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CAPACITIES OF SETS AND HARMONIC ANALYSIS ON THE GROUP 2^{ω}

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1. Introduction. In this paper we shall work on the dyadic group 2^{ω} which consists of all sequences $x=(x_1, x_2, \cdots)$, $x_i=0$ or 1, where addition is defined coordinatewise mod 2. The topology is the product topology which is the same as that given by an invariant metric $\delta(x,y)$, where if $x=(x_1, x_2, \cdots)$ and $y=(y_1, y_2, \cdots)$ are in 2^{ω}, then

$$\delta(x,y)=\sum_{n=1}^{\infty}|x_n-y_n|/2^n.$$

After this we shall write $|x-y|$ instead of $\delta(x,y)$.

In particular, we first define the Rademacher function $\varphi_0(\bar{x})$ by $\varphi_0(\bar{x})=1$ if $0\leqq\bar{x}<1/2$, $\varphi_0(\bar{x})=-1$ if $1/2\leqq\bar{x}<1$, and $\varphi_0(\bar{x})=\varphi_0(\bar{x}+1)$ for real $\bar{x}$. Next, we define $\varphi_n(\bar{x})=\varphi_0(2^n\bar{x})$ for every nonnegative integer n. Then the Walsh function $\psi_n(\bar{x})$ is defined by setting $\psi_0(\bar{x})=1$, $\psi_n(\bar{x})=\varphi_{n_1}(\bar{x})\cdots\varphi_{n_r}(\bar{x})$ where $n=2^{n_1}+\cdots+2^{n_r}$ and the n_i are uniquely determined by $n_{i+1}<n_i$. As is well known, $\{\psi_n\}_{n=0}^{\infty}$ form a complete orthonormal system and every function $f(\bar{x})$ which is integrable on $(0,1)$ may be expanded in a Walsh-Fourier series;

$$f(\bar{x})\sim\sum_{n=0}^{\infty}a_n\psi_n(\bar{x}),\ \text{where}\ a_n=\int_0^1 f(\bar{x})\psi_n(\bar{x})d\bar{x},\quad n=0,1,2,\cdots.$$

$\varphi_n(x)$ is defined on 2^{ω} with $x=(x_1, x_2, \cdots)$ by setting $\varphi_n(x)=1$ if $x_{n+1}=0$, $\varphi_n(x)=-1$ if $x_{n+1}=1$. $\psi_n(x)$ is defined on 2^{ω} by setting $\psi_0(x)=1$, $\psi_n(x)=\varphi_{n_1}(x)\cdots\varphi_{n_r}(x)$ where as before $n=2^{n_1}+\cdots+2^{n_r}$ and the n_i are uniquely determined. We note that $\{\psi_n\}_{n=0}^{\infty}$ gives us the full set of characters of 2^{ω}. N. J. Fine in his paper on the Walsh functions, [3], shows that the natural map $\lambda: 2^{\omega}\to[0,1]$ defined by

$$\lambda(x)=\sum_{n=1}^{\infty}x_n/2^n$$

is continuous, one-to-one except for a countable set, preserves Haar measure and carries the characters of 2^ω onto the Walsh functions.

The main purposes of this paper depend on the note [1] by L. H. Harper. As regard terminology and notations we shall follow it as a rule. In order to facilitate progress we set up some results of L. H. Harper which are needed in the sequel.

For $x \in 2^\omega$, let $\{x\} = 2^{-n}$, where n is the number of zeroes in x preceding the first one ($\{\dot{0}\} = 0$). Then

$$(1.1) \qquad |x| = \sum_{n=1}^{\infty} x_n 2^{-n} \leqq \{x\} \leqq 2|x|.$$

Fix $0 \leqq \alpha < 1$. Let

$$(1.2) \qquad K(x) = \{x\}^{-\alpha} \text{ if } 0 < \alpha < 1 \text{ or } \log 1/\{x\} \text{ if } \alpha = 0.$$

(All logarithms shall be taken to the base two.)
Then K is continuous except at zero and nonnegative so that a potential theory with respect to K is valid.

If E is a closed subset of 2^ω, then $\mathfrak{M}(E)$ is the set of all nonnegative, Borel measures of norm one on 2^ω supported on E. Fix $0 \leqq \alpha < 1$, let $\nu \in \mathfrak{M}(E)$ and form the energy integral

$$(1.3) \qquad I(\nu) = \int_{2^\omega} \int_{2^\omega} K(x-y) d\nu(x) d\nu(y).$$

Then there are two cases : Either $I(\nu) = +\infty$ for all ν in $\mathfrak{M}(E)$ or

$$(1.4) \qquad V = \inf I(\nu) < +\infty, \qquad \nu \in \mathfrak{M}(E).$$

E is said to be of capacity zero if $I(\nu) = +\infty$ for all ν in $\mathfrak{M}(E)$, or if $V < +\infty$, the capacity of E is

$$(1.5) \qquad C \equiv V^{-1/\alpha} \text{ if } 0 < \alpha < 1 \text{ or } C \equiv 2^{-V} \text{ if } \alpha = 0.$$

$$(1.6) \qquad U(x\,;\,\nu) = \int_{2^\omega} K(x-y) d\nu(y)$$

is the potential function associated with ν. The following two statements are standard results in potential theory (See [8] and [9]).

(1.7) If E is of positive capacity, then there exists a unique μ in $\mathfrak{M}(E)$ such that $I(\mu) = V$.

(1.8) The potential function $U(x; \mu)$, of the equilibrium distribution has the following properties;

(i) $U(x; \mu) \geqq V$ except for a set which is of measure zero with respect to every measure of finite energy.

(ii) $U(x; \mu) \leqq V$ for all x in the support of μ.

(iii) $U(x; \mu)$ is bounded on 2^{ω}.

The nth Dirichlet kernel for the Walsh functions is defined by

$$D_n(x) = \sum_{k=0}^{n-1} \psi_k(x). \tag{1.9}$$

If $f(x) \sim \sum_{n=0}^{\infty} a_n \psi_n(x)$, then partial sums can be written as

$$\sum_{k=0}^{n-1} a_k \psi_k(x) = \int_{2^{\omega}} f(x+t) D_n(t) dt. \tag{1.10}$$

The size of $D_n(x)$ is given by

$$|D_n(x)| \leqq \frac{2}{|x|} \quad (0 < |x| < 1). \tag{1.11}$$

Moreover, for some constants A and B independent of x and n,

$$(2^{1-\alpha}-1) \sum^{n} \frac{1}{2^{k(1-\alpha)}} D_{2^k}(x) \leqq AK(x) + B. \tag{1.12}$$

Let $[n]$ denote the greatest power of 2 in n ($[0]=1$ for convenience) then we have

$$\frac{1}{[k]^{1-\alpha}} |D_k(x)| \leqq 2K(x). \tag{1.13}$$

Henceforth, the letter A will be reserved to denote positive constant independent of x and n, which is not always the same number.

Now we arrive at the main theorem of L. H. Harper:

THEOREM. *Let* $f(x) \sim \sum_{n=0}^{\infty} a_n \psi_n(x)$ *be such that*

$$\sum_{n=0}^{\infty} a_n^2[n]^{1-\alpha} < \infty, \qquad 0 \leqq \alpha < 1.$$

Then if $s_n(x\,;f) = \sum_{k=0}^{n-1} a_k\psi_k(x)$ *diverges on a closed set* E, *the* α*-capacity of* E *is zero.*

This is a variant of the results for the trigonometric series which are summarized in Chapter 4 of Kahane-Salem [8].

Recently, in connection with the result of L. H. Harper, Professor Sh. Yano proposed the following problem : Let $\sum_{n=0}^{\infty} a_n\psi_n(x)$ be the Walsh-Fourier series of a function $f(x) \in L^p(2^\omega)$, $1 \leqq p < \infty$, and let $\sum_{k=0}^{n-1} \frac{a_k\psi_k(x)}{[k]^{\frac{1-\alpha}{p}}}$ $(0 \leqq \alpha < 1)$ diverge on a closed set E. Then what can we say about the α-capacity of E ?

In the present paper, we shall give some partial answers to the above problem. Main results are as follows :

THEOREM I. *Suppose that* $1 \leqq p \leqq 2$, $0 \leqq \alpha < 1$ *and that* $\sum_{n=0}^{\infty} a_n\psi_n(x)$ *is the Walsh-Fourier series of a function* $f(x) \in L^p(2^\omega)$. *Then if* $\sum_{k=0}^{n-1} \frac{a_k\psi_k(x)}{[k]^{\frac{1-\alpha}{p}}}$ *diverges on a closed set* E, *the* α*-capacity of* E *is zero.*

For the case $p=2$, this is reduced to Harper's theorem mentioned above.

THEOREM II. *Suppose that* $p > 2$, $0 \leqq \alpha < 1$ *and that* $\sum_{n=0}^{\infty} a_n\psi_n(x)$ *is the Walsh-Fourier series of a function* $f(x) \in L^p(2^\omega)$. *Then if* $\sum_{k=0}^{n-1} \frac{a_k\psi_k(x)}{[k]^{\frac{1-\alpha}{p}}}$ *diverges on a closed set* E, *the* $\alpha+\varepsilon$*-capacity of* E *is zero, where* ε *is any positive number.*

In both theorems, the trigonometric-Fourier series analogues have already been established in [6].

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2. In order to prove Theorem I and II, we need the following lemmas.

LEMMA 1. *Let* $\sum_{n=0}^{\infty} a_n\psi_n(x)$ *be the Walsh-Fourier series of a function* $f(x)\in L^p(2^\omega)$, *where* $1\leqq p<\infty$. *Then*

$$\int_{2^\omega} |\sigma_n(x\,;\,f)-f(x)|^p dx\to 0 \qquad (n\to\infty),$$

where $$\sigma_n(x\,;\,f)=\frac{s_1+s_2+\cdots+s_n}{n},\qquad s_i=\sum_{j=0}^{i-1} a_j\psi_j(x).$$

(*For the case* $p=1$, *the result was proved in* [2].)

PROOF. Let $p(x)$ be a Walsh polynomial, that is, a linear combination $\sum_{k=0}^{N-1} c_k\psi_k(x)$ such that

$$\int_{2^\omega} |f(x)-p(x)|^p dx<\varepsilon^p.$$

For $p(x)$, we can show that

$$|\sigma_n(x\,;\,p)-p(x)|=o(1)$$

in essentially the same way as Fine proved Theorem XVII in [3]. Then

$$\begin{aligned}\left[\int_{2^\omega} |\sigma_n(x\,;\,f)-f(x)|^p dx\right]^{\frac{1}{p}} &\leqq \left[\int_{2^\omega} |\sigma_n(x\,;\,f)-\sigma_n(x\,;\,p)|^p dx\right]^{\frac{1}{p}}\\ &\quad+\left[\int_{2^\omega} |\sigma_n(x\,;\,p)-p(x)|^p dx\right]^{\frac{1}{p}}+\left[\int_{2^\omega} |f(x)-p(x)|^p dx\right]^{\frac{1}{p}}\\ &\leqq\left[\int_{2^\omega} |\sigma_n(x\,;\,f-p)|^p dx\right]^{\frac{1}{p}}+o(1)+\left[\int_{2^\omega} |f(x)-p(x)|^p dx\right]^{\frac{1}{p}}\\ &\leqq 3\left[\int_{2^\omega} |f(x)-p(x)|^p dx\right]^{\frac{1}{p}}+o(1),\end{aligned}$$

since for any $h(x)\in L^p(2^\omega)$, by Minkowski's inequality,

$$\left[\int_{2^\omega} |\sigma_n(x\,;\,h)|^p dx\right]^{\frac{1}{p}}=\left[\int_{2^\omega}\left|\int_{2^\omega} h(t)K_n(x+t)dt\right|^p dx\right]^{\frac{1}{p}}$$

$$=\left[\int_{2^\omega}\left|\int_{2^\omega} h(x+t)K_n(t)dt\right|^p dx\right]^{\frac{1}{p}} \leqq \int_{2^\omega} |K_n(t)|\cdot\left[\int_{2^\omega} |h(x+t)|^p dx\right]^{\frac{1}{p}} dt$$

$$\leqq 2\left[\int_{2^\omega} |h(x)|^p dx\right]^{\frac{1}{p}},$$

where

$$K_n(x)=\frac{1}{n}\sum_{k=1}^{n}\left(\sum_{j=0}^{k-1}\psi_j(x)\right)$$

and it is known ([5]) that

$$\int_{2^\omega} |K_n(x)|\, dx \leqq 2.$$

This proves the lemma.

LEMMA 2. *If* $\sum_{k=0}^{\infty} a_k\psi_k(x)$ *satisfies*

$$\int_{2^\omega} |\sigma_n(x)-\sigma_m(x)|^p dx \to 0 \ (m,n\to\infty), \text{ where } 1\leqq p<\infty,$$

there exists a function $f(x)\in L^p(2^\omega)$ *such that* $f(x)\sim\sum_{k=0}^{\infty} a_k\psi_k(x)$.

For the case $p=1$, the result was proved in [2] and we can easily extend it for any p, $1\leqq p<\infty$, so we omit the proof.

LEMMA 3. *Let* $\sum_{k=0}^{\infty} a_k\psi_k(x)$ *be the Walsh-Fourier series of a function* $f(x)$ $\in L^p(2^\omega)$, $1\leqq p<\infty$. *Then there exist* $g(x)\in L^p(2^\omega)$ *and a function* $Q(n)$, $n=0,1,2,\cdots$, *which is positive, nondecreasing and tending to infinity with* n, *while* $\sum_{k=0}^{\infty} a_kQ(k)\psi_k(x)$ *is the Walsh-Fourier series of the function* $g(x)$.

For the case $p=1$, the result was proved by R. Salem in [4]. By the aids of Lemma 1, 2 and Minkowski's inequality, the assertion for p, $1<p<\infty$, is proved in an entirely similar way.

3. Proof of Theorem I. Suppose the α-capacity of E is not zero. Then we have an equilibrium distribution μ for E and constant M such that

$$\int_{2^\omega} K(x+t)d\mu(t) = U(x\,;\,\mu) \leqq M \tag{1}$$

on 2^ω.

From Lemma 3 we can find a function $g(x) \in L^p(2^\omega)$ and $Q(n)$, $n=0,1,2,\cdots$, where $Q(n)$ is positive, nondecreasing and tending to infinity with n, such that $\sum_{k=0}^{\infty} a_k Q(k)\psi_k(x)$ is the Walsh-Fourier series of $g(x)$. Then the partial sums

$$S_n(x) = \sum_{k=0}^{n-1} \frac{a_k Q(k)\psi_k(x)}{[k]^{\frac{1-\alpha}{p}}} \tag{2}$$

of the series $\sum_{k=0}^{\infty} \frac{a_k Q(k)\psi_k(x)}{[k]^{\frac{1-\alpha}{p}}}$ are unbounded on E. For, if not,

$$\begin{aligned} s_n(x\,;\,f) &= \sum_{k=0}^{n-1} \frac{a_k \psi_k(x)}{[k]^{\frac{1-\alpha}{p}}} = \sum^{n-1} \frac{a_k Q(k)\psi_k(x)}{[k]^{\frac{1-\alpha}{p}}} \cdot \frac{1}{Q(k)} \\ &= \sum_{k=0}^{n-2} \sum_{j=0}^{k} \frac{a_j Q(j)\psi_j(x)}{[j]^{\frac{1-\alpha}{p}}} \left[\frac{1}{Q(k)} - \frac{1}{Q(k+1)} \right] \\ &\quad + \sum_{j=0}^{n-1} \frac{a_j Q(j)\psi_j(x)}{[j]^{\frac{1-\alpha}{p}}} \cdot \frac{1}{Q(n-1)} \qquad \text{(by Abel)} \end{aligned} \tag{3}$$

and so $s_n(x\,;\,f)$ would converge. Define

$$E^+ = \{x \in 2^\omega\,;\, \overline{\lim}\, S_n(x) = +\infty\},\ E^- = \{x \in 2^\omega\,;\, \underline{\lim}\, S_n(X) = -\infty\}. \tag{4}$$

Either $\mu(E^+) > 0$ or $\mu(E^-) > 0$, so without loss of generality we assume the former. Also for $n=1,2,\cdots$ let $n(x) \equiv$ the least $k \leqq n$ such that

$$S_k(k) = \max_{1 \leqq j \leqq n} S_j(x). \tag{5}$$

Then $S_{n(x)}(x) = \max_{1\leqq j \leqq n} S_j(x)$ is a Borel measurable function, $S_{n(x)}(x) \geqq a_0 Q(0)$ and goes to $+\infty$ for all x in E^+. The upshot of all this then is that

$$(6) \qquad I=\int_{2^\omega} S_{n(x)}(x)d\mu(x)\to +\infty \quad (n\to +\infty).$$

However, if $p=1$,

$$(7) \qquad S_{n(x)}(x)=\sum_{k=0}^{n(x)-1}\frac{a_k Q(k)\psi_k(x)}{[k]^{1-\alpha}}=\int_\omega g(t)\sum_{k=0}^{n(x)-1}\frac{\psi_k(x+t)}{[k]^{1-\alpha}}\,dt,$$

where

$$(8) \qquad \sum_{k=0}^{n(x)-1}\frac{\psi_k(x+t)}{[k]^{1-\alpha}}=\sum_{k=0}^{n(x)-2}\left(\frac{1}{[k]^{1-\alpha}}-\frac{1}{[k+1]^{1-\alpha}}\right)\sum_{j=0}^{k}\psi_j(x+t)$$

$$+\frac{1}{[n(x)-1]^{1-\alpha}}\sum_{j=0}^{n(x)-1}\psi_j(x+t) \quad \text{(by partial summation)}$$

$$=(2^{1-\alpha}-1)\sum_{k=0}^{\log[n(x)-1]}\frac{1}{2^{k(1-\alpha)}}\sum_{j=0}^{2^k-1}\psi_j(x+t)$$

$$+\frac{1}{[n(x)-1]^{1-\alpha}}\sum_{j=0}^{n(x)-1}\psi_j(x+t).$$

Then applying relations (1.12) and (1.13) to the first part and the second part respectively we can find constants A and B such that

$$(9) \qquad \left|\sum_{k=0}^{n(x)-1}\frac{\psi_k(x+t)}{[k]^{1-\alpha}}\right|\leqq AK(x+t)+B.$$

Therefore, it follows that

$$(10) \qquad I=\int_{2^\omega} S_{n(x)}(x)d\mu(x)\leqq\int_{2^\omega}\int_{2^\omega}|g(t)|\cdot[AK(x+t)+B]dtd\mu(x)$$

$$=B\int_{2^\omega}|g(t)|\,dt+A\int_{2^\omega}|g(t)|\cdot\left[\int_{2^\omega}K(x+t)d\mu(x)\right]dt$$

$$\leqq(B+AM)\int_{2^\omega}|g(t)|\,dt<+\infty.$$

But this contradicts (6) so that the assumption that E is of positive α-capacity must be false.

Now we consider the case $1<p<2$. We express I in the following way ;

$$I=\int_{2^\omega} S_{n(x)}(x)d\mu(x)=\int_{2^\omega} g(t)\left[\int_{\omega} G_{n(x)}(x+t)d\mu(x)\right] dt \tag{11}$$

where

$$G_{n(x)}(x+t)=\sum_{k=0}^{n(x)-1}\frac{\psi_k(x+t)}{[k]^{\frac{1-\alpha}{p}}}.$$

From Hölder's inequality it follows that

$$I\leqq\left[\int_{2^\omega}|g(t)|^p dt\right]^{\frac{1}{p}}\cdot\left[\int_{2^\omega}\left|\int_{2^\omega} G_{n(x)}(x+t)d\mu(x)\right|^q dt\right]^{\frac{1}{q}}, \tag{12}$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1.$

Since $\int_{2^\omega}|g(t)|^p dt<\infty$, it is enough to estimate

$$I'=\int_{2^\omega}\left|\int_{2^\omega} G_{n(x)}(x+t)d\mu(x)\right|^q dt. \tag{13}$$

Here we prove that

$$|G_n(x)|\leqq A|x|^{\frac{1-\alpha}{p}-1},\quad\text{where } |x|=\sum_{n=1}^{\infty}\frac{x_n}{2^n}\ \text{ if } x=(x_1,x_2,\cdots). \tag{14}$$

For $n\leqq\dfrac{1}{|x|}$ we have

$$|G_n(x)|\leqq\sum_{k=0}^{n-1}\frac{1}{[k]^{\frac{1-\alpha}{p}}}\leqq\sum_{k=0}^{n-1}\frac{2^{\frac{1-\alpha}{p}}}{k^{\frac{1-\alpha}{p}}}=O(n^{1-\frac{1-\alpha}{p}})=O(|x|^{\frac{1-\alpha}{p}-1}). \tag{15}$$

For $n>\dfrac{1}{|x|}$,

$$G_n(x)=\sum_{k=0}^{l}\frac{\psi_k(x)}{[k]^{\frac{1-\alpha}{p}}}+\sum_{k=l+1}^{n-1}\frac{\psi_k(x)}{[k]^{\frac{1-\alpha}{p}}}=S_1(x)+S_2(x),\quad\text{say,} \tag{16}$$

where l is the integral part contained in $\frac{1}{|x|}$. By the fact proved above, we have $|S_1(x)| < O(|x|^{\frac{1-\alpha}{p}-1})$. By Abel's transformation,

$$(17)\qquad S_2(x) = \sum_{k=l+1}^{n-2}\left(\frac{1}{[k]^{\frac{1-\alpha}{p}}} - \frac{1}{[k+1]^{\frac{1-\alpha}{p}}}\right)\sum_{j=0}^{k}\psi_j(x) + \frac{1}{[n-1]^{\frac{1-\alpha}{p}}}\sum_{j=0}^{n-1}\psi_j(x) - \frac{1}{[l+1]^{\frac{1-\alpha}{p}}}\sum_{j=0}^{l}\psi_j(x).$$

Since

$$(18)\qquad \left|\sum_{j=0}^{k}\psi_j(x)\right| \leqq \frac{A}{|x|} \text{ by (1.11),}$$

we have

$$(19)\qquad |S_2(x)| \leqq \frac{A}{|x|}\cdot|x|^{\frac{1-\alpha}{p}} = A|x|^{\frac{1-\alpha}{p}-1}.$$

From (15), (16) and (19) the proof of (14) is completed. (We may prove (14) also applying the relation which is used in the proof of Lemma 1 in [10].)

Returning to the estimation of I', from (14), we have

$$(20)\qquad I' = \int_{2^\omega}\left|\int_\omega G_{n(x)}(x+t)d\mu(x)\right|^q dt \leqq A\int_\omega\left[\int_\omega |x+t|^{\frac{1-\alpha}{p}-1}d\mu(x)\right]^q dt.$$

Remembering the condition for p, we have $q>2$ and so,

$$(21)\qquad \left[\int_{2^\omega}|x+t|^{\frac{1-\alpha}{p}-1}d\mu(x)\right]^q = \left[\int_{2^\omega}|x+t|^{\frac{q-2}{q}(-\alpha)}\cdot|x+t|^{\frac{-\alpha-1}{q}}d\mu(x)\right]^q$$
$$\leqq\left[\int_{2^\omega}|x+t|^{-\alpha}d\mu(x)\right]^{q-2}\cdot\left[\int_{2^\omega}|x+t|^{\frac{-\alpha-1}{2}}d\mu(x)\right]^2.$$

We know from (1.1) and (1.2) that

$$(22)\qquad |x+t|^{-\alpha} \leqq AK(x+t)+1.$$

Hence we have

$$\left[\int_{2^{\omega}} |x+t|^{-\alpha} d\mu(x)\right]^{q-2} \leqq A \tag{23}$$

on 2^{ω}. Consequently

$$I' \leqq A \int_{2^{\omega}} \left[\int_{2^{\omega}} |x+t|^{\frac{1-\alpha}{2}-1} d\mu(x)\right]^{2} dt. \tag{24}$$

We define functions $\widehat{G}_{2^{p(x)}}(x)$ and $G_{2^{p(x)}-1}(x)$ as follows. Let $\frac{1}{2^{p}} \leqq |x| < \frac{1}{2^{p-1}}$; then we write

$$\widehat{G}_{2^{p(x)}}(x) = \sum_{k=0}^{2^{p}-1} \frac{\psi_k(x)}{k^{\frac{1-\alpha}{2}}}, \quad G_{2^{p(x)}-1}(x) = \sum_{k=0}^{2^{p-1}-1} \frac{\psi_k(x)}{[k]^{\frac{1-\alpha}{2}}}. \tag{25}$$

We denote by $x_p = (0, 0, \cdots 0, 1, 0, \cdots)$ an element of 2^{ω} consisted of zeroes except the p-th number. Then

$$\begin{aligned} \widehat{G}_{2^{p(x_p)}}(x_p) &= \sum_{k=0}^{2^{p-1}-1} \frac{1}{k^{\frac{1-\alpha}{2}}} - \sum_{k=2^{p-1}}^{2^{p}-1} \frac{1}{k^{\frac{1-\alpha}{2}}} \\ &= \sum_{k=0}^{2^{p-1}-1} (C_k - C_{2^{p}-k-1}) \\ &\geqq \sum_{k=0}^{2^{p-1}-1} (k+1)\Delta C_k, \end{aligned} \tag{26}$$

where $C_k = \frac{1}{k^{\frac{1-\alpha}{2}}}$ and $\Delta C_k = C_k - C_{k+1}$.

From the fact that

$$\psi_k(x) = \psi_k(x_p), \quad \text{if } \frac{1}{2^{p}} \leqq |x| < \frac{1}{2^{p-1}} \text{ and } 0 \leqq k < 2^{p},$$

we have

$$\widehat{G}_{2^{p(x)}}(x) = \widehat{G}_{2^{p(x_p)}}(x_p) \geqq \sum_{k=0}^{2^{p-1}-1} (k+1)\Delta C_k, \quad \text{if } \frac{1}{2^{p}} \leqq |x| \leqq \frac{1}{2^{p-1}}. \tag{27}$$

We know (See [7, p. 228]) that for any sufficiently large n, say, $n \geqq N$, there exists a constant A such that

$$(28)\qquad A\sum_{k=0}^{l}(k+1)\Delta C_k \geqq |x|^{\frac{1-\alpha}{2}-1}, \text{ if } |x|<\frac{1}{2^n},$$

where l is the integral part contained in $\frac{1}{|x|}$. Therefore, combining this fact with (27), we have

$$(29)\qquad A\widehat{G}_{2^{p(x)}}(x)\geqq |x|^{\frac{1-\alpha}{2}-1} \text{ if } p-1\geqq N, \text{ that is, } |x|<\frac{1}{2^N}.$$

Consequently, we have

$$(30)\qquad AG_{2^{p(x)-1}}(x)=A\sum_{k=0}^{2^{p-1}-1}\frac{\psi_k(x)}{[k]^{\frac{1-\alpha}{2}}}=A\sum_{k=0}^{2^{p-1}-1}\frac{1}{[k]^{\frac{1-\alpha}{2}}}\geqq A\sum_{k=0}^{2^{p-1}-1}\frac{1}{k^{\frac{1-\alpha}{2}}}$$

$$>A\widehat{G}_{2^{p(x)}}(x)\geqq |x|^{\frac{1-\alpha}{2}-1}, \text{ if } |x|<\frac{1}{2^N}.$$

Here $G_{2^{p(x)-1}}(x)$ is a Borel-measurable and nonnegative function. Now we set

$$(31)\qquad E_N(t)=\left\{x\in 2^\omega\ ;\ |x+t|<\frac{1}{2^N}\right\}.$$

Then on the complement of $E_N(t)$, we have

$$|x+t|\geqq\frac{1}{2^N} \text{ and so } |x+t|^{\frac{1-\alpha}{2}-1}\leqq 2^{N(1-\frac{1-\alpha}{2})}.$$

Therefore, returning to (24), since

$$(32)\qquad \left[\int_{2^\omega}|x+t|^{\frac{1-\alpha}{2}-1}d\mu(x)\right]^2\leqq\left[\int_{E_N(t)}|x+t|^{\frac{1-\alpha}{2}-1}d\mu(x)+2^{N(1-\frac{1-\alpha}{2})}\right]^2$$

$$\leqq 2\left[\int_{E_N(t)}|x+t|^{\frac{1-\alpha}{2}-1}d\mu(x)\right]^2+2^{2N(1-\frac{1-\alpha}{2})+1},$$

we have

$$\text{(33)} \qquad I' \leqq A\int_{2^\omega}\left[\int_{E_N(t)} |x+t|^{\frac{1-\alpha}{2}-1} d\mu(x)\right]^2 dt + A$$

$$\leqq A\int_{2^\omega}\left[\int_{E_N(t)} G_{2^{p(x+t)}-1}(x+t)d\mu(x)\right]^2 dt + A$$

$$\leqq A\int_{2^\omega}\left[\int_{2^\omega} G_{2^{p(x+t)}-1}(x+t)d\mu(x)\right]^2 dt + A.$$

Then

$$\text{(34)} \qquad I' \leqq A\int_{2^\omega}\int_{2^\omega}\int_{2^\omega} G_{2^{p(x+t)}-1}(x+t)\cdot G_{2^{p(y+t)}-1}(y+t)dtd\mu(x)d\mu(y) + A$$

$$= A\int_{2^\omega}\int_{2^\omega}\sum_{k=0}^{2^{q(x,y)}-1}\frac{\psi_k(x+y)}{[k]^{1-\alpha}}d\mu(x)d\mu(y) + A$$

(where $q(x, y) = \min(p(x+t), p(y+t))$)

$$\leqq A\int_{2^\omega}\int_{2^\omega} K(x+y)d\mu(x)d\mu(y) + A \qquad \text{(from (1.12) and (1.13))}$$

$$\leqq AM + A.$$

Consequently, from (12), we have $I < +\infty$. But this contradicts (6), so that Theorem I is also established for the case $1 < p < 2$.

4. **Proof of Theorem II.** Suppose $\alpha+\varepsilon$-capacity of E is not zero. Then we have a positive constant M and an equilibrium distribution μ for E such that

$$\int_{2^\omega} \{x+t\}^{-\alpha-\varepsilon} d\mu(x) \leqq M$$

on 2^ω, and so by the relation (1.1), we have

$$\text{(35)} \qquad \int_{2^\omega} |x+t|^{-\alpha-\varepsilon} d\mu(x) \leqq AM$$

on 2^ω. Arguing in an entirely similar way as before, we arrive at (6):

(36) $$I=\int_{2^\omega} S_{n(x)}(x)d\mu(x)\to+\infty \quad (n\to+\infty),$$

On the other hand, since

(37) $$\int_{2^\omega}|x+t|^{\frac{1-\alpha}{p}-1}d\mu(x)=\int_{2^\omega}|x+t|^{\frac{-\alpha-\varepsilon}{p}}\cdot|x+t|^{\frac{1-p+\varepsilon}{p}}d\mu(x)$$
$$\leqq\left[\int_{2^\omega}|x+t|^{-\alpha-\varepsilon}d\mu(x)\right]^{\frac{1}{p}}\cdot\left[\int_{2^\omega}|x+t|^{\frac{1-p+\varepsilon}{p}\cdot q}d\mu(x)\right]^{\frac{1}{q}}$$
$$\leqq A\left[\int_{2^\omega}|x+t|^{\frac{q}{p}\varepsilon-1}d\mu(x)\right]^{\frac{1}{q}},$$

we have

(38) $$I=\int_{2^\omega}g(t)\left[\int_{2^\omega}G_{n(x)}(x+t)d\mu(x)\right]dt$$
$$\leqq A\int_{2^\omega}\left[|g(t)|\int_{2^\omega}|x+t|^{\frac{1-\alpha}{p}-1}d\mu(x)\right]dt$$
$$\leqq A\int_{2^\omega}|g(t)|\cdot\left[\int_{2^\omega}|x+t|^{\frac{q}{p}\varepsilon-1}d\mu(x)\right]^{\frac{1}{q}}dt$$
$$\leqq A\left[\int_{2^\omega}|g(t)|^p dt\right]^{\frac{1}{p}}\cdot\left[\int_\omega\int_\omega|x+t|^{\frac{q}{p}\varepsilon-1}d\mu(x)dt.\right]^{\frac{1}{q}}.$$

Remembering that $\int_{2^\omega}|g(t)|^p dt<\infty$, it is enough to estimate

(39) $$\int_{2^\omega}\int_{2^\omega}|x+t|^{\frac{q}{p}\varepsilon-1}d\mu(x)dt.$$

We set

(40) $$E_k=\left\{t\in 2^\omega;\ \frac{1}{2^{k+1}}<|x+t|\leqq\frac{1}{2^k},\quad k=0,1,2,\cdots.\right.$$

Then $$2^\omega=\bigcup_{k=0}^{\infty}E_k$$

and the measure of E_k is $\frac{1}{2^{k+1}}$. Therefore it follows that

$$\int_{2^\omega} |x+t|^{\frac{q}{p}\varepsilon-1} dt = \sum_{k=0}^{\infty} \int_{E_k} |x+t|^{\frac{q}{p}\varepsilon-1} dt \tag{41}$$

$$\leqq \sum_{k=0}^{\infty} 2^{(k+1)(1-\frac{q}{p}\varepsilon)} \cdot 2^{-(k+1)}$$

$$= \frac{1}{2^{\frac{q}{p}\varepsilon}-1}.$$

Consequently, we have $\int_{2^\omega} |x+t|^{\frac{q}{p}\varepsilon-1} dt \leqq A$ on 2^ω and so I is finite which clearly contradicts (36). This completes the proof of Theorem II.

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