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ON THE CONVOLUTION ALGEBRA OF BEURLING

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1. Introduction. Let $f(x)$ be an integrable function with period 2π and its Fourier series be

$$S(f) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

We write $\widehat{A}$ for the class of functions with absolute convergent Fourier series. $\widehat{A}$ is a Banach algebra under usual operations. In this algebra, spectral synthesis is impossible and operating functions are analytic. A. Beurling [1] considered a subclass of $\widehat{A}$ such that

$$A_0 = \{f \mid A(f) < \infty\}$$

where

$$A(f) = \int_0^1 t^{-3/2}\left\{\int_0^{2\pi} |f(x+t) - f(x-t)|^2\, dx\right\}^{1/2} dt .$$

The algebra A_0 has remarkable properties, that is to say, that spectral synthesis is possible and the functions which satisfy the Lipschitz condition of order 1 are operating.

In this note, we extend slightly $A(f)$ to

$$A_\beta(f) = \int_0^1 t^{-2+\beta/2}\left\{\int_0^{2\pi} |f(x+t) - f(x-t)|^2\, dx\right\}^{\beta/2} dt \tag{1}$$

for $1 \leqq \beta < 2$ and show that $A_\beta(f) < \infty$ is equivalent to $B_\beta(f) < \infty$ or $C_\beta(f) < \infty$, where

$$B_\beta(f) = \sum_{n=1}^{\infty} n^{-\beta/2}\left\{\sum_{|k|=n+1}^{\infty} |c_k|^2\right\}^{\beta/2} \tag{2}$$

and

$$(3)\qquad C_\beta(f) = \sum_{n=1}^{\infty} n^{-3\beta/2}\left\{\sum_{|k|=1}^{n} k^2 |c_k|^2\right\}^{\beta/2}.$$

Denote by $s_n(x)$ the partial sum of $S(f)$, then

$$B_\beta(f) = \sum_{n=1}^{\infty} n^{-\beta/2}\left\{\int_0^{2\pi} |f(x) - s_n(x)|^2\, dx\right\}^{\beta/2}$$

and

$$C_\beta(f) = \sum_{n=1}^{\infty} n^{-3\beta/2}\left\{\int_0^{2\pi} |s_n'(x)|^2\, dx\right\}^{\beta/2}.$$

Therefore the above equivalency has an interpretation for approximation theory.

The above equivalence relation throws also lights on papers of Boas [3] and Kinukawa [5] on the absolute convergence of trigonometric series. We discuss this and related problems in the last section.

2. Equivalence relations. We begin with the equivalency of $B_\beta(f) < \infty$ and $C_\beta(f) < \infty$.

THEOREM 1. *For $1 \leqq \beta < 2$, the finiteness of $B_\beta(f)$ is equivalent to the finiteness of $C_\beta(f)$.*

PROOF. By a principle of the condensation test, $B_\beta(f) < \infty$ is equivalent to the finiteness of

$$(5)\qquad B_\beta'(f) = \sum_{k=0}^{\infty} 2^{k(1-\beta/2)}\left\{\sum_{|\nu|=2^k+1}^{\infty} |c_\nu|^2\right\}^{\beta/2}$$

and $C_\beta(f) < \infty$ is equivalent to the finiteness of

$$(6)\qquad C_\beta'(f) = \sum_{k=0}^{\infty} 2^{k(1-3\beta/2)}\left\{\sum_{|\nu|=1}^{2^{k+1}} \nu^2 |c_\nu|^2\right\}^{\beta/2}.$$

Concerning $C_\beta'(f)$, we have

$$C_\beta'(f) = \sum_{k=0}^{\infty} 2^{k(1-3\beta/2)}\left\{\sum_{|\nu|=1}^{2^{k+1}} \nu^2 |c_\nu|^2\right\}^{\beta/2}$$

$$= \sum_{k=0}^{\infty} 2^{k(1-3\beta/2)} \left\{ \sum_{m=0}^{k} \sum_{|\nu|=2^m+1}^{2^{m+1}} \nu^2 |c_\nu|^2 \right\}^{\beta/2}$$

$$\leqq \sum_{k=0}^{\infty} 2^{k(1-3\beta/2)} \left\{ 4 \sum_{m=0}^{k} 2^{2m} \sum_{|\nu|=2^m+1}^{2^{m+1}} |c_\nu|^2 \right\}^{\beta/2} .$$

Since $\beta/2 < 1$, by Jensen's inequality, the last term is less than

$$K \sum_{k=0}^{\infty} 2^{k(1-3\beta/2)} \left\{ \sum_{m=0}^{k} 2^{m\beta} \left(\sum_{|\nu|=2^m+1}^{2^{m+1}} |c_\nu|^2 \right)^{\beta/2} \right\}$$

$$= K \sum_{m=0}^{\infty} 2^{m\beta} \left(\sum_{|\nu|=2^m+1}^{2^{m+1}} |c_\nu|^2 \right)^{\beta/2} \sum_{k=m}^{\infty} 2^{k(1-3\beta/2)}$$

$$= K \sum_{m=0}^{\infty} 2^{m(1-\beta/2)} \left\{ \sum_{|\;|=2^m+1}^{2^{m+1}} |c_\nu|^2 \right\}^{\beta/2}$$

$$\leqq K \sum_{m=0}^{\infty} 2^{m(1-\beta/2)} \left\{ \sum_{|\nu|=2^m+1}^{\infty} |c_\nu|^2 \right\}^{\beta/2}$$

$$= K B'_\beta(f) .$$

Concerning the converse part, we proceed with the same method.

$$B'_\beta(f) = \sum_{k=0}^{\infty} 2^{k(1-\beta/2)} \left\{ \sum_{|\nu|=2^k+1}^{\infty} |c_\nu|^2 \right\}^{\beta/2}$$

$$= \sum_{k=0}^{\infty} 2^{k(1-\beta/2)} \left\{ \sum_{m=k}^{\infty} \sum_{|\nu|=2^m+1}^{2^{m+1}} |c_\nu|^2 \right\}^{\beta/2}$$

$$\leqq \sum_{k=0}^{\infty} 2^{k(1-\beta/2)} \left\{ \sum_{m=k}^{\infty} 2^{-2m} \left(\sum_{|\nu|=2^m+1}^{2^{m+1}} \nu^2 |c_\nu|^2 \right) \right\}^{\beta/2} .$$

By Jensen's inequality, we have

$$\leqq \sum_{k=0}^{\infty} 2^{k(1-\beta/2)} \left\{ \sum_{m=k}^{\infty} 2^{-m\beta} \left(\sum_{|\nu|=2^m+1}^{2^{m+1}} \nu^2 |c_\nu|^2 \right)^{\beta/2} \right\}$$

$$= \sum_{m=0}^{\infty} 2^{-m\beta} \left(\sum_{|\nu|=2^m+1}^{2^{m+1}} \nu^2 |c_\nu|^2 \right)^{\beta/2} \sum_{k=0}^{m} 2^{k(1-\beta/2)}$$

$$\leqq K \sum_{m=0}^{\infty} 2^{m(1-3\beta/2)} \left\{ \sum_{|\nu|=1}^{2^{m+1}} \nu^2 |c_\nu|^2 \right\}^{\beta/2}$$

$$= K C'_\beta(f) .$$

THEOREM 2. *For $1 \leqq \beta < 2$, the finiteness of $A_\beta(f)$ is equivalent to the finiteness of $B_\beta(f)$ or $C_\beta(f)$.*

PROOF. Since

$$\int_0^{2\pi} |f(x+t) - f(x-t)|^2\, dx = 4\pi \sum_{k=-\infty}^{\infty} |c_k|^2 \sin^2 kt\,,$$

we have

$$\begin{aligned}
A_\beta(f) &= \int_0^1 t^{-2+\beta/2} \left\{ \int_0^{2\pi} |f(x+t) - f(x-t)|^2\, dx \right\}^{\beta/2} dt \\
&= K \int_0^1 t^{-2+\beta/2} \left\{ \sum_{|k|=1}^{\infty} |c_k|^2 \sin^2 kt \right\}^{\beta/2} dt \\
&\leqq K \sum_{n=2}^{\infty} \int_{n^{-1}}^{(n-1)^{-1}} t^{-2+\beta/2} \left\{ \left(\sum_{|k|=1}^{n} |c_k|^2 k^2 t^2 \right)^{\beta/2} + \left(\sum_{k=n+1}^{\infty} |c_k|^2 \right)^{\beta/2} \right\} dt \\
&\leqq K \sum_{n=2}^{\infty} \left(\sum_{|k|=1}^{n} k^2 |c_k|^2 \right)^{\beta/2} \int_{n^{-1}}^{(n-1)^{-1}} t^{-2+3\beta/2}\, dt + K \sum_{n=2}^{\infty} \left(\sum_{|k|=n+1}^{\infty} |c_k|^2 \right)^{\beta/2} \int_{n^{-1}}^{(n-1)^{-1}} t^{-2+\beta/2}\, dt \\
&\leqq K \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n} |c_k|^2 k^2 \right)^{\beta/2} n^{-3\beta/2} + K \sum_{n=2}^{\infty} \left(\sum_{k=n+1}^{\infty} |c_k|^2 \right)^{\beta/2} n^{-\beta/2} \\
&\leqq K C_\beta(f) + K B_\beta(f)\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
A_\beta(f) &= \int_0^1 t^{-2+\beta/2} \left\{ \int_0^{2\pi} |f(x+t) - f(x-t)|^2\, dx \right\}^{\beta/2} dt \\
&= K \int^1 t^{-2+\beta/2} \left\{ \sum_{|k|=1}^{\infty} |c_k|^2 \sin^2 kt \right\}^{\beta/2} dt \\
&\geqq K \sum_{n=2}^{\infty} \int_{n^{-1}}^{(n-1)^{-1}} t^{-2+\beta/2} \left\{ \sum_{k=1}^{n} |c_k|^2 \sin^2 kt \right\}^{\beta/2} dt\,.
\end{aligned}$$

When $kt \leqq 1$, we have $\sin kt \geqq Akt$, and the last term is greater than

$$K \sum_{n=2}^{\infty} \int_{n^{-1}}^{(n-1)^{-1}} t^{-2+3\beta/2} \left\{ \sum_{|k|=1}^{n} k^2 |c_k|^2 \right\}^{\beta/2} dt \geqq K \sum_{n=2}^{\infty} n^{-3\beta/2} \left\{ \sum_{|k|=1}^{n} k^2 |c_k|^2 \right\}^{\beta/2}.$$

Thus the theorem is proved.

The equivalency of $A_\beta(f)<\infty$ to $B_\beta(f)<\infty$ is proved by Leindler [9] already and M. and S. Izumi [4] gave a simple proof for $A_\beta(f)\geqq KB_\beta(f)$, also.

3. Convergence of $\sum|c_n|^\beta$ and related problems. Stechkin proved very simply that $B(f)<\infty$ implies the absolute convergence of $\sum c_n$. The same proof gives the following theorem.

THEOREM 3. *If $B_\beta(f)<\infty$ for $1\leqq\beta<2$, then $\sum|c_n|^\beta<\infty$.*

PROOF. Without loss of generality, we can suppose that $f(x)$ is even. By Hölder's inequality,

$$\begin{aligned}\sum_{k=1}^\infty |c_k|^\beta &= \sum_{k=1}^\infty\sum_{n=1}^k \frac{|c_k|^\beta}{k}\\ &= \sum_{n=1}^\infty\sum_{k=n}^\infty \frac{|c_k|^\beta}{k}\\ &\leqq \sum_{n=1}^\infty\left\{\sum_{k=n}^\infty k^{-2/(2-\beta)}\right\}^{(2-\beta)/2}\left\{\sum_{k=n}^\infty(|c_k|^\beta)^{2/\beta}\right\}^{\beta/2}\\ &\leqq \sum_{n=1}^\infty n^{-\beta/2}\left(\sum_{k=n}^\infty |c_k|^2\right)^{\beta/2}\\ &= B_\beta(f)<\infty .\end{aligned}$$

THEOREM 4. *If $f(x)$ and $g(x)$ are continuous even function, of period 2π, with Fourier coefficient c_n and d_n, if $f(x)$ is a contraction of $g(x)$, and if $|d_n|\leqq\gamma_n$ where*

$$\sum_{n=1}^\infty n^{-3\beta/2}\left\{\sum_{k=1}^n k^2\gamma_k^2\right\}^{\beta/2}<\infty \tag{4}$$

or

$$\sum_{n=1}^\infty n^{-\beta/2}\left\{\sum_{k=n+1}^\infty \gamma_k^2\right\}^{\beta/2}<\infty \tag{5}$$

then $\sum|c_n|^\beta<\infty$.

PROOF. That $f(x)$ is a contraction of $g(x)$ means

$$|f(x)-f(y)| \leqq |g(x)-g(y)|.$$

Since (4) or (5) is equivalent to $A_\beta(f)<\infty$, Theorem is immediate from Theorem 3 and the form of $A_\beta(f)$.

Boas [3] and Kinukawa [5] proved Theorem 4 under the conditions (4) and (5).

COROLLARY. *Theorem 4 remains true when the hypothesis* (4) *or* (5) *is replaced by* $\gamma_n \downarrow 0$ *and* $\sum|\gamma_n|^\beta<\infty$.

Boas [3] and Konyushkov [7] proved that the hypothesis of corollary implies (4) and (5) and, equivalent to $A_\beta(f)<\infty$ by Theorem 2.

Kinukawa [6, 7] also discussed the problem of spectral synthesis under the conditions (4) and (5). However it is sufficient under (4) or (5) and this turns to $A_\beta(f)<\infty$ and reduces to Beurling's idea.

In particular, when $\beta=1$, as Beurling shows,

$$A_0=\{f|A(f)<\infty\}^{*)}$$

is an algebra. In fact, we have

$$\begin{aligned}
A(fg) &= \int_0^1 t^{-3/2}\left\{\int_0^{2\pi}|f(x+t)\,g(x+t)-f(x-t)\,g(x-t)|^2\,dx\right\}^{1/2}dt\\
&\leqq \int_0^1 t^{-3/2}\left\{\int_0^{2\pi}|f(x+t)\,g(x+t)-f(x-t)\,g(x+t)\right.\\
&\qquad\qquad \left.+f(x-t)\,g(x+t)-f(x-t)\,g(x-t)|^2\,dx\right\}^{1/2}dt\\
&\leqq \max_{0\leqq x\leqq 2\pi}|g(x)|\,A(f)+\max_{0\leqq x\leqq 2\pi}|f(x)|\,A(g)\\
&\leqq 2A(f)\,A(g),
\end{aligned}$$

because

$$\max_{0\leqq x\leqq 2\pi}|f(x)| \leqq \sum_{n=-\infty}^{\infty}|c_n| \leqq B(f) \leqq A(f).$$

The equivalency of $A(f)<\infty$, $B(f)<\infty$ and $C(f)<\infty$ gives the following inequalities between the formal products of Fourier coefficients.

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*) $A(f)$ means $A_\beta(f)$ when $\beta=1$ and $B(f)$, $C(f)$ are the same.

Let

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$g(x) \sim \sum_{n=-\infty}^{\infty} d_n e^{inx}$$

and

$$f(x)\, g(x) \sim \sum_{n=-\infty}^{\infty} \gamma_n e^{inx}$$

where

$$\gamma_n = \sum_{k=-\infty}^{\infty} c_k d_{n-k}.$$

THEOREM 5. *We have the following inequalities*

$$\sum_{n=1}^{\infty} n^{-1/2}\left(\sum_{|k|=n+1}^{\infty} |\gamma_k|^2\right)^{1/2}$$
$$\leqq K\left\{\sum_{n=1}^{\infty} n^{-1/2}\left(\sum_{|k|=n+1}^{\infty} |c_k|^2\right)^{1/2}\right\}\left\{\sum_{n=1}^{\infty} n^{-1/2}\left(\sum_{|k|=n+1}^{\infty} |d_k|^2\right)^{1/2}\right\}$$

and

$$\sum_{n=1}^{\infty} n^{-3/2}\left(\sum_{|k|=1}^{n} k^2 |\gamma_k|^2\right)^{1/2}$$
$$\leqq K\left\{\sum_{n=1}^{\infty} n^{-3/2}\left(\sum_{|k|=1}^{n} k^2 |c_k|^2\right)^{1/2}\right\}\left\{\sum_{n=1}^{\infty} n^{-3/2}\left(\sum_{|k|=1}^{n} k^2 |d_k|^2\right)^{1/2}\right\}$$

where K is a constant.

REMARK. Actually Beurling considers Fourier transforms. In Fourier integral case, $f(x)$ does not necessarily belong to the class $L^2(-\infty, \infty)$, but only

$$\int_0^x t^2\{|f(t)|^2 + |f(-t)|^2\}\, dt < \infty, \qquad \int_x^{\infty} \{|f(t)|^2 + |f(-t)|^2\}\, dt < \infty$$

for any fixed positive x. Hence calculation is somewhat troublesome, but we get analogous propositions.

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