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SIMPLY INVARIANT SUBSPACE THEOREMS FOR ANTISYMMETRIC FINITE SUBDIAGONAL ALGEBRAS

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1. Introduction. Recently, in [1] W. B. Arveson presented a theory of non-self-adjoint operator algebras which he called the subdiagonal algebra. His theory of subdiagonal algebra was motivated from the earlier work of H. Helson and D. Lowdenslager on matrix-valued analytic functions [3] and of R. V. Kadison and I. M. Singer on triangular operator algebras [4]. And subdiagonal algebra theory added some unity to these very different theories.

Let Φ be a faithful normal positive idempotent linear map of a von Neumann algebra M into itself. A subalgebra $\mathfrak{A}$ of M is said to be subdiagonal with respect to Φ if (i) $\mathfrak{A}+\mathfrak{A}^*$ is σ-weakly dense in M; (ii) $\Phi(AB)=\Phi(A)\Phi(B)$ $A, B\in\mathfrak{A}$; (iii) $\Phi(\mathfrak{A})\subset M$; (iv) $(\mathfrak{A}\cap\mathfrak{A}^*)^2$ is non-degenerate.

The definition of subdiagonal algebra resembles to that of weak*-Dirichlet algebra. From this point of view, W. B. Arveson generalized some properties of weak*-Dirichlet algebras to subdiagonal algebras. For instances, factorization theorem, Jensen's inequality and Szegö's theorem were shown to be valid in some examples.

In this note, we shall show that the analogue of simply invariant subspace theorem, which is an another fundamental property for weak*-Dirichlet algebras, is valid in the antisymmetric finite subdiagonal algebras.

2. Definitions. Throughout this note, M will be a von Neumann algebra on a separable Hilbert space $\mathfrak{H}$, Φ be a faithful normal positive idempotent linear map of M into itself, and $\mathfrak{A}$ be a subdiagonal subalgebra of M w.r.t. Φ.

The σ-weak closure $(\mathfrak{A}\cap\mathfrak{A}^*)^{\sim}$ of $(\mathfrak{A}\cap\mathfrak{A}^*)$ is a von Neumann algebra and Φ is an expectation on $(\mathfrak{A}\cap\mathfrak{A}^*)^{\sim}$. ([1], Prop. 2.1.4). A subdiagonal subalgebra $\mathfrak{A}$ of M is said to be finite if some faithful normal finite trace τ of M preserves the expectation associated with $\mathfrak{A}$ (i.e. $\tau\circ\Phi=\tau$); in this case, M is a finite von Neumann algebra necessarily. A subdiagonal algebra is said to be antisymmetric if $\mathfrak{A}\cap\mathfrak{A}^*=\{\lambda I\}$, scalar multiples of identity operator I.

Let $\mathfrak{A}$ be a finite subdiagonal subalgebra of M w.r.t. Φ, and τ be a Φ-preserving faithful normal finite trace of M. We write $L^1(M,\tau)$ (resp. $L^2(M,\tau)$)

the space of all integrable (resp. square integrable) operators for the gage space $(M, \mathfrak{H}, \tau)$ in the sense of I. E. Segal [5]. From the finiteness of τ, M is contained in $L^p(M, \tau)$ for $p = 1, 2$. It is well known that $L^1(M, \tau)$ becomes a Banach space and $L^2(M, \tau)$ becomes a Hilbert space. Let $\|\cdot\|_p$ denote the norm in $L^p(M, \tau)$, $(\cdot, \cdot)$ denote the inner product in $L^2(M, \tau)$. Then $\|X\|_1 = \tau(|X|)$ for $X \in L^1(M, \tau)$, $(X, Y) = \tau(Y^*X)$ for $X, Y \in L^2(M, \tau)$. For a subset $\mathcal{S}$ of $L^p(M, \tau)$ $p = 1, 2$. $[\mathcal{S}]_p$ will denote the closed subspace of $L^p(M, \tau)$ generated by $\mathcal{S}$.

Let $\mathfrak{T} = \{T \in \mathfrak{A}\,;\ \Phi(T) = 0\}$. Clearly $\mathfrak{T}$ is an ideal in $\mathfrak{A}$ and $\mathfrak{A} = \mathfrak{A} \cap \mathfrak{A}^* + \mathfrak{T}$. Hence $\mathfrak{A} + \mathfrak{A}^* = \mathfrak{A} \cap \mathfrak{A}^* + \mathfrak{T} + \mathfrak{T}^* = \mathfrak{A} + \mathfrak{T}^*$ is σ-weakly dense in M. A closed subspace $\mathfrak{M}$ of $L^p(M, \tau)$ $p = 1, 2$ is said to be left (resp. right) simply invariant if $[\mathfrak{T}\mathfrak{M}]_p \subsetneqq \mathfrak{M}$ (resp. $[\mathfrak{M}\mathfrak{T}]_p \subsetneqq \mathfrak{M}$). If $\mathfrak{M}$ is left (resp. right) simply invariant, then $\mathfrak{M}$ is left (resp. right) invariant i.e. $[\mathfrak{A}\mathfrak{M}]_p \subset \mathfrak{M}$ (resp. $[\mathfrak{M}\mathfrak{A}]_p \subset \mathfrak{M}$).

3. Simply invariant subspace theorems. In this section, we shall show the analogue of simply invariant subspace theorems for weak*-Dirichlet algebras for antisymmetric finite subdiagonal algebras.

Let $\mathfrak{A}$ be an antisymmetric finite subdiagonal subalgera of M w.r.t. Φ and τ be a Φ-preserving faithful normal finite trace of M. Notice that if $\mathfrak{A}$ is antisymmetric, then necessarily $\Phi(X) = \tau(X)I$ for every $X \in M$. Hence we can replace Φ by τ, and we call $\mathfrak{A}$ an antisymmetric subdiagonal subalgebra of M w.r.t. τ. Moreover τ is multiplicative on $\mathfrak{A}$ (i.e. $\tau(AB) = \tau(A)\tau(B)$ for $A, B \in \mathfrak{A}$).

THEOREM 1. *Let $\mathfrak{A}$ be an antisymmetric finite subdiagonal subalgebra of M w.r.t. τ. Then every left (resp. right) simply invariant subspace $\mathfrak{M}$ of $L^2(M, \tau)$ is of the form $[\mathfrak{A}U]_2$ (resp. $[U\mathfrak{A}]_2$) for some unitary operator U in M.*

PROOF. Suppose $\mathfrak{M}$ is left simply invariant. Since $[\mathfrak{T}\mathfrak{M}]_2$ is a proper subspace of $L^2(M, \tau)$, there exists a non-zero operator U in $\mathfrak{M} \ominus [\mathfrak{T}\mathfrak{M}]_2$. We may assume $\|U\|_2 = 1$. Let $A \in \mathfrak{A}$. Then $A - \tau(A)I \in \mathfrak{T}$ and

$$\begin{aligned}\tau(UU^*A) &= (AU, U) = ((A - \tau(A)I)U, U) + (\tau(A)U, U)\\ &= \tau(A)(U, U) = \tau(A).\end{aligned}$$

By taking the adjoint on both sides, we have $\tau(UU^*A^*) = \tau(A^*)$. Hence we have $\tau(UU^*A) = \tau(A)$ for all $A \in \mathfrak{A} + \mathfrak{A}^*$. That is

$$((I - UU^*), A) = 0 \qquad \text{for all} \quad A \in \mathfrak{A} + \mathfrak{A}^*.$$

Since $\mathfrak{A}+\mathfrak{A}^*$ is σ-weakly dense in M, $\mathfrak{A}+\mathfrak{A}^*$ is dense in $L^2(M,\tau)$. So $I-UU^*=0$ in $L^2(M,\tau)$. But τ is faithful, this implies that $UU^*=I$. Since M is finite, we have $U^*U=I$. Hence U is unitary.

Clearly $[\mathfrak{A}U]_2\subset\mathfrak{M}$ as $\mathfrak{A}U\subset\mathfrak{M}$ and $\mathfrak{M}$ is closed. Let $B\in\mathfrak{M}\ominus[\mathfrak{A}U]_2$. Then $\tau(BU^*A^*)=(B,AU)=0$ for all $A\in\mathfrak{A}$. Also since $\mathfrak{T}B\subset[\mathfrak{T}\mathfrak{M}]_2$, we have $\tau(BU^*T)=(TB,U)=0$ for all $T\in\mathfrak{T}$. So

$$\tau(BU^*X)=(X,UB^*)=0 \qquad \text{for all} \quad X\in\mathfrak{A}^*+\mathfrak{T}\,.$$

But $\mathfrak{A}^*+\mathfrak{T}=\mathfrak{A}+\mathfrak{A}^*$ and by the same reason as above, we conclude that $UB^*=0$. But U is unitary, so $B=0$. It follows that $\mathfrak{M}=[\mathfrak{A}U]_2$.

The assertion for right simply invariant subspace may be proved in just the same way.

In the following sentences, since we can discuss left and right symmetrically, we shall state the left case only.

LEMMA 1. *If* $X\in L^2(M,\tau)$ *and* $X\notin[X\mathfrak{T}]_2$, *then* $X=UA$ *where* $U\in[X\mathfrak{A}]_2$ *is unitary and* $[A\mathfrak{A}]_2=[\mathfrak{A}]_2$.

PROOF. Our assumption implies that $[X\mathfrak{A}]_2$ is a right simply invariant subspace of $L^2(M,\tau)$, and hence by Theorem 1,

$$[X\mathfrak{A}]_2=[U\mathfrak{A}]_2\,,\qquad U \text{ is unitary in } M.$$

So $X=UA$ where $A\in[\mathfrak{A}]_2$, since $[U\mathfrak{A}]_2=U[\mathfrak{A}]_2$. Since

$$U[A\mathfrak{A}]_2=[UA\mathfrak{A}]_2=[X\mathfrak{A}]_2=[U\mathfrak{A}]_2=U[\mathfrak{A}]_2\,,$$

we have $[A\mathfrak{A}]_2=[\mathfrak{A}]_2$. Clearly $U\in[U\mathfrak{A}]_2=[X\mathfrak{A}_2]$. This completes the proof.

LEMMA 2. *If* $X\in L^1(M,\tau)$ *and* $X\notin[X\mathfrak{T}]_2$, *then* $|X|^{1/2}\notin[|X|^{1/2}\mathfrak{T}]_2$. *where* $|X|=(X^*X)^{1/2}$.

PROOF. Let $X=V|X|$ be the polar decomposition of X, and put $X_1=V|X|^{1/2}$. Assume that $|X|^{1/2}\in[|X|^{1/2}\mathfrak{T}]_2$, then

$$X=X_1|X|^{1/2}\in X_1[|X|^{1/2}\mathfrak{T}]_2\subset[X_1|X|^{1/2}\mathfrak{T}]_1=[X\mathfrak{T}]_1\,.$$

This is a contradiction.

LEMMA 3. *If* $X \in L^1(M,\tau)$ *and* $X \notin [X\mathfrak{T}]_1$, *then* $X=YZ$ *where* $Y \in [X\mathfrak{A}]_1 \cap L^2(M,\tau)$ *and* $Z \in [\mathfrak{A}]_2$.

PROOF. By Lemma 1 and 2, $|X|^{1/2} = UZ$ where $U \in [|X|^{1/2}\mathfrak{A}]_2$, $Z \in [\mathfrak{A}]_2$ and $[Z\mathfrak{A}]_2 = [\mathfrak{A}]_2$. Let $X = V|X|$ be the polar decomposition of X and put $Y=V|X|^{1/2}U$. Clearly $Y \in L^2(M,\tau)$ and

$$YZ = V|X|^{1/2}UZ = V|X|^{1/2}|X|^{1/2} = V|X| = X.$$

Since $[Z\mathfrak{A}]_2 = [\mathfrak{A}]_2$. For arbitrary $\varepsilon > 0$, there exists a $A \in \mathfrak{A}$ such that $\|ZA-I\|_2 < \varepsilon/\|Y\|_2$. From

$$\|XA-Y\|_1 = \|Y(ZA-I)\|_1 < \|Y\|_2\|ZA-I\|_2 < \varepsilon,$$

one has $Y \in [X\mathfrak{A}]_1$. This completes the proof.

THEOREM 2. *Let* $\mathfrak{A}$ *be an antisymmetric finite subdiagonal subalgebra of* M *w.r.t.* τ. *Then every left simply invariant subspace* $\mathfrak{M}$ *of* $L^1(M,\tau)$ *is of the form* $\mathfrak{M}=[\mathfrak{A}U]_1$ *for some unitary* U *in* M.

PROOF. Put $\mathfrak{N}=\mathfrak{M}\cap L^2(M,\tau)$. By the Schwarz inequality, $\mathfrak{N}$ is a closed subspace of $L^2(M,\tau)$. We begin with showing that $\mathfrak{N}$ is a left simply invariant subspace of $L^2(M,\tau)$.

At first sight it is even not clear that $\mathfrak{N}$ should contain any non-zero operator at all; this is actually shown as follows: So by the assumption of simple invariance there exists an operator X in $\mathfrak{M}\backslash[\mathfrak{T}\mathfrak{M}]_1$. In particular $X \notin [\mathfrak{T}X]_1$, and by Lemma 3 we can write $X = ZY$ where $Z \in [\mathfrak{A}]_2$, $Y \in [\mathfrak{A}X]_1 \cap L^2(M,\tau)$. As $\mathfrak{A}X \subset \mathfrak{M}$, $Y \in [\mathfrak{A}X]_1 \subset \mathfrak{M}$. Thus $\mathfrak{N}$ is non-empty.

Now we claim that $Y \notin [\mathfrak{T}\mathfrak{N}]_2$. Indeed, if $Y \in [\mathfrak{T}\mathfrak{A}]_2$, there exists a sequence $\{A_n\}$ in $\mathfrak{T}\mathfrak{N}$ such that $\|A_n-Y\|_2 \to 0$. As $Z \in [\mathfrak{A}]_2$, there also exists a sequence $\{B_n\}$ in $\mathfrak{A}$ such that $\|B_n-Z\|_2 \to 0$. But then $B_nA_n \in \mathfrak{A}\mathfrak{T}\mathfrak{N} \subset \mathfrak{T}\mathfrak{N} \subset \mathfrak{T}\mathfrak{M}$, and

$$\begin{aligned}\|B_nA_n-ZY\|_1 &= \|B_n(A_n-Y) + (B_n-ZY)\|_1 \\ &< \|B_n\|_2\|A_n-Y\|_2 + \|B_n-Z\|_2\|Y\|_2 \\ &\to 0.\end{aligned}$$

Hence $X = ZY \in [\mathfrak{T}\mathfrak{M}]_1$. This contradicts our selection of X. This shows that $\mathfrak{N}$ is a left simply invariant subspace of $L^2(M,\tau)$. By the Theorem 1, there is a unitary U in M such that $\mathfrak{N}=[\mathfrak{A}U]_2$.

Thus we proved that $\mathfrak{A}U \subset [\mathfrak{A}U]_2 = \mathfrak{N} \subset \mathfrak{M}$. Taking the L^1-closure on bothe sides, we get $[\mathfrak{A}U]_1 \subset \mathfrak{M}$. Our perpose is to show the converse inclution $\mathfrak{M} \subset [\mathfrak{A}U]_1$. In fact, it is at least true that $\mathfrak{M} \backslash [\mathfrak{T}\mathfrak{M}]_1 \subset [\mathfrak{A}U]_1$, because if $X \in \mathfrak{M} \backslash [\mathfrak{T}\mathfrak{M}]_1$, then $X = ZY$ with $Z \in [\mathfrak{A}]_2$, $Y \in \mathfrak{N}$, as we showed in a previous paragraph. Now if $Y \in [\mathfrak{T}\mathfrak{M}]_1$, then $X+Y \in \mathfrak{M} \backslash [\mathfrak{T}\mathfrak{M}]_1$. Hence $X+Y \in [\mathfrak{A}U]_1$. However, $X \in [\mathfrak{A}U]_1$. Hence $Y \in [\mathfrak{A}U]_1$, and the proof is completed.

4. Maximality of antisymmetric finite subdiagonal algebras. A subdiagonal subalgebra of M w.r.t. Φ is said to be maximal if it is contained properly in no larger subdiagonal algebra of M w.r.t. Φ. For a given finite subdiagonal subalgebra $\mathfrak{A}$ of M w.r.t. Φ, we put

$$\mathfrak{A}_m = \{X \in M;\ \Phi(XT) = 0 \quad \text{for all } T \in \mathfrak{T}\}\ .$$

Then it was shown in [1] that $\mathfrak{A}_m$ is the maximal subdiagonal subalgebra of M containing $\mathfrak{A}$. ([1], Cor. 2.2.4).

In this section, using the results obtained in previous section we shall show that for an antisymmetric finite subdiagonal algebra $\mathfrak{A}$, the σ-weak closure $\mathfrak{A}^{\sim}$ of $\mathfrak{A}$ is the maximal subdiagonal algebra containing $\mathfrak{A}$ i.e. $\mathfrak{A}_m = \mathfrak{A}^{\sim}$.

LEMMA 4. $[\mathfrak{A}]_1 = \{X \in L^1(M, \tau) : \tau(XT) = 0$ *for all* $T \in \mathfrak{T}\}$.

PROOF. If $X \in [\mathfrak{A}]_1$, then clearly $\tau(XT)=0$ for all $T \in \mathfrak{T}$. Conversely let $X \in L^1(M, \tau)$ satisfies $\tau(XT) = 0$ for all $T \in \mathfrak{T}$. We have to show that $X \in [\mathfrak{A}]_1$. We may assume that $\tau(X) \neq 0$ by adding some constant if necessary. So $X \notin [X\mathfrak{T}]_1$ and by the Lemma 3, we have $X = YZ$ where $Y \in [X\mathfrak{A}]_1 \cap L^2(M, \tau)$, $Z \in [\mathfrak{A}]_2$. Since $\mathfrak{A}\mathfrak{T} \subset \mathfrak{T}$, $Y \in [X\mathfrak{A}]_1$ and $\tau(XT) = 0$ for all $T \in \mathfrak{T}$. One has $\tau(YT) = (Y, T^*) = 0$ for all $T \in \mathfrak{T}$. But $L^2(M, \tau) = [\mathfrak{A}]_2 \oplus [\mathfrak{T}^*]_2$. We have $Y \in [\mathfrak{A}]_2 \subset [\mathfrak{A}]_1$, by the Schwarz inequality. Hence $X = YZ \in [\mathfrak{A}]_2[\mathfrak{A}]_2 \subset [\mathfrak{A}]_1$. This completes the proof.

THEOREM 3. *Let $\mathfrak{A}$ be an antisymmetric finite subdiagonal subalgebra of M w.r.t. τ. Then the maximal subdiagonal subalgebra of M which contains $\mathfrak{A}$ is the σ-weak closure $\mathfrak{A}^{\sim}$ of $\mathfrak{A}$.*

PROOF. Put $\mathfrak{A}_m = \{X \in M;\ \tau(XT)=0$ for all $T \in \mathfrak{T}\}$. As we pointed out in the first part of this section $\mathfrak{A}_m$ is the maximal subdiagonal subalgebra of M containing $\mathfrak{A}$. So it suffices to show that $\mathfrak{A}_m = \mathfrak{A}^{\sim}$. It is clear that $\mathfrak{A}^{\sim} \subset \mathfrak{A}_m$ by the σ-weak continuity of τ. Since the σ-weak topology on M is the $\sigma(M, L^1(M, \tau))$-topology by the map $M \times L^1(M, \tau) \in \langle X, Y \rangle \to \tau(XY)$.

To establish the reverse inclusion it suffices to show that the polar of $\mathfrak{A}$

in $L^1(M,\tau)$ is contained in the polar of $\mathfrak{A}_m$. i.e. If $Y \in L^1(M,\tau)$ and $\tau(\mathfrak{A}Y)=0$, then $\tau(\mathfrak{A}_m Y)=0$. Let $Y \in L^1(M,\tau)$ and $\tau(AY)=0$ for all $A \in \mathfrak{A}$. Then by the Lemma 4, we have $Y \in [\mathfrak{A}]_1$ and $\tau(Y)=0$. Hence there exists a sequence $\{A_n\}$ in $\mathfrak{A}$ such that $\|A_n - Y\|_1 \to 0$.

The Lemma 4 implies that $\mathfrak{A}_m \subset [\mathfrak{A}]_1 \cap M$. Hence for every X in $\mathfrak{A}_m$, there exists a sequence $\{B_m\}$ in $\mathfrak{A}$ such that $\|B_m - X\|_1 \to 0$. Then

$$\begin{aligned}\|B_m A_n - XY\|_1 &\leqq \|B_m(A_n - Y)\|_1 + \|(B_m - X)Y\|_1 \\ &\leqq \|B_m\| \|A_n - Y\|_1 + \|Y\| \|B_m - X\|_1 .\end{aligned}$$

Hence for arbitrary $\varepsilon > 0$, there exists a m_0 such that

$$\|B_{m_0} A_n - XY\|_1 < \|B_{m_0}\| \|A_n - Y\|_1 + \varepsilon$$

Fix m_0, and letting $n \to \infty$, there exists a n_0 such that

$$\|B_{m_0} A_{n_0} - XY\|_1 < \varepsilon + \varepsilon = 2\varepsilon .$$

Hence we can choose the subsequences $\{A_{n_i}\}$, $\{B_{m_i}\}$ such that

$$\|B_{m_i} A_{n_i} - XY\|_1 \to 0 .$$

Hence we have $XY \in [\mathfrak{A}]_1$. Since

$$\tau(B_{m_i} A_{n_i}) \to \tau(XY), \text{ and } \tau(B_{m_i} A_{n_i}) = \tau(B_{m_i})\tau(A_{n_i}) \to \tau(X)\tau(Y) .$$

We have $\tau(XY) = \tau(X)\tau(Y) = 0$. Hence Y is in the polar of $\mathfrak{A}_m$. This completes the proof.

The above Theorem also implies that σ-weakly closed antisymmetric finite subdiagonal algebra is always maximal.

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