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SASAKIAN MANIFOLDS WITH CONSTANT ϕ-HOLOMORPHIC SECTIONAL CURVATURE

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1. Introduction. In a previous paper [8] we classified almost contact Riemannian manifolds which admit automorphism groups of the maximum dimensions. In this note we clarify the situations $(i-1)\sim(i-3)$ of the main theorem of [8], especially, the spaces $S^{2n+1}[H]$, $E^{2n+1}[-3]$ and $(L, CD^n)[H]$.

In the last section we give examples of compact Sasakian manifolds which are not regular.

2. Preliminary. Let (ϕ, ξ, η, g) be an almost contact metric structure on a connected C^∞-manifold M. That is, they satisfy

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \eta(\xi) = 1, \tag{2.1}$$

$$\phi\phi X = -X + \eta(X)\xi, \tag{2.2}$$

$$g(\xi, X) = \eta(X), \qquad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

where X and Y are vector fields on M. If $d\eta(X, Y) = 2g(X, \phi Y)$ is satisfied, then M is called a contact Riemannian manifold. If ξ is a Killing vector field, M is called a K-contact Riemannian manifold. Then we have

$$\nabla_X \xi = -\phi X, \tag{2.4}$$

where ∇ is the Riemannian connection. If we have the relation

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \tag{2.5}$$

then M is called a Sasakian manifold. A Sasakian manifold is a K-contact Riemannian manifold.

Denote by $K(X_p, Y_p)$ the sectional curvature for 2-plane spanned by X_p and Y_p, $p \in M$. M is said to have constant ϕ-holomorphic sectional curvature if $K(X_p, \phi X_p)$ is constant for any point p and for any $X_p \neq 0$ such that $\eta(X) = 0$. A Sasakian manifold M has constant ϕ-holomorphic sectional curvature H if and only if the Riemannian curvature tensor R satisfied ([2])

$$4R^a{}_{bcd} = (H+3)(\delta^a_d g_{bc} - \delta^a_c g_{bd}) + (H-1)(\eta_b\eta_d\delta^a_c + \xi^a\eta_c g_{bd} - \xi^a\eta_d g_{bc} - \eta_b\eta_c\delta^a_d - \phi^a_c\phi_{bd} + \phi^a_d\phi_{bc} - 2\phi^a_b\phi_{cd}). \tag{2.6}$$

3. Model spaces of Sasakian manifolds with constant ϕ-holomorphic sectional curvature. (i) $S^{2n+1}[H]$, $H > -3$. Let S^{2n+1} be the unit hypersphere in a Euclidean space E^{2n+2}. Consider $x \in S^{2n+1}$ as a unit vector from the origin to the point x and denote by J the natural complex structure of $E^{2n+2} = CE^{n+1}$. We consider $\xi = Jx$ as a tangent vector at x to S^{2n+1}. Let g be the metric on S^{2n+1} induced from the Euclidean metric in E^{2n+2}. Then g and ξ determine η and ϕ by $\eta = g(\xi,\cdot)$ and $d\eta(X,Y) = 2g(X,\phi Y)$. The structure defined above is Sasakian ([5], [6]).

Now consider the following deformed structure:

$$\phi^* = \phi, \quad \xi^* = \alpha^{-1}\xi,$$
$$\eta^* = \alpha\eta, \ g^* = \alpha g + (\alpha^2 - \alpha)\eta\otimes\eta,$$

where $\alpha = 4/(H+3) > 0$. We call this deformation D-homothetic deformation. Then $(\phi^*, \xi^*, \eta^*, g^*, \alpha)$ is a Sasakian structure with constant ϕ-holomorphic sectional curvature $H > -3$ (cf. [7], p. 709) and we denote S^{2n+1} with this structure by $S^{2n+1}[H]$. By (12. 1) and Lemma 6. 4 in [7], $S^{2n+1}[H]$ is δ-pinched: $\delta = H$, if $-3 < H < 1$ (and $\delta = H^{-1}$, if $H > 1$).

(ii) $E^{2n+1}[-3]$. Let $(x^1, \cdots, x^n, y^1, \cdots, y^n, z)$ be the natural coordinate system of E^{2n+1}. Then ξ, η, g and ϕ defined by

$$\xi = (0, \cdots, 0, 2),$$
$$2\eta = (-y^1, \cdots, -y^n, 0, \cdots, 0, 1),$$

$$g: \begin{cases} 4g_{\alpha\beta} = \delta_{\alpha\beta} + y^\alpha y^\beta, \quad 4g_{\alpha^*\beta^*} = \delta_{\alpha\beta}, \\ 4g_{\alpha\triangle} = 4g_{\triangle\alpha} = -y^\alpha, \quad 4g_{\triangle\triangle} = 1, \\ \text{the other types of components } = 0, \end{cases}$$

$$\phi: \begin{cases} \phi^\alpha_{\beta^*} = \delta^\alpha_\beta, \qquad \phi^{\alpha^*}_\beta = -\delta^\alpha_\beta, \qquad \phi^\triangle_{\beta^*} = y^\beta, \\ \text{the other types of components } = 0, \end{cases}$$

define a Sasakian structure on E^{2n+1}, where α, $\beta \in (1, \cdots, n)$ and $\alpha^* = \alpha + n$, etc. The Riemannian curvature tensor R has the following components (cf. [4], [5]):

$$
\begin{aligned}
R^{\alpha}_{\beta\gamma\delta} &= (-\delta_{\alpha\gamma}y^{\beta}y^{\delta}+\delta_{\alpha\delta}y^{\beta}y^{\gamma})/4,\\
R^{\alpha^*}_{\beta\gamma\delta^*} &= (-2\delta_{\alpha\beta}\delta_{\gamma\delta}-\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}y^{\beta}y^{\gamma})/4,\\
R^{\triangle}{}_{\beta\gamma\triangle} &= (y^{\beta}y^{\gamma}+\delta_{\beta\gamma})/4,\\
R^{\alpha}_{\beta^*\gamma^*\delta} &= -(2\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta})/4,\\
R^{\triangle}{}_{\beta^*\gamma^*\triangle} &= (\delta_{\beta\gamma})/4,\\
R^{\alpha}_{\beta\triangle\delta} &= R^{\alpha^*}{}_{\beta\triangle\delta^*} = -(\delta_{\alpha\delta}y^{\beta})/4,\\
R^{\alpha}{}_{\triangle\triangle\delta} &= R^{\alpha^*}{}_{\triangle\triangle\delta^*} = (\delta^{\alpha}_{\delta})/4,
\end{aligned}
$$

the other types of components $=0$.

Now it is checked that the relation (2.6) holds for $H=-3$. We denote E^{2n+1} with this structure by $E^{2n+1}[-3]$.

(iii) $(L, CD^n)[H]$, $H<-3$. Let (J, G) be a Kählerian structure of a simply connected homogeneous complex domain CD^n with constant holomorphic sectional curvature $k<0$. Since the fundamental 2-form W is a closed form, we have a real analytic 1-form w (not necessarily unique) such that $W=dw$. (In fact, since W is real analytic and closed, we have an open set U_1 and a real analytic 1-form w_1 on U_1 such that $W=dw_1$ on U_1. Let U_2 be another open set on which we have a real analytic 1-form w_2 such that $W=dw_2$ on U_2. Assume that $U_1\cap U_2$ is non-empty and simply connected. Then, as w_1-w_2 is a closed form, we have a real analytic function f on $U_1\cap U_2$ such that $w_1-w_2=df$ on $U_1\cap U_2$. f is extendable to a real analytic function f on U_2 and w_2+df on U_2 is the extension of w_1 on $U_1\cap U_2$. Since CD^n is an open disk, w_1 is uniquely extendable to w on CD^n.) Then we have a 1-form $\eta=2w+dt$ on a product space $L\times CD^n$, L being a real line with coordinate t. If we consider L as an additive group, then η is an infinitesimal connection form on the product bundle (L, CD^n). We have $\xi=\partial/\partial t$ and $g=\pi^*G+\eta\otimes\eta$ where $\pi:(L, CD^n)\to CD^n$ is the projection. η is written also as $\eta=2\pi^*w+dt$. And we have $d\eta=2\pi^*W$. Therefore, these tensors define a Sasakian structure on (L, CD^n) with constant ϕ-holomorphic sectional curvature H, where $H=k-3<-3$ (cf. [3]). We denote this space by $(L, CD^n)[H]$.

Three types of model spaces above are all real analytic and the structure tensors are also real analytic. Furthermore, spaces are simply connected and complete.

4. Uniqueness. We show that three types of model spaces are unique up to isomorphisms, where an isomorphism means a C^∞-diffeomorphism which maps the structure tensors into the corresponding structure tensors.

PROPOSITION 4.1. *Let M^{2n+1} be a complete and simply connected (C^∞-) Sasakian manifold with constant ϕ-holomorphic sectional curvature H.*

(i) *If $H > -3$, M is isomorphic to $S^{2n+1}[H]$;*

or M is D-homothetic to $S^{2n+1}[1]$;

(ii) *If $H = -3$, M is isomorphic to $E^{2n+1}[-3]$;*

(iii) *If $H < -3$, M is isomorphic to $(L, CD^n)[H]$.*

PROOF. Since M is of constant ϕ-holomorphic sectional curvature H, M admits local ϕ-holomorphic free mobility ([2]). Since M is complete and simply connected, M admits global ϕ-holomorphic free mobility, that is, M admits an automorphism group $\mathrm{Aut}(M)$ such that, for any points p and q, any ϕ-holomorphic plane at p is carried to any other ϕ-holomorphic plane at q by some element of $\mathrm{Aut}(M)$. $\mathrm{Aut}(M)$ is of $(n+1)^2$ dimension. Especially, M is (C^∞-) diffeomorphic to a homogeneous space $\mathrm{Aut}(M)/(\text{isotropy group})$ and hence we can assume that M is real analytic, and also that g is real analytic. Denote by $^*M$ one of the model spaces corresponding to $H > -3$, $= -3$ or < -3, and denote by $(^*\phi, {}^*\xi, {}^*\eta, {}^*g)$ the structure tensors. For arbitrary points p of M and $^*p$ of $^*M$, let $(e_1, \cdots, e_n, \phi e_1, \cdots, \phi e_n, \xi)$ and $(^*e_1, \cdots, {}^*e_n, {}^*\phi^*e_1, \cdots, {}^*\phi^*e_n, {}^*\xi)$ be orthonormal ϕ-basis at p and $^*p$, respectively. We define a linear isomorphism F of the tangent space at p to M onto that at $^*p$ to $^*M$ by $Fe_\alpha = {}^*e_\alpha$, $F\phi e_\alpha = {}^*\phi^*e^\alpha$ $(\alpha = 1, \cdots, n)$ and $F\xi = {}^*\xi$. Then we have $F\phi = {}^*\phi F$ and F is isometric at p. That is, F is isomorphic at p. Since both ϕ- and $^*\phi$-holomorphic sectional curvatures are equal to H, F maps R into $^*R$ by (2.6), F being considered as a map of tensor algebras. The covariant derivatives of ϕ and ξ are also written in terms of ϕ, ξ, g by (2.4) and (2.5). Consequently, the covariant derivative of R is expressed by ϕ, ξ and g. That is, we see that F maps the tensor $(\nabla R)_p$ into the tensor $(^*\nabla^*R)_{*p}$. Likewise, we see that F maps the tensors $(\nabla^k R)_p$ into the tensors $(^*\nabla^{k*}R)_{*p}$ for every positive integer k. Then we have an isometry f of M onto $^*M$ such that $f(p) = {}^*p$ and the differential of f at p is F (cf. [1], p. 259-261). By (2.4) and $F\phi = {}^*\phi\, F$ we see that $(\nabla\xi)_p$ is mapped to $(^*\nabla^*\xi)_{*p}$. Thus, we have

$$(^*\nabla(f\xi))_{*p} = f\cdot(\nabla\xi)_p = F\cdot(\nabla\xi)_p = (^*\nabla^*\xi)_{*p}.$$

Since f is an isometry, $f\xi$ is also a Killing vector field. By $(f\xi)_{*p} = {}^*\xi_{*p}$ and

$(^*\nabla(f\xi))_{*p}=(^*\nabla{}^*\xi)_{*p}$, we have $f\xi={}^*\xi$ on $^*M$. Because ϕ and η ($^*\phi$ and $^*\eta$, resp.) are determined by g and ξ($^*g$ and $^*\xi$, resp.), f is an isomorphism between two Sasakian manifolds M and $^*M$.

REMARK 4.2. Even if (the space of) a complete and simply connected Sasakian manifold with constant ϕ-holomorphic sectional curvature is real analytic, the structure tensors are not generally real analytic. For example, in the construction of $(L, CD^n)[H]$, let h be a non-real analytic C^∞-function and replace w by $w'=w+dh$. Then we have η' (or g') which is C^∞ and is not real analytic. Therefore analyticity of the space we need in the proof is analyticity as a homogeneous space Aut(M)/(isotropy group).

5. Complete Sasakian manifolds with constant $H>-3$.

PROPOSITION 5.1. *Every complete Sasakian manifold with constant ϕ-holomorphic sectional curvature $H>-3$ is obtained by a D-homothetic deformation of a complete Riemannian manifold of constant curvature* 1. *Conversely, every odd dimensional complete Riemannian manifold of constant curvature* 1 *is a Sasakian manifold.*

PROOF. For the first part, see [7], p. 715. For the second part, we apply J. A. Wolf's result [10] that every odd dimensional complete Riemannian manifold M of constant curvature 1 inherits a contact structure η'' from η on $S^{2n+1}[1]$. That is, η'' and the induced metric g'' from g of $S^{2n+1}[1]$ define a Sasakian structure on M. Q. E. D.

J. A. Wolf [9] classified (odd dimensional) homogeneous Riemannian manifolds $M=S^{2n+1}/\Gamma$ of constant curvature 1:

(a) *If* $(2n+1)+1=2r$(r : *odd*), *then*

$$S^{2n+1}=\{(z^1,\cdots,z^r)\in C^r;\ |z^1|^2+\cdots+|z^r|^2=1\}$$

and Γ is a finite group of matrices of the form λI_r, where $\lambda\in C$ with $|\lambda|=1$ and I_r is the $r\times r$ identity matrix;

(b) *If* $(2n+1)+1=4r$, *then*

$$S^{2n+1}=\{(q^1,\cdots,q^r)\in Q;\ |q^1|^2+\cdots+|q^r|^2=1\}$$

where Q is the field of quaternions and Γ is a finite group of matrices of the form ρI_r, where $\rho\in Q$ with $|\rho|=1$.

Conversely, if Γ is a finite group of the type described in (a) and (b), then $M=S^{2n+1}/\Gamma$ is homogeneous.

Therefore, a homogeneous Sasakian manifold with constant ϕ-holomorphic sectional curvature $H > -3$ is D-homothetic to one of the spaces of the type (a) or (b).

The converse is not true in general, as is seen in Remark 6.1.

6. Examples of compact Sasakian manifolds which are not regular. An almost contact manifold M is said regular if any point of M has a neighborhood U such that each trajectory of ξ through $p \in U$ meets U once as a slice. Otherwise we say that M is not regular.

Consider S^3 defined by

$$S^3 = \{(q = a+bi+cj+dk) \in Q \; ; \; |q| = 1\}.$$

For a point $q = (a, b, c, d) \in S^3$, we consider q as a vector. On the other hand, the structure of $S^3[1]$ is induced from the (almost) complex structure J: $(a, b, c, d) \to (-b, a, -d, c)$, where $(a, b, c, d) \in E^4 = CE^2$. If we define a map $\varphi : S^3[1] \to S^3$ (or $\varphi : E^4 \to Q$) by $\varphi(a, b, c, d) = (a+bi+dj+ck)$, then we have a transformation J^* of S^3 or Q so that $J^*\varphi = \varphi J$. That is, $J^*(a, b, c, d) = (-b, a, d, -c)$ and the action of J^* is just J^*: $q \to qi$. Since ξ^* at q is determined by J^*q, ξ^* at $q = (a, b, c, d)$ is a vector which has the components $(-b, a, d, -c)$.

Let $\rho = (\alpha, \beta, \gamma, \delta) \in Q$ with $\rho^s = 1$ for some integer s, and let Γ be a finite group generated by ρI. Every trajectory of ξ^* is given by the intersection of S^3 and 2-plane spanned by q and $J^*q = \xi^*$. Since $(\rho I) J^* q = \rho q i = J^*(\rho I) q$, for a real number t we have

$$(\rho I)(q + t\xi^*) = (\rho I) q + t J^*(\rho I) q.$$

Therefore, ρI maps ξ^* at q to ξ^* at $(\rho I) q$. Let $[\rho] = \varphi^{-1} \cdot \rho I \cdot \varphi$. Then $[\rho]$ is a transformation of $S^3[1]$ and preserves g and ξ. Hence $[\rho]$ is an automorphism of the Sasakian manifold $S^3[1]$. Identifying Γ with a finite group generated by $[\rho]$ we have a Sasakian manifold $M = S^3[1]/\Gamma$.

On the other hand, we have the Boothby-Wang's fibering $\pi : S^3[1] \to S^3[1]/\xi = CP^1$, where CP^1 is a complex projective space. Since $[\rho]$ is an automorphism of $S^3[1]$, $[\rho]$ induces an automorphism of CP^1(cf. [8]). Easily we see that ρ can be chosen so that the induced automorphism is not trivial on CP^1. By the fact that every automorphism of CP^1 has fixed points, we have trajectories of ξ in $S^3[1]$ which are invariant by $[\rho]$. These great circles are factorized in $M = S^3[1]/\Gamma$, and therefore, M is not regular.

REMARK 6.1. $M = S^3[1]/\Gamma$ defined above is not homogeneous Sasakian manifold in the sense that the automorphism group of M is not transitive, since every homogeneous contact manifold is regular.

REMARK 6.2. The similar arguments show that $S^{4r+3}[1]$ $(r=1,2,\cdots)$ is factorized by some finite group Γ so that $S^{4r+3}[1]/\Gamma$ is non-regular Sasakian manifold with constant curvature.

REMARK 6.3. Every $(4r+1)$-dimensional homogeneous Riemannian manifold of constant curvature 1 is of the form $S^{4r+1}[1]/F(t)$, where $F(t)$ is a finite cyclic group generated by $\exp t\xi$, $2\pi/t$ being an integer.

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