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A STUDY ON ALMOST CONTACT MANIFOLDS

KANJI MOTOMIYA

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Introduction. The notion of an almost contact structure (ϕ, ξ, η) was given by S. Sasaki [7]. The main purpose of this paper is to study certain almost contact manifolds which are similar to Kählerian manifolds.

Let M be an almost contact manifold with structure tensors (ϕ, ξ, η). We shall prove that there exists a linear connection such that ϕ, ξ and η are parallel with respect to it, and whose torsion tensor field T is given by:

$$T(X, Y) = 2d\eta(X, Y)\cdot\xi + \eta(Y)\phi X - \eta(X)\phi Y$$
$$- \frac{1}{4}\{\Phi(X, Y) - 2d\eta(\phi X, \phi Y)\cdot\xi + \eta(X)\phi(L_\xi\phi)Y - \eta(Y)\phi(L_\xi\phi)X\}$$

for all vector fields X and Y on M, where Φ is the Nijenhuis' tensor field of ϕ (Th. 1).

Let us consider the tensor field g of type (0, 2) on M given by:

$$g(X, Y) = -2d\eta(X, \phi Y) + \eta(X)\eta(Y)$$

for all vector fields X and Y on M. We shall say that the structure (ϕ, ξ, η) is non-degenerate if the tensor g is a (pseudo) Riemannian metric on M. Then, we shall prove that if the almost contact structure is non-degenerate, there exists a unique linear connection such that ϕ, ξ, η and g are parallel with respect to it, and whose torsion tensor field is the same tensor field as given in Th. 1 (Th. 2).

By Th. 2, it follows that if the non-degenerate almost contact structure on M is normal, then there exists a unique connection $\widetilde{\nabla}$ such that ϕ, ξ, η and g are parallel with respect to it, and whose torsion tensor field $\widetilde{T}$ is given by:

$$\widetilde{T}(X, Y) = 2d\eta(X, Y)\cdot\xi + \eta(Y)\phi X - \eta(X)\phi Y$$

for all vector fields X and Y on M (Th. 3, [6], [8]). This linear connection is closely related to the almost contact structure and will play an important role in the subsequent paper [4]. Non-degenerate normal almost contact manifolds are similar to Kählerian manifolds. We shall study the geometric properties on non-degenerate normal almost contact manifolds with respect to the linear connection $\widetilde{\nabla}$ obtained in Th. 3.

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1. On almost contact manifolds. In this paper the differentiability of manifolds and tensor fields means always the differentiability of class C^∞.

Let M be a differentiable manifold of dimension $2n+1$ $(n \geqq 1)$. We denote by $\mathfrak{X}(M)$ the Lie algebra of all vector fields on M. An almost contact structure on M is, by definition [7], a triple $\Sigma=(\phi, \xi, \eta)$, where ϕ is a tensor field of type $(1,1)$ on M, ξ is a vector field on M and η is a 1-form on M, which satisfies the following conditions:

$$\phi^2(X) = -X + \eta(X)\cdot\xi \qquad \text{for all} \quad X \in \mathfrak{X}(M)\,; \tag{1.1}$$

$$\eta(\xi) = 1\,. \tag{1.2}$$

We have the following equalities from equalities (1.1) and (1.2):

$$\phi(\xi) = 0\,; \tag{1.3}$$

$$\eta(\phi(X)) = 0 \qquad \text{for all} \quad X \in \mathfrak{X}(M)\,. \tag{1.4}$$

On any almost complex manifold, we have known that there exists a linear connection such that the almost complex structure is parallel with respect to it. We shall find such a connection on a manifold with (ϕ, ξ, η)-structure (cf. [8]).

THEOREM 1. *Let M be an almost contact manifold with structure tensors (ϕ, ξ, η). Then, there exists a linear connection such that the fundamental tensor fields ϕ, ξ and η are parallel with respect to it, and whose torsion tensor field $\widetilde{T}$ is given by*:

$$\begin{aligned}\widetilde{T}(X,Y) &= 2d\eta(X,Y)\cdot\xi + \eta(Y)\phi X - \eta(X)\phi Y \\ &\quad - \frac{1}{4}\{\Phi(X,Y) - 2d\eta(\phi X, \phi Y)\cdot\xi + \eta(X)\phi(L_\xi\phi)Y - \eta(Y)\phi(L_\xi\phi)X\}\end{aligned} \tag{1.5}$$

for all $X, Y \in \mathfrak{X}(M)$, *where* L_ξ *means the Lie derivation with respect to* ξ *and* Φ *is the Nijenhuis' tensor field of the tensor field* ϕ *of type* $(1,1)$, *i.e.*,

$$\Phi(X,Y) = -\phi^2[X,Y] + \phi[\phi X, Y] + \phi[X, \phi Y] - [\phi X, \phi Y]$$

for all $X, Y \in \mathfrak{X}(M)$.

LEMMA 1.1. *Let* M *be the same manifold as in Theorem* 1. *There exists a linear connection such that the two tensor fields* ξ *and* η *are parallel with respect to it, and whose torsion tensor field* $\overline{T}$ *is given by*:

$$\overline{T}(X,Y) = 2d\eta(X,Y)\cdot\xi \tag{1.6}$$

for all $X, Y \in \mathfrak{X}(M)$.

PROOF. We have known that there exists a torsion-free linear connection ∇ such that the vector field ξ is parallel with respect to it [3]. Let $\overline{A}$ be the tensor field of type $(1,2)$ given by:

$$\overline{A}(X)Y = (\nabla_X\eta)(Y)\cdot\xi \qquad \text{for all} \quad X, Y \in \mathfrak{X}(M)\,.$$

Making use of the above tensor field $\overline{A}$, we define a new linear connection $\overline{\nabla}$ by:

$$\overline{\nabla}_X Y = \nabla_X Y + \overline{A}(X)Y \qquad \text{for all} \quad X, Y \in \mathfrak{X}(M).$$

By condition (1.2) and the properties of the connection ∇, we have

$$(\nabla_X\eta)(\xi) = 0\,,$$

$$(\nabla_X\eta)(Y) - (\nabla_Y\eta)(X) = 2d\eta(X,Y) \qquad \text{for all} \quad X, Y \in \mathfrak{X}(M)\,.$$

Therefore, we have

$$\overline{\nabla}_X\xi = \nabla_X\xi + \overline{A}(X)\xi = (\nabla_X\eta)(\xi) = 0\,,$$

$$(\overline{\nabla}_X\eta)(Y) = (\nabla_X\eta)(Y) - \eta(\overline{A}(X)Y) = 0\,,$$

and

$$\overline{T}(X,Y) = \overline{A}(X)Y - \overline{A}(Y)X = \{(\nabla_X\eta)(Y) - (\nabla_Y\eta)(X)\}\cdot\xi = 2d\eta(X,Y)\cdot\xi\,.$$

Thus Lemma 1.1 is proved.

PROOF OF THEOREM 1. Let $\widetilde{A}$ be the tensor field of type (1, 2) given by:

$$\widetilde{A}(X)Y = (\overline{\nabla}_{\phi Y}\phi)X - (\overline{\nabla}_X\phi)\phi Y + \phi(\overline{\nabla}_X\phi)Y + \phi(\overline{\nabla}_Y\phi)X$$
$$- \eta(Y)\,\phi(\overline{\nabla}_\xi\phi)X - 4\eta(X)\phi Y$$

for all $X, Y \in \mathfrak{X}(M)$, where $\overline{\nabla}$ is the linear connection in Lemma 1.1. Then, we have the following four equalities.

(1.7) $\widetilde{A}(X)\,\xi = 0$;

(1.8) $\eta(\widetilde{A}(X)\,Y) = 0$;

(1.9) $\widetilde{A}(X)\,\phi Y - \phi(\widetilde{A}(X)\,Y) = 4(\widetilde{\nabla}_X\phi)\,Y$;

(1.10) $\widetilde{A}(X)Y - \widetilde{A}(Y)\,X = \Phi(X, Y) - 2d\eta(\phi X, \phi Y)\cdot\xi + \eta(X)\,\phi(L_\xi\phi)\,Y$
$- \eta(Y)\,\phi(L_\xi\phi)\,X + 4\eta(Y)\,\phi X - 4\eta(X)\,\phi Y$

for all $X, Y \in \mathfrak{X}(M)$. In fact, by equalities (1.1) ~ (1.4) and the properties of the connection $\overline{\nabla}$, we obtain

(1.11) $$(\overline{\nabla}_X\phi)\,\xi = 0;$$

(1.12) $$\eta((\overline{\nabla}_X\phi)\,Y) = 0;$$

(1.13) $$\phi(\overline{\nabla}_X\phi)\,Y + (\overline{\nabla}_X\phi)\,\phi Y = 0$$

for all $X, Y \in \mathfrak{X}(M)$. Therefore, (1.7), (1.8) follow from equalities (1.3), (1.4) and equalities (1.11), (1.12). Using equality (1.1) we have

$$\begin{aligned}
&\widetilde{A}(X)\,\phi Y - \phi(\widetilde{A}(X)\,Y)\\
&\quad = \{(\overline{\nabla}_{\phi^2 Y}\phi)\,X - (\overline{\nabla}_X\phi)\phi^2 Y + \phi(\overline{\nabla}_X\phi)\phi Y + \phi(\overline{\nabla}_{\phi Y}\phi)\,X - 4\eta(X)\,\phi^2 Y\}\\
&\qquad - \{\phi(\overline{\nabla}_{\phi Y}\phi)\,X - \phi(\overline{\nabla}_X\phi)\phi Y + \phi^2(\overline{\nabla}_X\phi)\,Y + \phi^2(\overline{\nabla}_Y\phi)\,X - \eta(Y)\,\phi^2(\overline{\nabla}_\xi\phi)\,X\\
&\qquad - 4\eta(X)\,\phi^2 Y\}\\
&\quad = 2(\overline{\nabla}_X\phi)\,Y + 2\phi(\overline{\nabla}_X\phi)\phi Y - \eta(Y)(\overline{\nabla}_X\phi)\,\xi - \eta((\overline{\nabla}_X\phi)\,Y)\cdot\xi - \eta((\overline{\nabla}_Y\phi)\,X)\cdot\xi\\
&\qquad + \eta(Y)\,\eta((\overline{\nabla}_\xi\phi)\,X)\cdot\xi\,.
\end{aligned}$$

By equalities (1.11) ~ (1.13), we obtain

$$\widetilde{A}(X)\,\phi Y - \phi(\widetilde{A}(X)Y) = 2(\overline{\nabla}_X\phi)\,Y - 2\phi^2(\overline{\nabla}_X\phi)\,Y$$
$$= 4(\overline{\nabla}_X\phi)\,Y - 2\eta((\overline{\nabla}_X\phi)Y)\cdot\xi$$
$$= 4(\overline{\nabla}_X\phi)\,Y\,.$$

$$\begin{aligned}\widetilde{A}(X)Y - \widetilde{A}(Y)X &= \{(\overline{\nabla}_{\phi Y}\phi)\,X - (\overline{\nabla}_X\phi)\phi Y + \phi(\overline{\nabla}_X\phi)Y + \phi(\overline{\nabla}_Y\phi)X \\ &\quad - \eta(Y)\phi(\overline{\nabla}_\xi\phi)\,X - 4\eta(X)\phi Y\} - \{(\overline{\nabla}_{\phi X}\phi)\,Y - (\overline{\nabla}_Y\phi)\phi X + \phi(\overline{\nabla}_Y\phi)\,X \\ &\quad + \phi(\overline{\nabla}_X\phi)\,Y - \eta(X)\phi(\overline{\nabla}_\xi\phi)\,Y - 4\eta(Y)\,\phi X\} \\ &= -\,\phi^2(\overline{\nabla}_X Y - \overline{\nabla}_Y X) + \phi(\overline{\nabla}_{\phi X}Y - \overline{\nabla}_Y\phi X) + \phi(\overline{\nabla}_X\phi Y - \overline{\nabla}_{\phi Y}X) \\ &\quad -(\overline{\nabla}_{\phi X}\phi Y - \overline{\nabla}_{\phi Y}\phi X) + \eta(X)\phi(\overline{\nabla}_\xi\phi)\,Y - \eta(Y)\,\phi(\overline{\nabla}_\xi\phi)\,X \\ &\quad - 4\eta(X)\,\phi Y + 4\eta(Y)\,\phi X \\ &= -\,\phi^2[X,Y] + \phi[\phi X, Y] + \phi[X,\phi Y] - [\phi X,\phi Y] \\ &\quad - \phi^2\cdot\overline{T}(X,Y) + \phi\cdot\overline{T}(\phi X,Y) + \phi\cdot\overline{T}(X,\phi Y) - \overline{T}(\phi X,\phi Y) \\ &\quad + \eta(X)\phi(\overline{\nabla}_\xi\phi)\,Y - \eta(Y)\,\phi(\overline{\nabla}_\xi\phi)\,X - 4\eta(X)\phi Y + 4\eta(Y)\phi X \\ &= \Phi(X,Y) - 2d\eta(\phi X,\phi Y)\cdot\xi + \eta(X)\,\phi(\overline{\nabla}_\xi\phi)\,Y - \eta(Y)\phi(\overline{\nabla}_\xi\phi)\,X \\ &\quad + 4\eta(Y)\,\phi X - 4\eta(X)\,\phi Y\,.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\phi(\overline{\nabla}_\xi\phi)\,X &= \phi\overline{\nabla}_\xi\phi X - \phi^2\overline{\nabla}_\xi X \\ &= \phi([\xi,\phi X]) - \phi^2([\xi,X]) + \phi\cdot\overline{T}(\xi,\phi X) - \phi^2\cdot\overline{T}(\xi,X) \\ &= \phi[\xi,\phi X] - \phi^2[\xi,X] = \phi(L_\xi\phi)\,X\,.\end{aligned}$$

Hence, (1.10) follows.

Now, making use of the tensor field $\widetilde{A}$ given above, we obtain the linear connection $\widetilde{\nabla}$ defined by:

$$\widetilde{\nabla}_X Y = \overline{\nabla}_X Y - \frac{1}{4}\widetilde{A}(X)Y \qquad \text{for all} \quad X,Y \in \mathfrak{X}(M)\,.$$

Then, since equalities (1.7) $\sim$ (1.10) hold, we can show that the linear connection $\widetilde{\nabla}$ satisfies the conditions in Theorem 1. Q.E.D.

2. On non-degenerate almost contact manifolds. Let M be an almost contact manifold with structure tensors (ϕ, ξ, η). We consider the tensor field g of type $(0, 2)$ on M defined by:

$$g(X, Y) = -2d\eta(X, \phi Y) + \eta(X)\eta(Y)$$

for all $X, Y \in \mathfrak{X}(M)$.

DEFINITION 1. *An almost contact structure (ϕ, ξ, η) on M is called non-degenerate if the tensor field g is a (pseudo) Riemannian metric on M, i.e., for each $p \in M$, g_p is a non-degenerate symmetric bilinear form on $T_p(M) \times T_p(M)$, where $T_p(M)$ is the tangent space at p.*

REMARK. If an almost contact structure (ϕ, ξ, η) is non-degenerate, the following two conditions hold:

(2.1) $$d\eta(\phi X, \phi Y) = d\eta(X, Y) \quad \text{for all} \quad X, Y \in \mathfrak{X}(M);$$

(2.2) η is a contact form, i.e., at each $p \in M$,
$$\underbrace{\eta_p \wedge (d\eta)_p \wedge \cdots \wedge (d\eta)_p}_{n\text{-times}} \neq 0, \quad \text{where} \quad \dim M = 2n+1.$$

In fact, (2.1) follows from the symmetry of g, and (2.2) follows from the fact that g is non-degenerate, and equalities (1.3) and (1.4). Conversely, if an almost contact structure (ϕ, ξ, η) satisfies conditions (2.1) and (2.2), the structure (ϕ, ξ, η) is non-degenerate.

On the other hand, we see that condition (2.1) is satisfied if the structure (ϕ, ξ, η) is normal, i.e.,

(2.3) $$\Phi(X, Y) - 2d\eta(X, Y) \cdot \xi = 0$$

for all $X, Y \in \mathfrak{X}(M)$, where Φ is the Nijenhuis' tensor field of the tensor field ϕ of type $(1, 1)$ [8].

Now, for the sake of simplicity, we denote by I the tensor field of type $(1,1)$ given by:

$$IX = X + (L_\xi \phi) X \qquad \text{for all} \quad X \in \mathfrak{X}(M).$$

THEOREM 2. *Let M be a non-degenerate almost contact manifold with structure tensors (ϕ, ξ, η). Then, there exists a unique linear connection $\tilde{\nabla}$ such that the fundamental tensor fields ϕ, ξ, η and g are parallel with respect to it, whose torsion tensor field $\tilde{T}$ is given by:*

$$\widetilde{T}(X, Y) = 2d\eta(X, Y)\cdot\xi + \eta(Y)\phi X - \eta(X)\phi Y$$
$$- \frac{1}{4}\{\Phi(X, Y) - 2d\eta(\phi X, \phi Y)\cdot\xi + \eta(X)\phi(L_\xi\phi)Y - \eta(Y)\phi(L_\xi\phi)X\}$$

for all $X, Y \in \mathcal{X}(M)$.

Let M be a non-degenerate almost contact manifold with structure tensors (ϕ, ξ, η). We have the Riemannian connection ∇ on M. With respect to the Riemannian connection ∇, we have

$$2g(\nabla_X Y, Z) = X\cdot g(Y, Z) + Y\cdot g(Z, X) - Z\cdot g(X, Y) + g([X, Y], Z) + g([Z, X]), Y) - g([Y, Z], X) \tag{2.4}$$

for all $X, Y, Z \in \mathcal{X}(M)$.

LEMMA 2.1. *Let M be a non-degenerate almost contact manifold with structure tensors (ϕ, ξ, η). We have*:

$$L_\xi\eta = 0; \tag{2.5}$$

$$(L_\xi g)(X, Y) = 2d\eta((L_\xi\phi)X, Y); \tag{2.6}$$

$$2\nabla_X\xi = -\phi\cdot IX; \tag{2.7}$$

$$(\nabla_X\eta)(Y) = d\eta(IX, Y) \tag{2.8}$$

for all $X, Y \in \mathcal{X}(M)$.

PROOF. By condition (2.1), we have

$$d\eta(\phi X, Y) + d\eta(X, \phi Y) = 0 \quad \text{for all} \quad X, Y \in \mathcal{X}(M).$$

This equation implies that

$$d\eta(X, \xi) = 0 \quad \text{for all} \quad X \in \mathcal{X}(M).$$

Hence (2.5) follows. Since the Lie derivation L_ξ commutes with the exterior differential d, (2.6) follows from (2.5) and the definition of g. Using equation (2.4), we obtain

$$2g(\nabla_X\xi, Z) = X\cdot g(\xi, Z) + \xi\cdot g(Z, X) - Z\cdot g(X, \xi) + g([X, \xi], Z) + g([Z, X], \xi) - g([\xi, Z], X)$$

$$= X \cdot \eta(Z) - Z \cdot \eta(X) - \eta([X, Z]) + (L_\xi g)(X, Z)$$
$$= 2d\eta(X, Z) + 2d\eta((L_\xi \phi) X, Z)$$
$$= g(-\phi X, Z) + g(-\phi(L_\xi \phi) X, Z) .$$

Hence, since g is non-degenerate, (2. 7) follows. Since $\nabla g = 0$ and $g(\xi, Y) = \eta(Y)$ for all $Y \in \mathcal{X}(M)$, we have

$$g(\nabla_X \xi, Y) = (\nabla_X \eta)(Y) \qquad \text{for all} \quad X, Y \in \mathcal{X}(M) .$$

Therefore, (2. 8) follows from (2. 7) and the definition of g. Q.E.D.

LEMMA 2.2. *Let M be a non-degenerate almost contact manifold with structure tensors (ϕ, ξ, η). Then, there exists a linear connection $\overline{\overline{\nabla}}$ such that the three tensor fields ξ, η and g are parallel with respect to it, whose torsion tensor field $\overline{T}$ is given by*:

$$\overline{T}(X, Y) = 2d\eta(X, Y) \cdot \xi + \frac{1}{2} \{\eta(Y) \phi \cdot IX - \eta(X) \phi \cdot IY\}$$

for all $X, Y \in \mathcal{X}(M)$.

PROOF. Let $\overline{A}$ be the tensor field of type (1, 2) given by:

$$\overline{A}(X) Y = \frac{1}{2} \{\eta(Y) \phi \cdot IX + 2d\eta(IX, Y) \cdot \xi\}$$

for all $X, Y \in \mathcal{X}(M)$. Making use of the Riemannian connection ∇, we have the linear connection $\overline{\overline{\nabla}}$ given by:

$$\overline{\overline{\nabla}}_X Y = \nabla_X Y + \overline{A}(X) Y \qquad \text{for all} \quad X, Y \in \mathcal{X}(M) .$$

By Lemma 2.1, we have

$$\overline{\overline{\nabla}}_X \xi = \nabla_X \xi + \overline{A}(X) \xi = \nabla_X \xi + \frac{1}{2} \phi \cdot IX = 0 ,$$

$$(\overline{\overline{\nabla}}_X \eta)(Y) = (\nabla_X \eta)(Y) - \eta(\overline{A}(X) Y) = (\nabla_X \eta)(Y) - d\eta(IX, Y) = 0 .$$

For all $X, Y, Z \in \mathcal{X}(M)$, we have

$$(\overline{\nabla}_Z g)(X, Y) = (\nabla_Z g)(X, Y) - g(\overline{A}(Z)X, Y) - g(X, \overline{A}(Z)Y)$$
$$= -\{g(\overline{A}(Z)X, Y) + g(X, \overline{A}(Z)Y)\}.$$

On the other hand, we obtain

$$g(\overline{A}(Z)X, Y) = \frac{1}{2}\eta(X)g(\phi\cdot IZ, Y) + g(d\eta(IZ, X)\cdot\xi, Y)$$
$$= -\eta(X)d\eta(IZ, Y) + \eta(Y)d\eta(IZ, X).$$

Hence, we obtain

$$g(\overline{A}(Z)X, Y) + g(X, \overline{A}(Z)Y) = 0.$$

Therefore, it follows that g is parallel with respect to the connection $\overline{\nabla}$. Since the Riemannian connection ∇ is torsion-free, we have

$$\overline{T}(X, Y) = \overline{A}(X)Y - \overline{A}(Y)X = \frac{1}{2}\{\eta(Y)\phi\cdot IX - \eta(X)\phi\cdot IY\}$$
$$+ d\eta(IX, Y)\cdot\xi - d\eta(IY, X)\cdot\xi$$
$$= \frac{1}{2}\{\eta(Y)\phi\cdot IX - \eta(X)\phi\cdot IY\} + 2d\eta(X, Y)\cdot\xi.$$

Thus Lemma 2.2 is proved.

PROOF OF THEOREM 2. We shall prove Theorem 2 in the same way that we proved Theorem 1. Let $\widetilde{A}$ be the tensor field given by:

$$\widetilde{A}(X)Y = (\overline{\nabla}_{\phi Y}\phi)X - (\overline{\nabla}_X\phi)\phi Y + \phi(\overline{\nabla}_X\phi)Y + \phi(\overline{\nabla}_Y\phi)X$$
$$- \eta(Y)\phi(\overline{\nabla}_\xi\phi)X + 2\eta(X)\phi Y$$

for all $X, Y \in \mathfrak{X}(M)$, where $\overline{\nabla}$ is the linear connection in Lemma 2.2. Making use of the tensor field $\widetilde{A}$ given above, we define a linear connection $\widetilde{\nabla}$ by:

$$\widetilde{\nabla}_X Y = \overline{\nabla}_X Y - \frac{1}{4}\widetilde{A}(X)Y \qquad \text{for all} \quad X, Y \in \mathfrak{X}(M).$$

Since ξ and η are parallel with respect to the linear connection $\overline{\nabla}$, it follows from the proof of Theorem 1 that the three tensor fields ϕ, ξ and η are parallel with respect to the linear connection $\widetilde{\nabla}$.

Next, we shall show that $\widetilde{\nabla} g = 0$. Since $\overline{\nabla} g = 0$, it is sufficient to prove the following equality:

$$g(\widetilde{A}(Z)X, Y) + g(X, \widetilde{A}(Z)Y) = 0 \tag{2.9}$$

for all $X, Y, Z \in \mathcal{X}(M)$. Since ξ, η and g are parallel with respect to the linear connection $\overline{\nabla}$, we obtain

$$g((\overline{\nabla}_Z\phi)X, Y) = -2(\overline{\nabla}_Z d\eta)(X, Y) \quad \text{for all} \quad X, Y, Z \in \mathcal{X}(M).$$

Therefore, we have

$$\begin{aligned}
&g(\widetilde{A}(Z)X, Y) + g(X, \widetilde{A}(Z)Y) \\
&= g((\overline{\nabla}_{\phi X}\phi)Z - (\overline{\nabla}_Z\phi)\phi X + \phi(\overline{\nabla}_Z\phi)X + \phi(\overline{\nabla}_X\phi)Z - \eta(X)\phi(\overline{\nabla}_\xi\phi)Z + 2\eta(Z)\phi X, Y) \\
&\quad + g((\overline{\nabla}_{\phi Y}\phi)Z - (\overline{\nabla}_Z\phi)\phi Y + \phi(\overline{\nabla}_Z\phi)Y + \phi(\overline{\nabla}_Y\phi)Z - \eta(Y)\phi(\overline{\nabla}_\xi\phi)Z + 2\eta(Z)\phi Y, X) \\
&= \{g((\overline{\nabla}_{\phi X}\phi)Z, Y) - g((\overline{\nabla}_Y\phi)Z, \phi X)\} + \{g((\overline{\nabla}_{\phi Y}\phi)Z, X) - g((\overline{\nabla}_X\phi)Z, \phi Y)\} \\
&\quad - 2\{g((\overline{\nabla}_Z\phi)X, \phi Y) + g(\phi Y, (\overline{\nabla}_Z\phi)Y)\} + 2\eta(Z)\{g(\phi X, Y) + g(\phi Y, X)\} \\
&\quad + \eta(X)g((\overline{\nabla}_\xi\phi)Z, \phi Y) - \eta(Y)g((\overline{\nabla}_\xi\phi)Z, \phi X) \\
&= 2\{\overline{\nabla}_{\phi X}(d\eta)(Y, Z) + \overline{\nabla}_Y(d\eta)(Z, \phi X)\} - 2\{\overline{\nabla}_{\phi Y}(d\eta)(Z, X) + (\overline{\nabla}_X d\eta)(\phi Y, Z)\} \\
&\quad - 2\eta(X)\overline{\nabla}_\xi(d\eta)(Z, \phi Y) + 2\eta(Y)\overline{\nabla}_\xi(d\eta)(Z, \phi X).
\end{aligned}$$

On the other hand, we have the following equality:

$$\overline{\nabla}_X(d\eta)(Y, Z) + \overline{\nabla}_Y(d\eta)(Z, X) + \overline{\nabla}_Z(d\eta)(X, Y) = 0 \tag{2.10}$$

for all $X, Y, Z \in \mathcal{X}(M)$. In fact, we obtain

$$\begin{aligned}
&\overline{\nabla}_X(d\eta)(Y, Z) + \overline{\nabla}_Y(d\eta)(Z, X) + \overline{\nabla}_Z(d\eta)(X, Y) \\
&= 3d(d\eta)(X, Y, Z) - d\eta(\overline{T}(X, Y), Z) - d\eta(\overline{T}(Y, Z), X) - d\eta(\overline{T}(Z, X), Y) \\
&= -\frac{1}{2}\{d\eta(\eta(Y)\phi\cdot IX - \eta(X)\phi\cdot IY, Z) + d\eta(\eta(Z)\phi\cdot IY - \eta(Y)\phi\cdot IZ, X) \\
&\quad + d\eta(\eta(X)\phi\cdot IZ - \eta(Z)\phi\cdot IX, Y)\}
\end{aligned}$$

$$= \eta(X)\{d\eta(\phi\cdot IY, Z) + d\eta(Y, \phi\cdot IZ)\} + \eta(Y)\{d\eta(Z, \phi\cdot IX) + d\eta(\phi\cdot IZ, X)\}$$
$$+ \eta(Z)\{d\eta(X, \phi\cdot IY) + d\eta(\phi\cdot IX, Y)\} = 0,$$

because it follows that

$$\phi\cdot(L_\xi\phi) + (L_\xi\phi)\phi = 0,$$

and $d\eta((L_\xi\phi)X, Y) + d\eta(X, (L_\xi\phi)Y) = 0$ for all $X, Y \in \mathfrak{X}(M)$. Hence, making use of equality (2.10), we obtain

$$\begin{aligned} g(\widetilde{A}(Z)X, Y) &+ g(X, \widetilde{A}(Z)Y) \\ &= -2\overline{\nabla}_Z(d\eta)(\phi X, Y) + 2\overline{\nabla}_Z(d\eta)(X, \phi Y) \\ &= g((\overline{\nabla}_Z\phi)\phi X, Y) - g((\overline{\nabla}_Z\phi)X, \phi Y) \\ &= g((\overline{\nabla}_Z\phi)\phi X + \phi(\overline{\nabla}_Z\phi)X, Y) = 0, \end{aligned}$$

which implies (2.9).

Now, we shall prove the following equality:

$$\begin{aligned} \widetilde{A}(X)Y - \widetilde{A}(Y)X &= \Phi(X, Y) - 2d\eta(X, Y)\cdot\xi \\ &- \eta(X)\phi(L_\xi\phi)Y + \eta(Y)\phi(L_\xi\phi)X + 2\eta(X)\phi Y - 2\eta(Y)\phi X \end{aligned} \tag{2.11}$$

for all $X, Y \in \mathfrak{X}(M)$. In fact, from the proof of Theorem 1, we have

$$\begin{aligned} \widetilde{A}(X)Y - \widetilde{A}(Y)X &= \Phi(X, Y) + \eta(X)\phi(L_\xi\phi)Y - \eta(Y)\phi(L_\xi\phi)X \\ &- \phi^2\cdot\overline{T}(X, Y) + \phi\cdot\overline{T}(\phi X, Y) + \phi\cdot\overline{T}(X, \phi Y) - \overline{T}(\phi X, \phi Y). \end{aligned}$$

On the other hand, we obtain

$$\begin{aligned} -\phi^2\cdot\overline{T}(X, Y) &+ \phi\cdot\overline{T}(\phi X, Y) + \phi\cdot\overline{T}(X, \phi Y) - \overline{T}(\phi X, \phi Y) \\ &= -\phi^2\{\eta(Y)\phi\cdot IX - \eta(X)\phi\cdot IY\} + \eta(Y)\phi^2\cdot I\cdot\phi X - \eta(X)\phi^2\cdot I\cdot\phi Y \\ &\quad - 2d\eta(\phi X, \phi Y)\cdot\xi \\ &= 2\eta(Y)\phi(L_\xi\phi)X - 2\eta(X)\phi(L_\xi\phi)Y - 2d\eta(X, Y)\cdot\xi, \end{aligned}$$

which implies (2.11). Since the torsion tensor field $\widetilde{T}$ of the linear connection $\widetilde{\nabla}$ is given by:

$$\widetilde{T}(X, Y) = \overline{T}(X, Y) - \frac{1}{4}(\widetilde{A}(X)Y - \widetilde{A}(Y)X) \qquad \text{for all} \quad X, Y \in \mathfrak{X}(M),$$

it follows from equality (2.11) that there exists the linear connection which satisfies the conditions in Theorem 2. Uniqueness of the linear connecion is proved in the same way that we prove uniqueness of the Riemannian connection. Thus Theorem 2 is proved.

REMARK. We have been orally communicated from N. Tanaka that every pseudo-convex hypersurface M in a complex manifold N admits an almost contact structure (ϕ, ξ, η) which safisfies the following conditions:

(a) (ϕ, ξ, η) is non-degenerate and the pseudo-Riemannian metric g is positive definite;
(b) $N^*(X, Y) = \Phi(X, Y) - 2d\eta(\phi X, \phi Y)\cdot\xi - \eta(X)\phi(L_\xi\phi)Y + \eta(Y)\phi(L_\xi\phi)X = 0$ for all $X, Y \in \mathfrak{X}(M)$ (cf. [9]).

It follows from Theorem 2 that the almost contact structure (ϕ, ξ, η) admits a unique connection $\tilde{\nabla}$ such that $\tilde{\nabla}\phi = \tilde{\nabla}\xi = \tilde{\nabla}\eta = \tilde{\nabla}g = 0$, and whose torsion tensor field $\tilde{T}$ is given by:

$$\tilde{T}(X, Y) = 2d\eta(X, Y)\cdot\xi + \eta(Y)\phi X - \eta(X)\phi Y$$
$$- \frac{1}{2}\{\eta(X)\phi(L_\xi\phi)Y - \eta(Y)\phi(L_\xi\phi)X\}$$

for all $X, Y \in \mathfrak{X}(M)$.

3. On non-degenerate normal almost contact manifolds. Let M be a non-degenerate normal almost contact manifold with structure tensors (ϕ, ξ, η), that is, a non-degenerate almost contact manifold such that the structure (ϕ, ξ, η) satisfies the condition (2.3) ([7]).

THEOREM 3. ([6], [8]). *Let M be a non-degenerate normal almost contact manifold with structure tensors (ϕ, ξ, η). Then there exists a unique linear connection $\tilde{\nabla}$ such that the fundamental tensor fields ϕ, ξ, η and g are parallel with respect to it, whose torsion tensor field $\tilde{T}$ is given by:*

$$\tilde{T}(X, Y) = 2d\eta(X, Y)\cdot\xi + \eta(Y)\phi X - \eta(X)\phi Y$$

for all $X, Y \in \mathfrak{X}(M)$.

PROOF. Since the (ϕ, ξ, η)-structure is normal, equailty (2.3) holds. Hence it follows that $L_\xi\phi = 0$. Therefore, Theorem 3 is proved from Theorem 2.

REMARK. Conversely, it follows that if there exists a linear connection on a non-degenerate almost contact manifold with structure tensors (ϕ, ξ, η) which satisfies the conditions in Theorem 3, then the (ϕ, ξ, η)-structure is normal.

A non-degenerate normal almost contact structure is similar to a Kählerian structure. We have the following equality which holds on Kählerian manifolds.

PROPOSITION 1. *Let M be a non-degenerate normal almost contact manifold with structure tensors (ϕ, ξ, η). Let $\widetilde{R}$ denote the curvature tensor field of the linear connection $\widetilde{\nabla}$ obtained in Theorem* 3. *Then we have*

$$\widetilde{R}(\phi X, \phi Y) = \widetilde{R}(X, Y)$$

for all $X, Y \in \mathfrak{X}(M)$.

To prove Proposition 1, we shall show that the following equalities hold.

LEMMA 3.1.

$$\phi \cdot \widetilde{R}(X, Y) = \widetilde{R}(X, Y) \cdot \phi\,; \tag{3.1}$$

$$\widetilde{R}(X, Y)\,\xi = 0\,; \tag{3.2}$$

$$\eta \cdot \widetilde{R}(X, Y) = 0 \tag{3.3}$$

for all $X, Y \in \mathfrak{X}(M)$.

PROOF. Since ϕ, ξ and η are parallel with respect to the linear connection $\widetilde{\nabla}$, we obtain the above three equalities.

LEMMA 3.2.

$$\mathfrak{S}\{\widetilde{R}(X, Y) Z\} = \mathfrak{S}\{-2d\eta(X, Y)\phi Z\} \tag{3.4}$$

for all $X, Y, Z \in \mathfrak{X}(M)$, *where* $\mathfrak{S}\{P(X, Y, Z)\}$ *denotes the sum of the expression* $P(X, Y, Z)$ *over the cyclic permutation* (X, Y, Z), (Y, Z, X) *and* (Z, X, Y).

PROOF. By $\widetilde{\nabla}\widetilde{T} = 0$ and the Bianchi's identity: for all $X, Y, Z \in \mathfrak{X}(M)$, $\mathfrak{S}\{\widetilde{R}(X, Y) Z\} = \mathfrak{S}\{\widetilde{T}(\widetilde{T}(X, Y), Z)\} + \mathfrak{S}\{(\widetilde{\nabla}_X T)(Y, Z)\}$, it is sufficient to show the following equality:

$$\mathfrak{S}\{\widetilde{T}(\widetilde{T}(X,Y),Z)\} = \mathfrak{S}\{-2d\eta(X,Y)\phi Z\} \tag{3.5}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Since $\widetilde{T}(X,Y) = 2d\eta(X,Y)\cdot\xi + \eta(Y)\phi X - \eta(X)\phi Y$, we obtain

$$\begin{aligned}\widetilde{T}(\widetilde{T}(X,Y),Z) &= 2d\eta(\widetilde{T}(X,Y),Z)\cdot\xi + \eta(Z)\phi\cdot\widetilde{T}(X,Y) - \eta(\widetilde{T}(X,Y))\cdot\phi Z \\ &= 2\{\eta(Y)\,d\eta(\phi X,Z)\cdot\xi - \eta(X)\,d\eta(\phi Y,Z)\cdot\xi\} \\ &\quad + 2\{\eta(Y)\,\eta(Z)\phi^2X - \eta(Z)\,\eta(X)\phi^2Y\} - 2d\eta(X,Y)\cdot\phi Z,\end{aligned}$$

which implies (3.5). Q.E.D.

LEMMA 3.3.

$$g(\widetilde{R}(X,Y)U,V) = -g(U,\widetilde{R}(X,Y)V); \tag{3.6}$$

$$\begin{aligned}&g(\widetilde{R}(X,Y)U,V) - g(\widetilde{R}(U,V)X,Y) \\ &\quad = -2d\eta(X,Y)\,g(\phi U,V) + 2d\eta(U,V)\,g(\phi X,Y)\end{aligned} \tag{3.7}$$

for all $X, Y, U, V \in \mathfrak{X}(M)$, *where* g *is the pseudo-Riemannian metric given in* §2.

PROOF. By $\widetilde{\nabla}g = 0$, we have

$$Z\cdot g(X,Y) = g(\widetilde{\nabla}_Z X,Y) + g(X,\widetilde{\nabla}_Z Y)$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Making use of this equality succesively, we obtain (3.6). By Lemma 3.2, we have

$$g(\mathfrak{S}\{\widetilde{R}(X,Y)\,U\},V) = g(\mathfrak{S}\{-2d\eta(X,Y)\,\phi U\},V),$$

$$g(\mathfrak{S}\{\widetilde{R}(Y,U)\,V\},X) = g(\mathfrak{S}\{-2d\eta(Y,U)\,\phi V\},X),$$

$$-g(\mathfrak{S}\{\widetilde{R}(U,V)\,X\},Y) = -g(\mathfrak{S}\{-2d\eta(U,V)\,\phi X\},Y),$$

and

$$-g(\mathfrak{S}\{\widetilde{R}(V,X)\,Y\},U) = -g(\mathfrak{S}\{-2d\eta(V,X)\,\phi Y\},U).$$

Summing up the four equalities and using equality (3.6), we have

$$2g(\widetilde{R}(X,Y)\,U,V) - 2g(\widetilde{R}(U,V)\,X,Y)$$
$$= -4\{d\eta(X,Y)\,g(\phi U,V) - d\eta(U,V)\,g(\phi X,Y)\}\,.$$

Hence (3. 7) follows. Q.E.D.

PROOF OF PROPOSITION 1. Making use of equality (3. 7) repeatedly, it follows from Lemma 3.1 that

$$\begin{aligned} &g(\widetilde{R}(\phi X,\phi Y)\,U,V) \\ &\quad= g(\widetilde{R}(U,V)\,\phi X,\phi Y) - 2\{d\eta(\phi X,\phi Y)\,g(\phi U,V) - d\eta(U,V)\,g(\phi^2 X,\phi Y)\} \\ &\quad= g(\phi\widetilde{R}(U,V)\,X,\phi Y) - 2\{d\eta(X,Y)\,g(\phi U,V) - d\eta(U,V)\,g(\phi X,Y)\} \\ &\quad= g(\widetilde{R}(U,V)\,X,Y) - 2\{d\eta(X,Y)\,g(\phi U,V) - d\eta(U,V)\,g(\phi X,Y)\} \\ &\quad= g(\widetilde{R}(X,Y)\,U,V)\,. \end{aligned}$$

Since g is a pseudo Riemannian metric, Proposition 1 is thereby proved.

A non-degenerate almost contact manifold with structure tensors (ϕ, ξ, η) has the pseudo Riemannian metric g which is related with the almost contact structure (ϕ, ξ, η). We have studied Riemannian geometric properties on non-degenerate normal contact manifolds (cf. [6], [8], $\cdots$).

Now we shall study relation between the linear connection $\widetilde{\nabla}$ obtained in Theorem 3 and the Riemannian connection ∇ with respect to the pseudo Riemmanian metric g. From the construction of the linear connection $\widetilde{\nabla}$, we have

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2} A(X)\,Y$$

for all $X, Y \in \mathfrak{X}(M)$, where A is the tensor field of type (1, 1) given by:

$$A(X)\,Y = 2d\eta(X,Y)\cdot\xi + \eta(X)\,\phi Y - \eta(Y)\,\phi X$$

for all $X, Y \in \mathfrak{X}(M)$ [6].

Let $\widetilde{R}$ (resp. R) denote the curvature tensor field of the linear connection $\widetilde{\nabla}$ (resp. ∇). We have

PROPOSITION 2.

$$\widetilde{R}(X,Y) = R(X,Y) + \frac{1}{4} B(X,Y)$$

for all $X, Y \in \mathfrak{X}(M)$, *where* B *is the tensor field of type* (1, 3) *given by*:

$$B(X,Y)Z = 2\{\eta(Y)\,d\eta(\phi X, Z) - \eta(X)\,d\eta(\phi Y, Z)\}\cdot\xi - \eta(Z)\,\eta(Y)X$$
$$+ \eta(Z)\,\eta(X)Y - 2d\eta(Y,Z)\cdot\phi X + 2d\eta(X,Z)\cdot\phi Y - 4d\eta(X,Y)\cdot\phi Z.$$

PROOF. Since $\widetilde{\nabla}A = 0$, it follows that, for all $X, Y \in \mathfrak{X}(M)$,

$$\begin{aligned}
\widetilde{R}(X,Y) &= \widetilde{\nabla}_X\widetilde{\nabla}_Y - \widetilde{\nabla}_Y\widetilde{\nabla}_X - \widetilde{\nabla}_{[X,Y]} \\
&= \widetilde{\nabla}_X\nabla_Y + \widetilde{\nabla}_X\Big(\frac{1}{2}A(Y)\Big) - \widetilde{\nabla}_Y\nabla_X - \widetilde{\nabla}_Y\Big(\frac{1}{2}A(X)\Big) - \widetilde{\nabla}_{[X,Y]} \\
&= \nabla_X\nabla_Y + \frac{1}{2}A(X)\,\nabla_Y + \frac{1}{2}A(\widetilde{\nabla}_X Y) + \frac{1}{2}A(Y)\widetilde{\nabla}_X \\
&\quad - \nabla_Y\nabla_X - \frac{1}{2}A(Y)\,\nabla_X - \frac{1}{2}A(\widetilde{\nabla}_Y X) - \frac{1}{2}A(X)\widetilde{\nabla}_Y \\
&\quad - \nabla_{[X,Y]} - A([X,Y]) \\
&= R(X,Y) + \frac{1}{4}\{A(A(X)Y) - A(A(Y)X) - A(X)A(Y) + A(Y)A(X)\}.
\end{aligned}$$

On the other hand, we have

$$A(A(X)Y)Z = 2\{\eta(Y)\,d\eta(\phi X, Z) - \eta(X)\,d\eta(\phi Y, Z)\}\cdot\xi$$
$$- \eta(Z)\,\eta(Y)X + \eta(Z)\,\eta(X)Y - 2d\eta(X,Y)\cdot\phi Z,$$

and

$$A(X)A(Y)Z = 2\{\eta(Z)\,d\eta(X, \phi Y) - \eta(Y)\,d\eta(X, \phi Z)\}\cdot\xi$$
$$+ \eta(X)\,\eta(Z)Y - \eta(X)\,\eta(Y)Z + 2d\eta(Y,Z)\cdot\phi X.$$

Hence we see that for all $X, Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned}
B(X,Y)Z &= A(A(X)Y)Z - A(A(Y)X)Z - A(X)A(Y)Z + A(Y)A(X)Z \\
&= 2\{\eta(Y)\,d\eta(\phi X, Z) - \eta(X)\,d\eta(\phi Y, Z)\}\cdot\xi \\
&\quad - \eta(Z)\,\eta(Y)X + \eta(Z)\,\eta(X)Y - 2d\eta(Y,Z)\cdot\phi X + 2d\eta(X,Z)\cdot\phi Y \\
&\quad - 4d\eta(X,Y)\cdot\phi Z.
\end{aligned}$$

This completes the proof of Proposition 2.

Let $\widetilde{S}$ (resp. S) be the Ricci tensor field of the linear connection $\widetilde{\nabla}$ (resp. ∇). We have

PROPOSITION 3. *For all* $X, Y \in \mathfrak{X}(M)$,

$$\widetilde{S}(X,Y) = S(X,Y) + \frac{1}{2}g(X,Y) + \frac{(n-1)}{2}\eta(X)\eta(Y),$$

where $\dim M = 2n+1$.

COROLLARY. $\widetilde{S}$ *is also symmetric, i.e.*,

$$\widetilde{S}(X,Y) = \widetilde{S}(Y,X) \qquad \textit{for all} \quad X,Y \in \mathfrak{X}(M).$$

PROPOSITION 4. *For all* $X, Y, Z \in \mathfrak{X}(M)$,

$$(\widetilde{\nabla}_Z\widetilde{R})(X,Y) = (\nabla_Z R)(X,Y) + A(Z)R(X,Y) - R(X,Y)A(Z) \\ - R(A(Z)X,Y) - R(X,A(Z)Y).$$

PROOF. Since $\widetilde{\nabla}B = 0$, we have

$$\begin{aligned}(\widetilde{\nabla}_Z\widetilde{R})(X,Y) &= (\widetilde{\nabla}_Z R)(X,Y) \\ &= [\widetilde{\nabla}_Z, R(X,Y)] - R(\widetilde{\nabla}_Z X, Y) - R(X, \widetilde{\nabla}_Z Y) \\ &= (\nabla_Z R)(X,Y) + [A(Z), R(X,Y)] - R(A(Z)X,Y) - R(X,A(Z)Y).\end{aligned}$$

Q.E.D.

COROLLARY. *If* $\widetilde{\nabla}\widetilde{R} = 0$, *then we have*

$$(\nabla_Z R)(X,Y) = [R(X,Y),A(Z)] + R(A(Z)X,Y) + R(X,A(Z)Y)$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES
KYOTO UNIVERSITY
KYOTO, JAPAN