.

Now we show that $B \xrightarrow{i} C_f \xrightarrow{p} \Sigma A$ fullfil the conditions in Definition 2.1. $\phi : C_f \to \Sigma A \vee C_f$ is defined by

$$\phi(b) = (*,b) \quad b \in B \subset C_f$$

$$\phi(a,t) = \begin{cases} (<a,2t>,*) & 0 \leqq t \leqq 1/2 \\ (*,(a,2t-1)) & 1/2 \leqq t \leqq 1 , \end{cases}$$

and $h : \Sigma A \to (C_f)_*$ is defined as in Example 1. Then conditions 1) and 2) in Definition 2.1 hold evidently and so we prove only 3).

$$(h \nabla k_1)\,\phi(a,t) = \begin{cases} h<a,2t> & 0 \leqq t \leqq 1/2 \\ k_1(a,2t-1) & 1/2 \leqq t \leqq 1 , \end{cases}$$

$$= \begin{cases} (*, (a, 4t)) & 0 \leqq t \leqq 1/4 \\ ((a, 2-4t), *) & 1/4 \leqq t \leqq 1/2 \\ ((a, 2t-1), *) & 1/2 \leqq t \leqq 1 . \end{cases}$$

Hence we have $(h \bigtriangledown k_1) \circ \phi | CA \simeq k_2 | CA$. Also $(h \bigtriangledown k_1) \phi(b) = (b, *) = (*, b) = k_2(b)$ for $b \in B$. Thus $(h \bigtriangledown k_1) \circ \phi \simeq k_2$.

The following Lemma 2.2 is a generalization of Proposition 4.6 in [2] and a dual of Lemma 4.1 in [6].

LEMMA 2.2. *Let $B \xrightarrow{q} E \xrightarrow{p} F$ be a principal cofibration. Let X be any space and let $[v]$, $[v'] \in \pi(E, X)$. Then*

$$q^* [v] = q^* [v']$$

if and only if there exists a map $w: F \to X$ such that $\phi^* [w \bigtriangledown v] = [v']$.

PROOF. Suppose that $v' \simeq (w \bigtriangledown v) \circ \phi$. Then by 2) in Definition 2.1,

$$v' \circ q \simeq (w \bigtriangledown v) \circ \phi \circ q = (w \bigtriangledown v) \circ (1 \vee q) \circ i_2 = v \circ q .$$

Conversely suppose that $vq \simeq vq'$. Then, by the lowering homotopy property, we may assume that $vq = vq'$. It is evident that $v \bigtriangledown v' : E \vee E \to X$ induces a map $\overline{v \bigtriangledown v'} : E_* \to X$ such $\overline{v \bigtriangledown v'} \circ \pi = v \bigtriangledown v'$. We set $w = \overline{v \bigtriangledown v'} \circ h$ and consider the map $(w \bigtriangledown v) \circ \phi : E \to F \vee E \to X$.

Since $v = (v \bigtriangledown v') \circ j_1 = \overline{v \bigtriangledown v'} \circ \pi \circ j_1 = \overline{v \bigtriangledown v'} \circ k_1$, we have

$$w \bigtriangledown v = (\overline{v \bigtriangledown v'} \circ h) \bigtriangledown (\overline{v \bigtriangledown v'} \circ k_1) = \overline{v \bigtriangledown v'} \circ (h \bigtriangledown k_1) \quad \text{by (1).}$$

Hence $(w \bigtriangledown v) \circ \phi = \overline{v \bigtriangledown v'} \circ (h \bigtriangledown k_1) \circ \phi = \overline{v \bigtriangledown v} \circ k_2$ (by 3) in Definition 2.1). But $\overline{v \bigtriangledown v} \circ k_2 = \overline{v \bigtriangledown v'} \circ \pi \circ j_2 = (v \bigtriangledown v') \circ j_2 = v'$. Thus we have $\phi^* [w \bigtriangledown v] = [v']$. Q. E. D.

Let $B \xrightarrow{q} E \xrightarrow{p} F$ be a principal cofibration and let X be any space. Let $w_i : F \to X$ be maps $(i = 1, 2)$. Then $w_1 + w_2 = \Delta' \circ (w_1 \vee w_2) \circ \mu$ induces a binary operation in $\pi(F, X)$.

Consider the diagram

$$\begin{array}{ccccc} E & \xrightarrow{\phi} & F \vee E & \xrightarrow{w_1 \bigtriangledown w_2 p} & X \\ \downarrow p & & \downarrow 1 \vee p & & \uparrow \Delta' \\ F & \xrightarrow{\mu} & F \vee F & \xrightarrow{w_1 \vee w_2} & X \vee X . \end{array}$$

By 1) in Definition 2.1, the left sequence is commutative and clearly $\Delta' \circ (w_1 \vee w_2) \circ (1 \vee p) = w_1 \bigtriangledown w_2 p$. Thus we have,

LEMMA 2.3. $p^* [w_1 + w_2] = \phi^* [w_1 \bigtriangledown w_2 p]$.

Lemma 2.3 generalizes Proposition 4.6′ in [2].

Following to [3], a diagram in the category of sets

$$
\begin{array}{ccc}
A_0 & \xrightarrow{j_1} & A_1 \\
{\scriptstyle j_2}\downarrow & & \downarrow{\scriptstyle k_1} \\
A_2 & \xrightarrow{k_2} & A_3
\end{array}
\tag{2}
$$

is called an exact square if it is commutative and if $k_1(a_1) = k_2(a_2)$ for $a_i \in A_i$ $(i = 1, 2)$ then there exists an $a_0 \in A_0$ such that $a_i = j_i(a_0)$ $(i = 1, 2)$.

The following Lemma 2.4 together with Lemma 2.2 have the key-roles for the later discussions.

LEMMA 2.4. *Let $B \xrightarrow{q} E \xrightarrow{p} F$ be a principal cofibration. Let $f: X \to Y$ be a map such that $f_*: \pi(F, X) \to \pi(F, Y)$ is a surjection. Then if $q^*: \pi(E, X) \to \pi(B, X)$ is a surjection, the diagram*

$$
\begin{array}{ccc}
\pi(E, X) & \xrightarrow{q^*} & \pi(B, X) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\pi(E, Y) & \xrightarrow{q^*} & \pi(B, Y)
\end{array}
$$

is an exact square.

PROOF. Let $[u] \in \pi(B, X)$ and $[v] \in \pi(E, Y)$ such that $f_*[u] = q^*[v]$. Since $q^*: \pi(E, X) \to \pi(B, X)$ is a surjection, there exists an $[s] \in \pi(E, X)$ such that $q^*[s] = [u]$. Then we have $q^* f_* [s] = q^*[v]$. Hence, by Lemma 2.2 there exists a map $w: F \to Y$ such that $\phi^* [w \bigtriangledown fs] = [v]$.

Also since $f_*: \pi(F, X) \to \pi(F, X)$ is a surjection, there exists $[d] \in \pi(F, X)$ such that $f_*[d] = [w]$. Then we have $[v] = \phi^*[fd \bigtriangledown fs] = \phi^* f_* [d \bigtriangledown s]$. If we define a map $l: E \to X$ by $l = (d \bigtriangledown s)\phi$, then $fl \simeq v$ and so $f_*[l] = v$. Now again applying Lemma 2.2 to the map $l = (d \bigtriangledown s)\phi$, we have $q^*[l] = q^*[s] = [v]$.

From now on, we work in the category of the spaces having the homotopy type of connected CW-complexes.

DEFINITION 2.5. Two cofibrations $B \xrightarrow{q} E \xrightarrow{p} F$ and $B \xrightarrow{q'} E' \xrightarrow{p'} F'$, are equivalent if there exists a homotopy equivalence $s: E \to E'$ such that $sq = q'$.

We remark that if cofibres F and F' are 1-connected, then an equivalence $s: E \to E'$ induces a homotopy equivalence $\bar{s}: F \to F'$. We call such an $\bar{s}$ an induced cofibre equivalence. This is proved as follows.

From [3; Theorem 3.6 and Corollary 3.7] we may assume that two cofibrations q and q' are inclusion cofibrations.

Then, from §7 in [3] and $\bar{s}p = p's$, we have a commutative diagram:

$$\begin{array}{ccccccccc}
H_r(B) & \xrightarrow{q_*} & H_r(E) & \xrightarrow{p_*} & H_r(F) & \xrightarrow{\partial} & H_{r-1}(B) & \xrightarrow{q_*} & H_{r-1}(E) \\
\downarrow 1 & & \downarrow s_* & & \downarrow \bar{s}_* & & \downarrow 1 & & \downarrow s_* \\
H_r(B) & \xrightarrow{q'_*} & H_r(E') & \xrightarrow{p'_*} & H_r(F') & \xrightarrow{\partial} & H_{r-1}(B) & \xrightarrow{q'_*} & H_{r-1}(E')
\end{array}$$

where upper and lower sequences are exact.

By Five Lemma, we have $\bar{s}_*: H_r(F) \approx H_r(F')$ for all r. Since F and F' are 1-connected, we may conclude that $\bar{s}: F \to F'$ is a homotopy equivalence.

LEMMA 2.6. (J. H. C. Whitehead) (cf. [7]) *Let X and Y be 0-connected spaces and $f: X \to Y$ a map. Then the following statememts are equivalent*;

a) $f_*: \pi_i(X) \to \pi_i(Y)$ *is isomorphic for $i < N$ and epimorphic for $i \leqq N$,*

b) *For any CW-complex K, $f_*: \pi(K, X) \to \pi(K, Y)$ is injective for* $\dim K < N$ *and surjective for* $\dim K \leqq N$.

THEOREM 2.7. *Let $B \xrightarrow{q'} E' \xrightarrow{p'} F'$ be a cofibration, where E' and F' are 1-connected, and $B \xrightarrow{q} E \xrightarrow{p} F$ a principal cofibration. Then the former is equivalent to the latter if the following condition are satisfied*:

i) *there exists a homotopy equivalence $\omega: F \to F'$,*

ii) *E, B are CW-complexes such that $q(B)$ is subcomplex of E and* dim *$F \leqq r$, and B is $(r-1)$-connected,*

iii) *$H^i(F', G) = 0$ for $i > s$ and an arbitrary abelian group G, and E' is $(s-1)$-connected.*

PROOF. We have the commutative diagram

$$\begin{array}{ccccc}
\pi(F, E') & \xrightarrow{p^*} & \pi(E, E') & \xrightarrow{q^*} & \pi(B, E') \\
\downarrow p'_* & & \downarrow p'_* & & \downarrow p'_* \\
\pi(F, F') & \xrightarrow{p^*} & \pi(E, F') & \xrightarrow{q^*} & \pi(B, F'),
\end{array}$$

where upper and lower sequences are exact.

The obstructions extending the map $q': B \to E'$ to a map of E into E are in $H^{i+1}(E, B; \pi_i(E'))$. Since the pair (E, B) satisfies the homotopy extension property, we have $H_i(E, B) \approx H_i(F)$ for all i. Thus, by virtue of the conditions i) and iii), it results that

$$H^{i+1}(E, B; \pi_i(E')) = 0 \quad \text{for } i \geqq s .$$

But E' is $(s-1)$-connected, and so $H^i(E, B; \pi_i(E')) = 0$ for $i \leqq s-1$. Hence the existence of a map $s' : E \to E'$ such that $s'q = q'$ is assured.

Since B is $(r-1)$-connected, $p'_* : H_i(E') \to H_i(F')$ is isomorphic for $i < r$ and epimorphic for $i \leqq r$ and $p'_* : \pi_i(E') \to \pi_i(F')$ is so. By Lemma 2.6 and the condition ii), it follows that $p'_* : \pi(F, E') \to \pi(F, F')$ is surjective.

Now if we take the maps $q' : B \to E'$ and $\omega p : E \to F'$, then $p'_*[q'] = 0 = q^*[\omega p]$. Thus Lemma 2.4 may be applied and we see that there exists a map $s : E \to E'$ such that $sq \simeq q'$ and $p's \simeq \omega p$.

Accordingly we have the diagram in which each ladder is homotopy-commutative;

$$\begin{array}{ccccc} B & \xrightarrow{q} & E & \xrightarrow{p} & F \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \omega} \\ B & \xrightarrow{q'} & E' & \xrightarrow{p'} & F' . \end{array}$$

Applying the (inclusion) cofibration homology exact sequence to the above diagram, we have a commutative diagram

$$\begin{array}{ccccccccccc} \longrightarrow & H_{i+1}(F) & \xrightarrow{\partial} & H_i(B) & \xrightarrow{q_*} & H_i(E) & \xrightarrow{p_*} & H_i(F) & \xrightarrow{\partial} & H_{i-1}(B) & \longrightarrow \\ & \downarrow{\scriptstyle r_*} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle s_*} & & \downarrow{\scriptstyle \omega_*} & & \downarrow{\scriptstyle 1} & \\ \longrightarrow & H_{i+1}(F') & \xrightarrow{\partial} & H_i(B) & \xrightarrow{q'_*} & H_i(E') & \xrightarrow{p'_*} & H_i(F') & \xrightarrow{\partial} & H_{i-1}(B) & \longrightarrow . \end{array}$$

By Five Lemma, $s_* : H_i(E) \to H_i(E')$ is isomorphic onto for each i and E, E' are 1-connected. Hence $s : E \to E'$ is a homotopy equivalence.

3. Induced cofibrations. Throughout the remainder we assume that all spaces have the homotopy type of connected CW-complexes.

An induced cofibration is a precise dual of the principal fibration in [5] and hence the results obtained in [5] can be dualize.

In the well known cofibration $A \xrightarrow{\iota} CA \xrightarrow{\rho} \Sigma A$, we consider

$$J : \pi(\Sigma A, \Sigma A) \to \pi_1(A, \rho) \text{ and } \iota_* : \pi(A, A) \to \pi_1(A, \rho) ,$$

where the definitions of J and ι_* are due to that of Eckmann-Hilton [3].

LEMMA 3.1. *Define* $\sigma : \Sigma A \to \Sigma A$, *by* $\sigma <a, t> = <a, 1-t>$, *then*

$$J[\sigma] = \iota_*[1],$$

where 1 *represents the identity map of* A.

LEMMA 3.2. *For a map* $f : \Sigma A \to B$, *let* $f_t : CA \to B$ *be a nullhomotopy of* $f\rho$. *If we define* $f' : \Sigma A \to B$ *by* $f' <a, s> = f_{s-1}\, \iota(a)$, *then*

$$f' \cong f\sigma .$$

Since the proofs of Lemma 3.1 and 3.2 are precise dual of that of Lemmas 2.1 and 2.2 in [5] respectively, we shall omit it.

Let $A_1 \xrightarrow{\alpha} A_2 \xrightarrow{\beta} A_3$ be a differential triple, i.e. $\beta\alpha = *$.
We set $A_2 \cap_\beta EA_3 = \{(a_2, u) \in A_2 \times EA_3 ;\ \pi(u) = \beta(a_2)\}$, where EA_3 is the path space in A_3 starting at the base point $*$ and $\pi : EA_3 \to A_3$ is defined by $\pi(u) = u(1)$.

Let P be any space and for any map $w : P \to A_2 \cap_\beta EA_3$, we set $w(x) = (u(x), \bar{u}(x))$, where $u(x) \in A_2$ and $\bar{u}(x) \in EA_3$. Then it is evident that w induces maps $u : P \to A_2$ and $\bar{u} : P \to EA_3$. Now if we define $v : CP \to A_3$ by $v(x, t) = \bar{u}(x)(t)$, then $\beta u(x) = v(x, 1)$. Thus to a map $w : P \to A_2 \cap_\beta EA_3$, we may correspond a pair of maps (u, v):

$$\begin{tikzcd} P \arrow[r, "u"] \arrow[d, "\iota"'] & A_2 \arrow[d, "\beta"] \\ CP \arrow[r, "v"] & A_3 . \end{tikzcd}$$

Let (u, v) be a pair of maps corresponding to another map $w' : P \to A_2 \cap_\beta EA_3$. Then it is easily verified that if $w \cong w'$, then $(u,v) \cong (u', v')$.

Conversely a homotopy class of map $w : P \to A_2 \cap_\beta EA_3$ corresponds to a homotopy class of pair (u, v).
Thus we have a one-to-one correspondence $\theta : \pi(P, A_2 \cap_\beta EA_3) \to \pi_1(P, \beta)$ defined by $\theta[w] = [(u, v)]$.

If we define $\bar{\alpha} : A_1 \to A_2 \cap_\beta EA_3$ by $\bar{\alpha}(a_1) = (\alpha a_1, *)$ where $* \in EA_3$ denotes a constant path based at $*$, then we have the commutative diagram:

$$\begin{tikzcd} \pi(P, A_2 \cap_\beta EA_3) \arrow[rr, "\theta"] & & \pi_1(P, \beta) \\ & \pi(P, A_1) . \arrow[ul, "\bar{\alpha}_*"] \arrow[ur, "\alpha_*"'] & \end{tikzcd}$$

THEOREM 3.3. Let $B \xrightarrow{q} E \xrightarrow{p} F$ *be a cofibration and* $l : \Sigma A \rightarrow F$ *a homotopy equivalence. Then the cofibration* q *is equivalent to an induced cofibration with induced cofibre equivalence in* $[l]$ *if and only if* $J[l\sigma] \in \operatorname{Im} q_*$.

The proof of Theorem 3.3 are obtained by dual discussion of Theorem 3.4 in [5] and we shall omit it.

COROLLARY 3.4. *Let* $B \xrightarrow{q} E \xrightarrow{p} F$ *be a cofibration and* $l : \Sigma A \rightarrow F$ *a homotopy equivalence. Then the cofibration* q *is equivalent to an induced cofibration with induced cofibre equivalence* $[l]$ *if and only if* $\theta^{-1} J[l\sigma] \in \operatorname{Im} \bar{q}_*$.

In the next Theorem all spaces are assumed to be connected CW-complexes.

THEOREM 3.5. *Let* $B \xrightarrow{q} E \xrightarrow{p} F$ *be a cofibration where* B *is* $(r-1)$*-connected and* F $(s-1)$*-connected* $(s > r > 1)$. *Suppose that* $\dim F \leqq r+s-1$. *Then the cofibration* q *is equivalent to an induced cofibration.*

PROOF. By the assupmtion F is $(s-1)$-connected and $\dim F \leqq 2s-1$. Then it is well known that F is homotopically equivalent to a suspension space, say, ΣA. Since $H^i(F) \approx H^i(\Sigma A) \approx H^{i-1}(A)$, we have $H^i(A)=0$ for $i \geqq r+s-1$. Clearly A is 1-connected. Hence applying Hilton's Theorem 1′ in [4],

$$\bar{q}_* : \pi(A, B) \rightarrow \pi(A, E \cap pEF)$$

is surjective. Therefore Theorem follows from Corollary 3.4.

Let $B \xrightarrow{q} E \xrightarrow{p} F$ be a cofibration where B, E are 1-connected. Suppose that F is a $K'(G, s)$-space, where $K'(G, s)$ is a polyhedron with abelian fundamental group such that $H_i(K'(G, s)) = 0$ for $i \neq s$ and $H_s(K'(G, s)) = G$. Then F is $(s-1)$-connected and may be considered as a $(s+1)$-dimensional polyhedron. Thus, we have Hilton's Theorem 7.1 in [3] as a Corollary.

COROLLARY 3.6. (*Hilton*) *Let* $B \xrightarrow{q} E \xrightarrow{p} F$ *be a cofibration where* B *and* F *are* 1*-connected and* F *is a* $K'(G,s)$*-space. Then the cofibration is equivalent to an induced cofibration.*

4. **Application to Lusternik-Schnirelmann category**. Let X^n be the Cartesian product of n-copies of X, and let $T^n(X)$ be the subspace of X^n consisting of points $(x_1, \cdots, x_n)$ such that $x_i = *$ for some i.

DEFINITION 4.1. X has category $\leqq n$ ($\operatorname{cat} X \leqq n$) if there exists a map

$\eta: X \to T^n(X)$ with $j\eta \simeq \Delta_X$ where $j: T^n(X) \to X^n$ is injection and $\Delta_X: X \to X^n$ is the diagonal map.

The map η is called the structure map.

THEOREM 4.2. *Let $B \xrightarrow{q} E \xrightarrow{p} F$ be a principal cofibration where F is a H'-space in the strong sense (see, § 1). If cat $B \leqq n$ and there exists a map $f: E \to T^n(E)$ such that $T^n(q) \circ \eta \simeq f \circ q$, where η is the structure map for B and $T^n(q): T^n(B) \to T^n(E)$ is induced by q, then cat $E \leqq n$.*

PROOF. We have a commutative diagram:

$$\begin{array}{ccccc}
\pi(F, T^n(E)) & \xrightarrow{p^*} & \pi(E, T^n(E)) & \xrightarrow{q^*} & \pi(B, T^n(E)) \\
\downarrow j_* & & \downarrow j_* & & \downarrow j_* \\
\pi(F, E^n) & \xrightarrow{p^*} & \pi(E, E^n) & \xrightarrow{q^*} & \pi(B, E)
\end{array}$$

in which the horizontal rows exact. By the assumption, we have $q^*[f] = [T^n(q) \circ \eta]$. Since F is a H'-space in the strong sense, by the same arguments as in Proposition 2.8 in [1] and the remark in the course of the proof of Theorem 3.4 in [1], it follows that

$$j_*: \pi(F, T^n(E)) \longrightarrow \pi(F, E^n)$$

is surjective.

Now we have $j_*[T^n(q) \circ \eta] = [j_E \circ T^n(q) \circ \eta] = [q^n \circ j_E \circ \eta] = [q^n \circ \Delta_B] = [\Delta_E \circ q] = q^*[\Delta_E]$. Thus Lemma 2.4 may be applied and the existence of a map $\zeta: E \to T^n[E]$ such that $j \circ \zeta \simeq \Delta_E$ is assured. Q. E. D.

REMARK. Theorem 4.2 is a generalization of Theorem 3.4 in [1]. In fact, let $f: A \to B$ be a map and let $B \xrightarrow{i} C_f \to \Sigma A$ be a cofibration induced by f. Suppose that f is n-quasiprimitive in the sense of Berstein and Hilton [1]. Then if cat $B \leqq n$ with the structure map $\eta: B \to T^n(B)$, there exists a map $\psi: A \to T^n(A)$ such that the diagram

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\psi \downarrow & & \downarrow \mu \\
T^n(A) & \xrightarrow{T^n(f)} & T^n(B)
\end{array}$$

is homotopy-commutative.

Consider a diagram;

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{q} & C_f \\ \psi\downarrow & & \downarrow\mu & & \downarrow \\ T^n(A) & \xrightarrow{T^n(f)} & T^n(B) & \xrightarrow{T^n(q)} & T^n(C_f) \end{array}$$

where $q: B \to C_f$ is the (inclusion) cofibration.

Then it is easily verified that the sequence

$$\pi(C_f, T^n(C_f)) \xrightarrow{i^*} \pi(B, T^n(C_f) \xrightarrow{f^*} \pi(A, T^n(C_f))$$

is exact. Since $T^n(i)\,\mu f \cong T^n(i)T^n(f)\psi \cong *$, there exists a map $u: C_f \to T^n(C_f)$ with $u\circ i \cong T^n(i)\circ\mu$.

Thus we see that if a cofibration $q: B \to C_f$ is induced by f and f is n-quasiprimitive, then the assumptions of Theorem 4.1 are satisfied and cat $C_f \leqq n$.

5. Appendix. Finally we shall define a dual of H-fibration in [5] which is a intermediate notion between arbtrary cofibration and principal cofibration.

DEFINITION 5.1. A cofibration $B \xrightarrow{q} E \xrightarrow{p} F$ is a H'-cofibration if there exists a co-operation $\phi: E \to F \vee E$ and a homotopy $H_t: E \to F\times E$ subject to the following conditions:

(a) the diagram

$$\begin{array}{ccc} B & \xrightarrow{i_2} & F\vee B \\ \downarrow q & & \downarrow 1\vee q \\ E & \xrightarrow{\phi} & F\vee E \end{array}$$

is commutative.

(b) $H_0 = j\circ\phi$ ($j: F\vee E \to F\times E$ injection) and $H_1 = (p\times 1)\circ\Delta_E$, $H_t q = *$ for all t.

PROPOSITION 5.2. *H_t induces an H'-structure on F.*

PROOF. By Definition 5.1, we have $H_t q = *$ for any t and especially $H_0 q = j\phi q = *$. Hence $(1\times p)j\phi q = *$.

In the diagram;

$$\begin{array}{ccccc} E & \xrightarrow{\phi} & F\vee E & \xrightarrow{j} & F\times E \\ \downarrow p & & \downarrow 1\vee p & & \downarrow 1\times p \\ F & \longrightarrow & F\vee F & \xrightarrow{j_F} & F\times F \end{array}$$

where $j_F: F\vee F \to F\times F$ is the injection, we have $(1\times p)j = j_F(1\vee p)$. Since j_F is the injection, $(1\vee p)\phi q = *$. Hence $(1\vee p)\phi$ induces a map $\mu: F\to F\vee F$ such that $\mu p = (1\vee p)\phi$. Also $(1\times p)H_1$ induces a homotopy $H'_t: F\to F\times F$ such that $H'_t p = (1\times p)H_t$. Then $H'_1 p = (1\times p)H_1 = (1\times p)(p\times 1)\Delta_E = \Delta_F p$, where Δ_E and Δ_F denote the diagonal maps in E and F respectively. Also $H'_0 p = (1\times p)H_0 = (1\times p)j\phi = j_F\mu p$.

Thus it follows that $H'_0 = j_F\mu$ and $H'_1 = \Delta_F$. Hence the map $\mu: F\to F\vee F$ defines an H'-structure. Q. E. D.

THEOREM 5.3. *Let $B \xrightarrow{q} E \xrightarrow{p} F$ be a H'-cofibration in which all spaces are CW-complexes. If $u\in H^n(E,Q)$ and $v\in H^m(F,Q)$, where Q is the field of rational numbers, then we have*

$$p^*(v)\cup u = 0\,.$$

PROOF. Let $E\# E$ be the quotient space $E\times E/E\vee E$ and let $\pi: E\times E\to E\# E$ be the projection. If we identify $H^r(E\# E, Q) = \sum\limits_{\substack{p+q=r\\ p,q>0}} H^p(E,Q)\otimes H^q(E,Q)$, by the definition of the cup product in terms of the diagonal map, we have $p^*(v)\cup u = (\pi\Delta_E)^*(p^*(v)\otimes u)$. Let $\overline{p\times 1}: E\# E\to F\# E$ be a map induced by $p\times 1: E\times E\to F\times E$, there exists a commutative diagram:

$$\begin{array}{ccc} H^m(F,Q)\otimes H^n(E,Q) & \xrightarrow{(p\times 1)^*} & H^m(E,Q)\otimes H^n(E,Q) \\ \downarrow & & \downarrow \\ H^{m+n}(F\# E,Q) & \xrightarrow{\overline{(p\times 1)}^*} & H^{n+m}(E\# E,Q). \end{array}$$

Hence $(\pi\Delta_E)^*(p^*(v)\otimes u) = (\pi\Delta_E)^*(\overline{p\times 1})^*(v\otimes u)$. But $j\phi\simeq (p\times 1)\Delta_E$ by the condition b) in Definition 5.1. Hence $(p\times 1)\pi\Delta_E = \pi(p\times 1)\Delta_E \simeq \pi j\phi\simeq *$.

Thus we may conclude that $p^*(v)\cup u = 0$. Q. E. D.

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