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EXTREME POINTS FOR REGULAR SUMMABILITY MATRICES*

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1. In this paper we shall be studying bounded sequences summed by a regular matrix. The usual norm for the space of bounded sequences is $\|s_n\| = \sup|s_n|$ and the *unit ball* is the set of sequences with $\|s_n\| \leqq 1$. If A is a regular matrix, then $\mathfrak{A}$ denotes the set of bounded sequences summed by A, $\mathfrak{A}$ is called the *summability field* of A. If $\{s_n\}$ is summed by A, A-lim s_n denotes the value to which it is summed.

The following result is due to Brudno, see [4] and [2].

THEOREM 1. *If* $\mathfrak{A} \supset \mathfrak{B}$, *then* $A-\lim s_n = B-\lim s_n$ *for every* $\{s_n\} \in \mathfrak{B}$.

By the *summability method* $\mathfrak{U}$, we denote the set of all matrices which have $\mathfrak{A}$ as their summability field. Let A be a regular matrix, $h(A)$ is called the *matrix norm* of A, where

$$h(A) = \sup_m \sum_{n=1}^{\infty} |a_{m,n}| < \infty$$

It is clear that

$$1 \leqq \sup|A-\lim s_n| \leqq h(A)$$

where the sup is taken over all bounded sequences in the unit ball summed by A. The value of $\sup|A-\lim s_n|$ is a function of the summability field and we define the *field norm* $N(\mathfrak{A})$ by

$$N(\mathfrak{A}) = \sup|A-\lim s_n|$$

where the sup is taken over the unit ball. We can also consider,

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$$\|\mathfrak{U}\| = \inf h(A)$$

where the infimum is taken over the matrices which have $\mathfrak{A}$ as their summability field. The following fundamental theorem is due to Brudno [3].

THEOREM 2. *For every regular matrix* A,

$$\|\mathfrak{U}\| = N(\mathfrak{A}).$$

Brudno [3] also showed that there is always a sequence $\{s_n\}$ in the unit ball such that

$$A-\lim s_n = N(\mathfrak{A}).$$

such sequences are called *extreme points* of $\mathfrak{A}$.

If there is an A' in $\mathfrak{U}$, such that

$$h(A') = |\mathfrak{U}| = N(\mathfrak{A}).$$

we shall say that the field norm of $\mathfrak{A}$ is *attained* by the matrix A ; otherwise, we shall say that the norm is *not attained.*

2. In [1] we discussed the extreme points of summability fields for which the norm is attained. We proved:

THEOREM 3. *If* $\mathfrak{A}$ *is a summability field, and if the norm of* $\mathfrak{A}$ *is attained, then there is a matrix* B *belonging to the method, such that,*

i) $h(B) = N(\mathfrak{A})$

ii) *in every column of* B, *all the non-zero elements have the same sign.*

The sequence $\{s_n\}$, where $s_n = +1$ or -1 according as the characteristic sign of the n th column of B is + or −, we call the *characteristic sequence.* Clearly the characteristic sequence is an extreme point of $\mathfrak{A}$.

If $\{s_n\} \in \mathfrak{A}$, $s_n = 0$, $n \in E_1$ and

$$\lim_{m\to\infty} \sum_{n \notin E_1} |a_{m,n}| = 0,$$

then $\{s_n\}$ is said to be a *sparse* sequence for $\mathfrak{A}$. Schur's theorem makes it plain that sparse sequences are independent of the particular matrix A representing $\mathfrak{A}$. We shall prove the following theorem:

THEOREM 4. *Let the norm of* $\mathfrak{A}$ *be attained and let* $\{t_n\}$ *be a characteristic sequence for* $\mathfrak{A}$. *Then* $\{s_n\}$ *is an extreme point of* $\mathfrak{A}$ *if and only if*

i) $\{s_n\}$ *is in the unit ball;*

ii) $s_n = t_n + v_n + r_n$, $(n = 1, 2, \cdots)$, *where* $\{v_n\}$ *is sparse for* $\mathfrak{A}$, *and* $\{r_n\}$

converges to zero.

PROOF. Clearly if the conditions are satisfied, then $\{s_n\}$ is an extreme point of $\mathfrak{A}$. On the other hand, let B be the matrix given in Theorem 3 and $\{t_n\}$ the characteristic sequence.

Now $B-\lim(t_n-s_n)=0$ and

$$\sum b_{m,n}(t_n-s_n)=\sum |b_{m,n}|\,|t_n-s_n| \quad (m=1, 2, \cdots).$$

If $|t_n-s_n|>a$, $n\in E(a)$, then

$$\lim_{m\to\infty} \sum_{n\in E(a)} |b_{m,n}|=0.$$

We can choose the functions $\lambda(m)$ and $\mu(m)$ so that $\lambda(m)-\lambda(m-1)\leqq 1$, $(m=2, 3, \cdots)$ and

$$\lim_{m\to\infty} \sum_{n=1}^{\lambda(m)} |b_{m,n}|=\lim_{m\to\infty} \sum_{n=\mu(m)+1}^{\infty} |b_{m,n}|=0.$$

If the matrix $B'=(b'_{m,n})$ is defined by the relations

$$b'_{m,n}=b_{m,n}, (\lambda(m)+1 \leqq n \leqq \mu(m))$$
$$b'_{m,n}=0, (1\leqq n\leqq \lambda(m);\ \mu(m)+1\leqq n\leqq\infty)$$

then B and B' sum the same set of bounded sequences and B' has all the properties associated with B, (including the same characteristic sequence). We choose a sequence of indices $\{m_k\}$ satisfying the following conditions:

1) $\lambda(m_{k+1})\leqq \mu(m_k)$, $(k=1, 2, \cdots)$

2) $\sum |b'_{m,n}|<\frac{1}{k} \qquad \left(n\in E\left(\frac{1}{k}\right)\right)$

for all m, $(m\geqq m_k)$. Let $\{v_n\}$ be defined by the relations:

$$v_n=s_n-t_n, \left(\lambda(m_{k+1})\leqq n<\lambda(m_{k+2});\ n\in E\left(\frac{1}{k}\right)\right),$$

$$v_n=0 \ \text{ elsewhere, } (k=1, 2, \cdots).$$

Let $\{r_n\}$ be the sequence defined by $s_n-t_n-v_n=r_n$, $(n=1, 2, \cdots)$. It is clear that $\{r_n\}$ converges to zero.

Let E be the set where $v_n\neq 0$, then if $m_k\leqq m<m_{k+1}$,

$$\sum_{n\in E} |b'_{m,n}|\leqq \sum_{n\in E\left(\frac{1}{k-1}\right)} |b'_{m,n}|+\sum_{n\in E\left(\frac{1}{k}\right)} |b'_{m,n}|\leqq \frac{1}{k}+\frac{1}{k-1}$$

and $\{v_n\}$ is sparse for $\mathfrak{A}$. This completes the proof of our theorem.

Brudno [2] noted that as a consequence of Theorem 1, the sum attached to a bounded sequence in $\mathfrak{A}$ depends only on which other sequences belong to $\mathfrak{A}$; the sum for each sequence is an *internal characteristic* of $\mathfrak{A}$. From the definition, it is clear that the extreme points are an internal characteristic of $\mathfrak{A}$. We note that if $\{s_n\}$ is not a convergent sequence, then $\{s_n\}$ is sparse if and only if $\{\xi_n s_n\}$ belongs to $\mathfrak{A}$, where $\{\xi_n\}$ runs through the bounded sequences. It is clear that sparse sequences are also an internal characteristic. Convergent sequences can also be recongnized. Hence, it is possible to examine $\mathfrak{A}$ and determine whether or not the conclusions of Theorem 4 are satisfied. If we could show that Theorem 4 was satisfied if and only if the norm is attained, then we would be able to tell whether or not the norm was attained by examining $\mathfrak{A}$ only. The property of attaining or not attaining the norm would be an internal characteristic of the summability field,

It would be a good result to know that Theorem 4 is true only when the norm is attained.

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