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ON PARALLEL HYPERSURFACES OF AN ELLIPTIC HYPERSURFACE OF THE SECOND ORDER IN E^{n+1}

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In [1] M. Berger stated a theorem which is equivalent to the following:

Let M be a complete Riemannian manifold whose sectional curvature K satisfies the inequality

$$(1)\qquad 0 < A \leqq K(\Pi) \leqq B,$$

where A and B are positive constants and Π is any tangent plane to M. Let X be any Jacobi field along a geodesic $x = \gamma(s)$ parameterized with arclength s such that

$$(2)\qquad \|X(0)\| = 1,\quad X'(0) = 0,\ <X(0), \gamma'(0)> = 1,$$

then

$$(3)\qquad \|X(s)\| \leqq \cos\sqrt{A}\, s \quad \text{for} \quad 0 \leqq s \leqq \frac{\pi}{2\sqrt{B}}.$$

This statement is made sure of its truth in the case dim M=2 or M is locally symmetric, using their properties. Regarding this theorem, the author will investigate the curvature of the following elementary spaces which are generally non-symmetric.

An elliptic hypersurface Q of order 2:

$$(4)\qquad \sum_{\lambda=1}^{n+1} \frac{1}{a_\lambda^2} x_\lambda^2 = 1 \quad (a_1, \cdots, a_{n-1} > 0)^{1)}$$

in the $(n+1)$-dimensional Euclidean space E^{n+1} with the orthogonal coordinates $x_1, \cdots, x_{n+1}$, is, as well known, an n-dimensional compact Riemannian manifold with positive sectional curvature. The parallel hypersurface Q_c of Q which is the locus of the points with distance c from each point on

1) In this paper, Greek indices run from 1 to $n+1$ and Latin indices from 1 to n.

the normal inner half line through it has the same property as Q for a suitable value c. If Q is not a sphere, these parallel hypersurfaces are not symmetric. In this paper, the author will mainly prove the following theorems.

THEOREM A. *Let Q be an elliptic hypersurface given by* (4) *with* $0 < a_1 \leqq a_2 \leqq \cdots \leqq a_{n+1}$. *Then, for any constant c such that*

$$0 \leqq c \leqq \frac{a_1^2}{2a_{n+1}}, \tag{5}$$

the sectional curvature $\overline{K}(\Pi)$ of the parallel hypersurface Q_c of Q satisfies the inequality:

$$\frac{a_1^2}{(a_n^2 - ca_1)(a_{n+1}^2 - ca_1)} \leqq \overline{K}(\Pi) \leqq \frac{a_{n+1}^2}{(a_1^2 - ca_{n+1})(a_2^2 - ca_{n+1})}, \tag{6}$$

where Π is any tangent plane element of Q_c.

We call any section curve of Q or Q_c by the coordinate planes of E^{n+1} a *principal section.*

THEOREM B. *Along any principal section γ of Q or Q_c, where* $c \leqq \frac{a_1^2}{2a_{n+1}}$, *the inequality* (3) *holds for any Jacobi field X satisfying the condition* (2).

1. Parallel hypersurfaces of a convex hypersurface in E^{n+1}. Let Q be any closed hypersurface in E^{n+1}. At each point x of Q, we take all orthonormal $(n+1)$-frame $(x, e_1, \cdots, e_n, e_{n+1})$ of E^{n+1} such that $(x, e_1, \cdots, e_n)$ is an orthonormal n-frame of Q at x, $e = e_{n+1}$ is the inner unit normal vector of Q at x. The set of these $(n+1)$-frames is a submanifold B of the orthonormal frame bundle of E^{n+1}. On B, we have the 1-forms $\omega_1, \cdots, \omega_n$, $\omega_{\lambda\mu} = -\omega_{\mu\lambda}$, $\lambda, \mu = 1, 2, \cdots, n+1$, such that

$$\begin{cases} dx = \sum_i \omega_i e_i, \qquad de_i = \sum_j \omega_{ij} e_j + \omega_{in+1} e_{n+1}, \\ de_{n+1} = -\sum_i \omega_{in+1} e_i \end{cases} \tag{1.1}$$

and

$$\begin{cases} d\omega_i = \sum_j \omega_j \wedge \omega_{ji}, \qquad \sum_i \omega_i \wedge \omega_{in+1} = 0, \\ d\omega_{\lambda\mu} = \sum_\nu \omega_{\lambda\nu} \wedge \omega_{\nu\mu}. \end{cases} \tag{1.2}$$

From the second of (1.2), we put

$$\omega_{in+1} = \sum_j A_{ij}\omega_j, \quad A_{ij} = A_{ji} \tag{1.3}$$

and we have the 2nd fundamental form of Q

$$\Phi = \sum_i \omega_i \omega_{in+1} = \sum_{i,j} A_{ij}\omega_i \omega_j . \tag{1.4}$$

Now, for a constant c, we define Q_c by

$$\bar{x} = x + ce, \quad x \in Q, \tag{1.5}$$

then $(\bar{x}, e_1, \cdots, e_{n+1})$ is an orthonormal $(n+1)$-frame for the immersed submanifold Q_c as $(x, e_1, \cdots, e_{n+1})$ for Q. The set B_c of all $(\bar{x}, e_1, \cdots, e_{n+1})$ is also a submanifold of the orthonormal frame bundle of E^{n+1}, when Q_c is an imbedded submanifold. In such case, we may identify B_c with B by (1.5). Then, for Q_c we have from (1.1) and (1.2)

$$\begin{gathered} d\bar{x} = \sum_i \bar{\omega}_i e_i = \sum_i (\omega_i - c\omega_{in+1}) e_i, \\ \bar{\omega}_i = \omega_i - c\,\omega_{in+1} = \sum_j (\delta_{ij} - cA_{ij})\omega_j . \end{gathered} \tag{1.6}$$

The line element $d\bar{s}^2$ of Q_c is given by

$$d\bar{s}^2 = \sum_i \bar{\omega}_i\bar{\omega}_i = \sum_{ij} \Big(\delta_{ij} - 2cA_{ij} + c^2 \sum_k A_{ik}A_{kj}\Big)\omega_i\omega_j. \tag{1.7}$$

As (1.3), we put for Q_c

$$\omega_{in+1} = \sum_j \bar{A}_{ij}\bar{\omega}_j, \quad \bar{A}_{ij} = \bar{A}_{ji}, \tag{1.8}$$

then we get from (1.6)

$$A_{ij} = \bar{A}_{ij} - c\sum_k \bar{A}_{ik}A_{kj}. \tag{1.9}$$

In matrix form, (1.9) can be written as

(1.9′) $$A = \bar{A}(1-cA)$$

or

(1.10) $$\bar{A} = \frac{A}{1-cA}$$

except the case $1/c$ is one of the eigen values of A. The 2nd fundamental form $\bar{\Phi}$ of Q_c is given by

(1.11) $$\bar{\Phi} = \sum_i \omega_i \, \omega_{in+1} = \sum_{i,j} \bar{A}_{ij}\omega_i \omega_j$$

and we have from (1.6)

(1.12) $$\bar{\Phi} = \Phi - c\sum_i \omega_{in+1}\,\omega_{in+1} = \Phi - cd\sigma^2 ,$$

where $d\sigma^2$ denotes the line element of the spherical representation of $Q \subset E^{n+1}$ or the 3rd fundamental form of Q.

The components of the curvature tensor of Q at x with respect to the frame $(x, e_1, \cdots, e_n)$ are

(1.13) $$R_{ijhk} = A_{ik}A_{jh} - A_{ih}A_{jk}$$

and the ones of Q_c at $\bar{x} = x + ce_{n+1}$ with respect to $(\bar{x}, e_1, \cdots, e_n)$ are

(1.14) $$\bar{R}_{ijhk} = \bar{A}_{ik}\bar{A}_{jh} - \bar{A}_{ih}\bar{A}_{jk} \, .$$

Now, we take two orthogonal unit tangent vectors $X = \sum_i X_i e_i$ and $Y = \sum_i Y_i e_i$ at $\bar{x} \in Q_c$, then the sectional curvature for the tangent plane element Π of Q_c spanned by X and Y is given by

(1.15) $$\bar{K}(\Pi) = \bar{K}(X, Y) = \Big(\sum_{i,j} \bar{A}_{ij}X_iX_j\Big)\Big(\sum_{i,j} \bar{A}_{ij}Y_iY_j\Big) - \Big(\sum_{i,j} \bar{A}_{ij}X_iY_j\Big)^2 .$$

In the following, we will express the right hand side of (1.15) by X_i, Y_i, c and A_{ij}. For simplicity, for any vectors $X = \sum_i X_i e_i$, $Y = \sum_i Y_i e_i$, we introduce the notations as follows

$$<X, Y> = \sum_i X_iY_i, \quad \|X\| = \sqrt{<X, X>}, \quad A(X) = \sum_{i,j} A_{ij}X_je_i \, .$$

Then, by (1. 4), (1. 7) and (1. 12) we have

$$\bar{K}(\Pi) = P/G,$$

where

$$P = \{<A(X), X> - c\|A(X)\|^2\}\{<A(Y), Y> - c\|A(Y)\|^2\} - \{<A(X), Y> - c<A(X), A(Y)>\}^2 \tag{1.16}$$

and

$$G = \{\|X\|^2 - 2c<A(X), X> + c^2\|A(X)\|^2\} \times \{\|Y\|^2 - 2c<A(Y), Y> + c^2\|A(Y)\|^2\} - \{<X, Y> - 2c<A(X), Y> + c^2<A(X), A(Y)>\}^2. \tag{1.17}$$

In these equations, as well known, we may consider X and Y as independent vectors on the plane element Π. As in (1. 15), we assume

$$\|X\| = \|Y\| = 1, \quad <X, Y> = 0,$$

then we have

$$\begin{aligned} P = K(X, Y) &- c\{\|A(X)\|^2<A(Y), Y> + \|A(Y)\|^2<A(X), X> \\ &- 2<A(X), Y><A(X), A(Y)>\}. \\ &+ c^2\{\|A(X)\|^2\|A(Y)\|^2 - <A(X), A(Y)>^2\}, \end{aligned} \tag{1.16'}$$

$$\begin{aligned} G = 1 &- 2c\{<A(X), X> + <A(Y), Y>\} \\ &+ c^2\{\|A(X)\|^2 + \|A(Y)\|^2 + 4K(X, Y)\} \\ &- 2c^3\{\|A(X)\|^2<A(Y), Y> + \|A(Y)\|^2<A(X), X> \\ &- 2<A(X), Y><A(X), A(Y)>\} \\ &+ c^4\{\|A(X)\|^2\|A(Y)\|^2 - <A(X), A(Y)>^2\}, \end{aligned} \tag{1.17'}$$

where $K(X, Y)$ denotes the sectional curvature of Q at x corresponding to X and Y.

Lastly, we choose such a frame $(x, e_1, \cdots, e_n)$ that

$$A = (A_{ij}) = \begin{pmatrix} \alpha_1 & & & & 0 \\ & \alpha_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & \alpha_n \end{pmatrix},$$

then we have easily from (1.16) and (1.17)

$$(1.16\ )\qquad P=\sum(1-c\alpha_i)\,\alpha_i X_i^2\sum_j(1-c\alpha_j)\,\alpha_j Y_j^2-\Big(\sum_i(1-c\alpha_i)\,\alpha_i X_i Y_i\Big)^2$$

$$=\sum_{i<j}(1-c\alpha_i)(1-c\alpha_j)\,\alpha_i\alpha_j(X_iY_j-X_jY_i)^2,$$

$$(1.17'')\qquad G=\sum_i(1-2c\alpha_i+c^2\alpha_i^2)\,X_i^2\sum_j(1-2c\alpha_j+c^2\alpha_j^2)\,Y_j^2$$

$$-\Big\{\sum_i(1-2c\alpha_i+c^2\alpha_i^2)X_iY_i\Big\}^2$$

$$=\sum_{i<j}(1-c\alpha_i)^2(1-c\alpha_j)^2(X_iY_j-X_jY_i)^2.$$

Hence

$$(1.18)\qquad \overline{K}(X,Y)=\frac{\sum_{i<j}(1-c\alpha_i)(1-c\alpha_j)\,\alpha_i\alpha_j(X_iY_j-X_jY_i)^2}{\sum_{i<j}(1-c\alpha_i)^2(1-c\alpha_j)^2(X_iY_j-X_jY_i)^2}.$$

Here, assuming that Q is convex at x and

$$(1.19)\qquad \alpha_1\geqq\alpha_2\geqq\cdots\geqq\alpha_n,$$

then we have for

$$(1.20)\qquad 0\leqq c\leqq\frac{1}{2\alpha_1}$$

$$(1-c\alpha_1)(1-c\alpha_2)\,\alpha_1\alpha_2\geqq P\geqq(1-c\alpha_{n-1})(1-c\alpha_n)\,\alpha_{n-1}\alpha_n,$$

$$(1-c\alpha_1)^2(1-c\alpha_2)^2\leqq G\leqq(1-c\alpha_{n-1})^2(1-c\alpha_n)^2,$$

and

$$(1.21)\qquad \frac{\alpha_1\alpha_2}{(1-c\alpha_1)(1-c\alpha_2)}\geqq\overline{K}(\Pi)\geqq\frac{\alpha_{n-1}\alpha_n}{(1-c\alpha_{n-1})(1-c\alpha_n)},$$

where the both equalities hold for $X=e_1$, $Y=e_2$ and $X=e_{n-1}$, $Y=e_n$ respectively.

2. The range of the sectional curvature of parallel hypersurfaces of an elliptic hypersurface. In this section, we assume that Q is an elliptic hypersurface of the 2nd order in E^{n+1} given by (4) and $0<a_1\leqq a_2\leqq\cdots\leqq a_{n+1}$. At a point $x\in Q$, we take a unit tangent vector $X=\sum_i X_ie_i$, then for the

section of Q by the plane through x and parallel to the normal unit vector e_{n+1} and X, we have

$$< \frac{d^2x(s)}{ds^2}, e_{n+1}(s) >_{s=0} = - < X, \frac{de_{n+1}(s)}{ds} >_{s=0} = \sum_{i,j} A_{ij} X_i X_j, \tag{2.1}$$

where $x(s)$ denotes the point of the section, s is the arclength of the section measured from $x=x(0)$, and $e_{n+1}(s)$ is the unit inner normal vector at $x(s)$. The components of e_{n+1} are clearly

$$l_\lambda = -p(x)\frac{x_\lambda}{a_\lambda^2}, \tag{2.2}$$

where

$$p(x) = 1/\sqrt{\sum_\lambda \frac{x_\lambda^2}{a_\lambda^4}}. \tag{2.3}$$

Considering x in (2.2) as the coordinates of $x(s)$, we have

$$\frac{dl_\lambda}{ds} = -p(x)\frac{1}{a_\lambda^2}\frac{dx_\lambda}{ds} + l_\lambda \frac{d}{ds}\log p(x).$$

Since $\left(\frac{dx_\lambda}{ds}\right)_{s=0}$ are the components of X with respect to the canonical coordinates of E^{n+1}, we get from (2.1)

$$\sum_{i,j} A_{ij}X_iX_j = p(x)\sum_\lambda \frac{1}{a_\lambda^2}\left(\frac{dx_\lambda}{ds}\right)^2_{s=0}. \tag{2.4}$$

Denoting the length of the radius of Q with the same direction of X by $r(X)$, we have easily

$$(r(X))^2 \sum_\lambda \frac{1}{a_\lambda^2}\xi_\lambda^2 = 1, \tag{2.5}$$

where ξ_λ are the components of X with respect to the canonical coordinates of E^{n+1}. Hence, from (2.4), we have

$$\sum_{i,j} A_{ij}X_iX_j = \frac{p(x)}{(r(X))^2}. \tag{2.6}$$

The section of Q by the hyperplane through the center of Q and parallel to the tangent hyperplane at $x \in Q$ is also an elliptic hypersurface of the 2nd order in this hyperplane. We denote the principal radii of this section by

$$0 < r_1(x) \leqq r_2(x) \leqq \cdots \leqq r_n(x).$$

Since we may consider that the directions of these principal radii are orthogonal to each other, we choose a frame $(x, e_1, \cdots, e_n)$ such that $e_1, \cdots, e_n$ are parallel to these directions. Then, we have

$$\alpha_i = \frac{p(x)}{(r_i(x))^2}, \quad i = 1, 2, \cdots, n, \tag{2.7}$$

and

$$\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n > 0. \tag{2.8}$$

By means of (2. 7), (2. 8) and (1. 21), for

$$0 \leqq c \leqq \frac{r_1(x)^2}{2p(x)} \tag{2.9}$$

we have

$$\frac{p(x)^2}{(r_{n-1}(x)^2 - cp(x))(r_n(x)^2 - cp(x))} \leqq \bar{K}(\Pi) \leqq \frac{p(x)^2}{(r_1(x)^2 - cp(x))(r_2(x)^2 - cp(x))}, \tag{2.10}$$

where Π denotes any tangent plane element to Q_c at $\bar{x} = x + ce_{n+1}$.

Now, in connection with (2. 1), we take an auxiliary function

$$f(p) = \frac{p^2}{(\alpha^2 - cp)(\beta^2 - cp)}$$

of p, where α, β, c are constants such that $0 < \alpha \leqq \beta$, $0 \leqq c$. Then, we have easily

$$f'(p) = \frac{p\{2\alpha^2\beta^2 - cp(\alpha^2 + \beta^2)\}}{(\alpha^2 - cp)^2(\beta^2 - cp)^2}.$$

Hence, for

$$0 \leqq c \leqq \frac{2\alpha^2\beta^2}{a_{n+1}(\alpha^2 + \beta^2)} \tag{2.11}$$

$f(p)$ is a non-decreasing function of p in the interval $a_1 \leqq p \leqq a_{n+1}$. Thus, we get for c in (2. 11)

$$\frac{a_1{}^2}{(\alpha^2-ca_1)(\beta^2-ca_1)} \leqq \frac{p^2}{(\alpha^2-cp)(\beta^2-cp)} \leqq \frac{a_{n+1}{}^2}{(\alpha^2-ca_{n+1})(\beta^2-ca_{n+1})} \tag{2.12}$$

Let us come back to the situation in (2.10). We have

$$\min_{x\in Q} \frac{r_1(x)^2}{2p(x)} = \frac{a_1{}^2}{2a_{n+1}}. \tag{2.13}$$

On the other hand, we suppose that $r_1(x)$ and $r_2(x)$ correspond to two unit vectors X and Y with components ξ_λ and η_λ with respect to the canonical coordinates of E^{n+1} which are orthogonal to each other. Then, we have

$$\frac{1}{r_1(x)^2} + \frac{1}{r_2(x)^2} \geqq \frac{1}{r_{n-1}(x)^2} + \frac{1}{r_n(x)^2}. \tag{2.14}$$

From (2.5)

$$\begin{aligned}
&\frac{1}{a_1{}^2} + \frac{1}{a_2{}^2} - \frac{1}{r_1(x)^2} - \frac{1}{r_2(x)^2} \\
&= \frac{1}{a_1{}^2} + \frac{1}{a_2{}^2} - \sum_\lambda \frac{1}{a_\lambda^2}\xi_\lambda{}^2 - \sum_\lambda \frac{1}{a_\lambda^2}\eta_\lambda{}^2 \\
&\geqq \frac{1}{a_1{}^2}(1-\xi_1^2-\eta_1^2) + \frac{1}{a_2{}^2}(1-\xi_2^2-\eta_2^2) - \frac{1}{a_3{}^2}\sum_{3\leqq\lambda}(\xi_\lambda^2+\eta_\lambda^2) \\
&= \left(\frac{1}{a_1{}^2} - \frac{1}{a_3{}^2}\right)(1-\xi_1^2-\eta_1^2) + \left(\frac{1}{a_2{}^2} - \frac{1}{a_3{}^2}\right)(1-\xi_2^2-\eta_2^2) \geqq 0,
\end{aligned}$$

hence we have

$$\frac{1}{r_1(x)^2} + \frac{1}{r_2(x)^2} \leqq \frac{1}{a_1{}^2} + \frac{1}{a_2{}^2}, \tag{2.15}$$

making use of the relations $\sum_\lambda \xi_\lambda^2 = \sum_\lambda \eta_\lambda^2 = 1$, $\sum_\lambda \xi_\lambda\eta_\lambda = 0$. Analogously we have

$$\frac{1}{r_{n-1}(x)^2} + \frac{1}{r_n(x)^2} \geqq \frac{1}{a_n{}^2} + \frac{1}{a_{n+1}{}^2}. \tag{2.16}$$

Regarding (2.11) and (2.13), we have

$$\frac{a_1{}^2}{2a_{n+1}} \leqq \frac{a_1{}^2a_2{}^2}{a_{n+1}(a_1{}^2+a_2{}^2)} \leqq \frac{r_1(x)^2r_2(x)^2}{a_{n+1}\{r_1(x)^2+r_2(x)^2\}}. \tag{2.17}$$

Thus for

$$0 \leqq c \leqq \frac{a_1^2}{2a_{n+1}} \tag{2.18}$$

we have

$$\frac{a_1^2}{(r_{n-1}(x)^2-ca_1)(r_n(x)^2-ca_1)} \leqq \frac{p(x)^2}{(r_{n-1}(x)^2-cp(x))(r_n(x)^2-cp(x))} \leqq \overline{K}(\Pi) \tag{2.19}$$

and

$$\begin{aligned} \overline{K}(\Pi) &\leqq \frac{p(x)^2}{(r_1(x)^2-cp(x))(r_2(x)^2-cp(x))} \\ &\leqq \frac{a_{n+1}^2}{(r_1(x)^2-ca_{n+1})(r_2(x)^2-ca_{n+1})}. \end{aligned} \tag{2.20}$$

Making use of (2.15), we have

$$\begin{aligned} &(r_1(x)^2-ca_{n+1})(r_2(x)^2-ca_{n+1}) - (a_1^2-ca_{n+1})(a_2^2-ca_{n+1}) \\ &= r_1(x)^2r_2(x)^2 - a_1^2a_2^2 - ca_{n+1}\{r_1(x)^2+r_2(x)^2-a_1^2-a_2^2\} \\ &\geqq r_1(x)^2r_2(x)^2 - a_1^2a_2^2 - ca_{n+1}(r_1(x)^2+r_2(x)^2)\left(1-\frac{a_1^2a_2^2}{r_1(x)^2r_2(x)^2}\right) \\ &= \{r_1(x)^2r_2(x)^2-a_1^2a_2^2\}\left\{1-ca_{n+1}\left(\frac{1}{r_1(x)^2}+\frac{1}{r_2(x)^2}\right)\right\}. \end{aligned}$$

From (2.17) and (2.18), we have

$$1-ca_{n+1}\left(\frac{1}{r_1(x)^2}+\frac{1}{r_2(x)^2}\right) \geqq 0.$$

On the other hand, making use of the relations

$$\sum_\lambda \xi_\lambda^2 = \sum_\lambda \eta_\lambda^2 = 1, \quad \sum_\lambda \xi_\lambda \eta_\lambda = 0 = \sum_\lambda \frac{1}{a_\lambda^2}\xi_\lambda\eta_\lambda,$$

we have

$$\begin{aligned} \frac{1}{a_1^2a_2^2} - \frac{1}{r_1(x)^2r_2(x)^2} &= \frac{1}{a_1^2a_2^2} - \sum_\lambda \frac{1}{a_\lambda^2}\xi_\lambda^2 \sum_\mu \frac{1}{a_\mu^2}\eta_\mu^2 \\ &= \sum_{\lambda<\mu}\left(\frac{1}{a_1^2a_2^2} - \frac{1}{a_\lambda^2a_\mu^2}\right)(\xi_\lambda\eta_\mu - \xi_\mu\eta_\lambda)^2 \geqq 0, \end{aligned}$$

that is

$$r_1(x)^2\, r_2(x)^2 - a_1^{\,2} a_2^{\,2} \geqq 0\,.$$

Thus we have

$$(r_1(x)^2 - ca_{n+1})(r_2(x)^2 - ca_{n+1}) \geqq (a_1^{\,2} - ca_{n+1})(a_2^{\,2} - ca_{n+1})\,. \tag{2.21}$$

Analogously, we get

$$(r_{n-1}(x)^2 - ca_1)(r_n(x)^2 - ca_1) \leqq (a_n^{\,2} - ca_1)(a_{n+1}^{\,2} - ca_1)\,. \tag{2.22}$$

From (2.19), (2.20), (2.21) and (2.22), for c in (2.18) we get the inequality

$$\frac{a_1^{\,2}}{(a_n^{\,2} - ca_1)(a_{n+1}^{\,2} - ca_1)} \leqq \overline{K}(\Pi) \leqq \frac{a_{n+1}^{\,2}}{(a_1^2 - ca_{n+1})(a_2^2 - ca_{n+1})}\,.$$

It is clear that the left equality holds for some Π tangent to Q_c at $(a_1 - c, 0, \cdots, 0)$ and the right one holds for some Π tangent to Q_c at $(0, 0, \cdots, a_{n+1} - c)$. Thus, the proof of Theorem A is completed.

3. The Jacobi equation along a principal section of Q. Let Q be an elliptic hypersurface of the 2nd order in E^{n+1} given by (4). In the domain of Q such that $x_{n+1} \neq 0$, we regard $x_1, \cdots, x_n$ as local coordinates of it, then

$$x_{n+1} = \pm\, a_{n+1} F\,, \tag{3.1}$$

$$F = \sqrt{1 - \sum_i \frac{x_i^{\,2}}{a_i^{\,2}}}\,. \tag{3.2}$$

In the coordinates, the line element of Q:

$$ds^2 = \sum_\lambda dx_\lambda dx_\lambda = \sum_{i,j} g_{ij}\, dx_i\, dx_j$$

gives

$$g_{ij} = \delta_{ij} + \frac{a_{n+1}^{\,2}}{F^2} \frac{x_i x_j}{a_i^{\,2} a_j^{\,2}} \tag{3.3}$$

and

$$g^{ij} = \delta_{ij} - p(x)^2 \frac{x_i x_j}{a_i^{\,2} a_j^{\,2}}\,. \tag{3.4}$$

From (3. 3), we have

$$\frac{\partial g_{ij}}{\partial x_k} = \frac{a_{n+1}^2}{F^2} \cdot \frac{1}{a_i^2 a_j^2} (\delta_{ik} x_j + \delta_{jk} x_i) + \frac{2a_{n+1}^2}{F^4} \frac{x_i x_j x_k}{a_i^2 a_j^2 a_k^2}$$

and

$$\Gamma_{ij,k} \equiv \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{kj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k}\right)$$

$$= \frac{a_{n+1}^2}{2F^2}\left\{\frac{1}{a_i^2 a_k^2}(\delta_{ij} x_k + \delta_{kj} x_i) + \frac{1}{a_j^2 a_k^2}(\delta_{ij} x_k + \delta_{ki} x_j)\right.$$

$$\left. - \frac{1}{a_i^2 a_j^2}(\delta_{ki} x_j + \delta_{kj} x_i)\right\} + \frac{a_{n+1}^2}{F^4} \frac{x_i x_j x_k}{a_i^2 a_j^2 a_k^2}.$$

Thus, the Christoffel's symbols of Q in the coordinates are given by

$$\text{(3. 5)} \qquad \Gamma^l_{ij} = \sum_k g^{lk} \Gamma_{ij,k} = \frac{p(x)^2}{2} \delta_{ij} \left(\frac{1}{a_i^2} + \frac{1}{a_j^2}\right) \frac{x_l}{a_l^2} + \frac{p(x)^2}{F^2} \frac{x_i x_j x_l}{a_i^2 a_j^2 a_l^2}.$$

Along the principal section γ given by

$$\text{(3. 6)} \qquad x_2 = x_3 = \cdots = x_n = 0,$$

we have from (3. 5)

$$\Gamma^1_{11} = \frac{p(x)^2 x_1}{F^2 a_1^4}, \quad \Gamma^\alpha_{ij} = \Gamma^1_{\alpha 1} = 0, \qquad \alpha = 2, 3, \cdots, n,$$

and so the equations of parallel displacement of a tangent vector ξ with components ξ^i along γ in Q are

$$\frac{d\xi^1}{ds} + \Gamma^1_{11} \xi^1 \frac{dx_1}{ds} = 0, \quad \frac{d\xi^\alpha}{ds} = 0, \qquad \alpha = 2, 3, \cdots, n.$$ [2]

2) In general, the equations of parallel displacement of a tangent vector ξ along a curve $x_i = x_i(s)$ are

$$\frac{d\xi^i}{ds} + \sum_{j,k} \Gamma^i_{jk} \xi^j \frac{dx_k}{ds} = 0$$

and the equations of a geodesic are

$$\frac{d^2 x_i}{ds^2} + \sum_{j,k} \Gamma^i_{jk} \frac{dx_j}{ds} \frac{dx_k}{ds} = 0.$$

Hence the vector e_α with components δ^i_α are parallel displaced along γ. Since $g_{i\beta} = \delta_{i\beta}$ along γ, $e_2, e_3, \cdots, e_n$ are orthogonal to each other and to γ. The equations above imply also that γ is a geodesic.

On the other hand, for any vectors X, Y in E^{n+1} with components X_λ, Y_λ with respect to the canonical coordinates, we define

$$Q(X, Y) = \sum_\lambda \frac{1}{a_\lambda^2} X_\lambda Y_\lambda \,. \tag{3.7}$$

Then, for the 2nd fundamental form Φ of Q, we have easily from (2.5) and (2.6) the equality

$$\Phi(X, Y) = p(x)\, Q(X, Y)\,, \tag{3.8}$$

where X, Y are tangent to Q at x, that is

$$Q(x, X) = Q(x, Y) = 0\,,$$

regarding x as the position vector.

By means of (1.13) and (3.8), for the curvature tensor R of Q and tangent vectors X, Y, Z to Q at x, we have

$$\begin{aligned} <Y, R(Z, XZ)> &= \Phi(Z, Z)\, \Phi(X, Y) - \Phi(Z, X)\, \Phi(Z, Y) \\ &= p(x)^2 \{Q(Z, Z)\, Q(X, Y) - Q(Z, X)\, Q(Z, Y)\}\,. \end{aligned} \tag{3.9}$$

In general, the equations of a Jacobi field along a geodesic σ is

$$\frac{D}{ds}\frac{DX}{ds} + R\left(\frac{d\sigma}{ds}, X\frac{d\sigma}{ds}\right) = 0\,. \tag{3.10}$$

Along the principal section γ, we get easily

$$Q\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right) = \frac{1}{a_1^2}\left(\frac{dx_1}{ds}\right)^2 + \frac{1}{a_{n+1}^2}\left(\frac{dx_{n+1}}{ds}\right)^2 = \frac{1}{a_1^2 F^2}\left(\frac{dx_1}{ds}\right)^2,$$

$$Q\left(\frac{d\gamma}{ds}, X\right) = \frac{1}{a_1^2}\frac{dx_1}{ds} X_1 + \frac{1}{a_{n+1}^2}\frac{dx_{n+1}}{ds} X_{n+1} = \frac{1}{a_1^2 F^2}\frac{dx_1}{ds} X_1\,,$$

$$Q(X, Y) = \frac{1}{a_1^2 F^2} X_1 Y_1 + \sum_{\alpha=2}^{n} \frac{1}{a_\alpha^2} X_\alpha Y_\alpha$$

using (3. 1), (3. 2) and $Q(x, Y)=0$. Now putting $X^i=X_i$, we have

$$\frac{DX^1}{ds}=\frac{dX_1}{ds}+\Gamma^1_{11}X^1\frac{dx_1}{ds}=\frac{dX^1}{ds}+\frac{p(x)^2}{F^2}\frac{x_1}{a_1{}^4}X^1\frac{dx_1}{ds},\ \frac{DX^\alpha}{ds}=\frac{dX^\alpha}{ds},$$

$$\frac{D}{ds}\frac{DX^1}{ds}=\frac{d^2X^1}{ds^2}+2\frac{p(x)^2}{F^2}\frac{x_1}{a_1{}^4}\frac{dx_1}{ds}\frac{dX^1}{ds}+$$

$$+\frac{p(x)^4}{a_1{}^4F^4}\left\{\frac{1}{a_{n+1}{}^2}-\left(\frac{1}{a_1{}^2}-\frac{2}{a_{n+1}^2}\right)\frac{x_1^2}{a_1^2}+3\left(\frac{1}{a_1{}^2}-\frac{1}{a_{n+1}{}^2}\right)\frac{x_1^4}{a_1^4}\right\}\left(\frac{dx_1}{ds}\right)^2X^1,$$

$$\frac{D}{ds}\frac{DX^\alpha}{ds}=\frac{d^2X^\alpha}{ds^2},\qquad \alpha=2,3,\cdots,n.$$

From (3. 9), (3. 10) and the calculations above, the Jacobi's equations along γ are

$$(3.11)\quad\begin{cases}\dfrac{d^2X_1}{ds^2}+2\dfrac{p(x)^2}{F^2}\dfrac{x_1}{a_1{}^4}\dfrac{dx_1}{ds}\dfrac{dX_1}{ds}\\[2ex] \quad+\dfrac{p(x)^4}{a_1{}^4F^4}\left\{\dfrac{1}{a_{n+1}{}^2}-\left(\dfrac{1}{a_1{}^2}-\dfrac{2}{a_{n+1}{}^2}\right)\dfrac{x_1{}^2}{a_1{}^2}+3\left(\dfrac{1}{a_1{}^2}-\dfrac{1}{a_{n+1}{}^2}\right)\dfrac{x_1{}^4}{a_1{}^4}\right\}\left(\dfrac{dx_1}{ds}\right)^2X_1=0,\\[2ex] \dfrac{d^2X_\alpha}{ds^2}+\dfrac{p(x)^2}{a_1{}^2a_\alpha{}^2F^2}\left(\dfrac{dx_1}{ds}\right)^2X_\alpha=0,\qquad \alpha=2,3,\cdots,n.\end{cases}$$

The second part of (3. 11) shows that the Jacobi's equations have $(n-1)$ solutions $\underset{(\alpha)}{X}$ orthogonal to γ such that $\underset{(\alpha)}{X}/\|\underset{(\alpha)}{X}\|=e_\alpha$, $\alpha=2,3,\cdots,n$.

4. Proof of Theorem B. Firstly, we show that the principal section γ_c:

$$(4.1)\qquad x_2=x_3=\cdots=x_{n+1}=0$$

of Q_c is also a geodesic as γ in Q. Making use of the frame $(x, e_1,\cdots,e_n)$ along γ defined in §3, we have

$$\Phi(e_1,e_1)=p(x)Q\left(\frac{d\gamma}{ds},\frac{d\gamma}{ds}\right)=\frac{p(x)}{a_1{}^2F^2}\left(\frac{dx_1}{ds}\right)^2=\frac{p(x)^3}{a_1{}^2a_{n+1}{}^2},$$

$$\Phi(e_1,e_\alpha)=0,\ \Phi(e_\alpha,e_\beta)=p(x)\,Q(e_\alpha,e_\beta)=\frac{p(x)\delta_{\alpha\beta}}{a_\alpha a_\beta}.$$

With respect to this frame, we have along γ

$$(4.2) \qquad (A_{ij}) = \begin{pmatrix} \frac{p(x)^3}{a_1^2 a_{n+1}^2} & & & & 0 \\ & \frac{p(x)}{a_1^2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \frac{p(x)}{a_n^2} \end{pmatrix}.$$

Then, from (1.6) and (4.2), we have for γ_c

$$\frac{d\bar{x}}{ds} = \frac{dx}{ds} + c\frac{de_{n+1}}{ds} = \left\{1 - \frac{c\,p(x)^3}{a_1^2 a_{n+1}^2}\right\} e_1 ,$$

$$\frac{d^2\bar{x}}{ds^2} = \frac{d}{ds}\left\{1 - \frac{c\,p(x)^3}{a_1^2 a_{n+1}^2}\right\} e_1 + \left\{1 - \frac{c\,p(x)^3}{a_1^2 a_{n+1}^2}\right\} \frac{de_1}{ds}$$

$$\equiv \frac{d}{ds}\left\{1 - \frac{c\,p(x)^3}{a_1^2 a_{n+1}^2}\right\} e_1 \qquad (\mathrm{mod}\ e_{n+1}),$$

for γ is a geodesic and so

$$\frac{de_1}{ds} \equiv 0 \qquad (\mathrm{mod}\ e_{n+1}).$$

The equation above shows that γ_c is a geodesic of Q_c. Along γ_c, we have from the consideration in §3

$$\frac{de_\alpha}{ds} \equiv 0 \qquad (\mathrm{mod}\ e_{n+1}),$$

hence

$$(4.3) \qquad \frac{\bar{D}e_i}{ds} = 0, \qquad i = 1, 2, \cdots, n,$$

where $\bar{D}$ denotes the covariant differentiation of the space Q_c. On the other hand, with respect to the frame $(\bar{x}, e_1, \cdots, e_n)$, the matrix $(\bar{A}_{ij})$ is of a diagonal form by virtue of (1.10) and (4.2). Then, the Jacobi's equations along γ_c can be written as

$$\frac{d^2\bar{X}^i}{d\bar{s}^2} + \sum_k \bar{R}_{1ik1}\bar{X}^k = 0,$$

where $\bar{s}$ denotes the arc length of γ_c and $\bar{X} = \sum_i \bar{X}^i e_i$ and they turn into the following

$$(4.4) \qquad \frac{d^2\bar{X}^1}{d\bar{s}^2} = 0, \quad \frac{d^2\bar{X}^\alpha}{d\bar{s}^2} + \frac{p(x)^4\bar{X}^\alpha}{\{a_1{}^2a_{n+1}{}^2 - cp(x)^3\}\{a_\alpha{}^2 - cp(x)\}} = 0.$$

The second part of (4.4) shows that γ_c has $(n-1)$ Jacobi fields $\underset{(\alpha)}{\bar{X}}$ orthogonal to γ_c such that $\underset{(\alpha)}{\bar{X}}/\|\underset{(\alpha)}{\bar{X}}\| = e_\alpha$, $\alpha = 2, 3, \cdots, n$, which are also parallel along γ_c.

According to Theorem 1 in [3], the above circumstance along any principal section of Q_c follows that Theorem B is true for the principal section.

REFERENCES

[1] M. BERGER, An extension of Rauch's metric comparison theorem and some applications, Illinois Journ. Math., 61 (1962), 700–712.

[2] H. E. RAUCH, Geodesics and curvature in differential geometry in the large, New York, Yeshiba Univ., 1959.

[3] T. ŌTSUKI, A note on Jacobi fields of δ-pinched Riemannian manifolds, (to appear).

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