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HOMOGENEITY AND SOME CURVATURE CONDITIONS FOR HYPERSURFACES*)

PATRICK J. RYAN

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Introduction. In this paper we investigate hypersurfaces of real space forms which satisfy certain conditions. Specifically, our results center around analogues for real space forms of non-zero curvature of the following theorems for Euclidean space.

THEOREM 0.1 (Nagano-Takahashi [1]). *A homogeneous hypersurface in Euclidean space is isometric to the Riemannian product of a sphere with a Euclidean space provided that the second fundamental form has rank different from 2 at some point.*

THEOREM 0.2 (Nomizu [2]). *A complete hypersurface in Euclidean space whose curvature operator when extended to act as a derivation on the tensor algebra at each point, satisfies $R(X,Y)\cdot R=0$ (for all tangent vectors X and Y) is congruent to the Riemannian product of a sphere with a Euclidean space (embedded as a cylinder over the sphere) provided that the second fundamental form has rank $\geqq 3$ at some point. (We allow the case when the Euclidean space has dimension zero in which case our hypersurface is a sphere.)*

In §1 we introduce the basic facts about hypersurfaces in Riemannian manifolds and in particular in spaces of constant curvature. The real space forms are mentioned together with the model hypersurfaces for the later classification theorems. Finally, rigidity of hypersurfaces in real space forms is discussed.

The main local decomposition theorems are proved in §2. The structure of the hypersurface is studied by looking at the distributions of principal vectors. Section 3 is devoted to a classification of complete Einstein hypersurfaces of real space forms and provides a global version of the work of Fialkow [3].

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In section 4 we prove an analogue of Theorem 0.2 for hypersurfaces in the sphere and also prove a local version for hypersurfaces in real hyperbolic space. The homogeneous case is discussed in §5 and a new proof of Theorem 0.1 is obtained, together with results for the sphere and hyperbolic space. Finally, in §6 we investigate the minimal hypersurfaces of real space forms which satisfy one of the conditions previously studied.

All manifolds and differentiable functions will be of class C^∞. Any notation not explicitly defined will be found in [4].

1. Hypersurfaces. Let $\widetilde{M}^{n+1}$ be a connected Riemannian manifold. A connected Riemannian manifold M^n together with an isometric immersion f of M into $\widetilde{M}$ is called a hypersurface of $\widetilde{M}$. If M admits two immersions f_1 and f_2 as hypersurfaces in $\widetilde{M}$, (M, f_1) and (M, f_2) are said to be congruent when there is an isometry τ of $\widetilde{M}$ such that $f_2 = \tau \circ f_1$.

For any immersion f of a manifold M^n into a manifold $\widetilde{M}^{n+k}$ it is a familiar notion that the tangent vectors to M may be considered as tangent vectors to $\widetilde{M}$ and it is possible to talk about covariant differentiation in $\widetilde{M}$ of vector fields along directions tangent to M. This idea is made precise by the following formalism due to Dombrowski.

We say that Z is a vector field in $\widetilde{M}$ along f if

(i) Z is a differentiable map of M into $T(\widetilde{M})$, and

(ii) the following diagram is commutative

$$\begin{tikzcd} & T(\widetilde{M}) \arrow[d, "\pi"] \\ M \arrow[ur, "Z"] \arrow[r, "f"'] & \widetilde{M} \end{tikzcd}$$

where π is the natural projection.

Let $\mathfrak{X}_f(\widetilde{M})$ denote the set of vector fields in $\widetilde{M}$ along f. $\mathfrak{X}_f(\widetilde{M})$ is a module over $\mathfrak{F}(\widetilde{M})$ in a natural way. If $X \in \mathfrak{X}(M)$, then $f_* X \in \mathfrak{X}_f(\widetilde{M})$. Also if $Y \in \mathfrak{X}(\widetilde{M})$, then the restriction of Y, which we denote by f^*Y, lies in $\mathfrak{X}_f(\widetilde{M})$.

Let $\widetilde{\nabla}$ be a Riemannian connection on $\widetilde{M}$. Then it can be proved that there is a unique mapping $\overline{\nabla} : \mathfrak{X}(M) \times \mathfrak{X}_f(\widetilde{M}) \to \mathfrak{X}_f(\widetilde{M})$ such that the following formulae hold:

(1) $\bar{\nabla}_{X_1+X_2}Z = \bar{\nabla}_{X_1}Z+\bar{\nabla}_{X_2}Z$

(2) $\bar{\nabla}_X(Z_1+Z_2) = \bar{\nabla}_X Z_1+\bar{\nabla}_X Z_2$

(3) $\bar{\nabla}_{(\phi\circ f)X}Z = \phi\bar{\nabla}_X Z$ for $\phi \in \mathfrak{F}(\widetilde{M})$

(4) $\bar{\nabla}_X(\phi Z) = X(\phi\circ f)Z+\phi\bar{\nabla}_X Z$ for $\phi \in \mathfrak{F}(\widetilde{M})$

(5) $\bar{\nabla}_X(f^*W)_p = f^*(\bar{\nabla}_V W)_{f(p)}$ for all $p \in M$ and $W \in \mathfrak{X}(\widetilde{M})$

provided that $f_*(X_p) = V_{f(p)}$.

In the case where (M^n, f) is a hypersurface in $\widetilde{M}^{n+1}$ we may define $\xi \in \mathfrak{X}_f(\widetilde{M})$ subject to the following conditions:

i) $g(\xi,\xi) = 1$

ii) $g(\xi, f_*X) = 0$ for all $X \in \mathfrak{X}(M)$.

Such a ξ can always be defined locally and is referred to as a field of unit normals. If M is simply connected and $\widetilde{M}$ is orientable, ξ may be defined globally on M. Wherever ξ is defined, it is uniquely determined up to a sign by i) and ii). For each $p \in M$, $T_{f(p)}(\widetilde{M})$ is the orthogonal direct sum of $f_*(T_p(M))$ and the span of ξ_p. Since f_* is one to one we can define, for X and Y in $\mathfrak{X}(M)$, $\nabla_X Y \in \mathfrak{X}(M)$ and $h(X, Y) \in \boldsymbol{R}$ by the formula

$$\bar{\nabla}_X(f_*Y) = f_*(\nabla_X Y)+h(X, Y)\xi.$$

It turns out that ∇ is just the Riemannian connection of M and that h is a tensor field on M of type $(0, 2)$. Furthermore if we define the operator A on tangent vectors to M by the equation

$$\bar{\nabla}_X\xi = -f_*(AX)$$

we can show that $h(X, Y) = g(AX, Y)$ for all X and $Y \in \mathfrak{X}(M)$. A is a symmetric tensor field of type $(1, 1)$ defined wherever ξ is defined and is called the second fundamental form. A^2 is well defined on all of M since any two normal fields agree up to a sign.

In the following we will consider $T_p(M)$ as a subspace of $T_{f(p)}(\widetilde{M})$ for all p whenever confusion is unlikely. We will replace $\bar{\nabla}_X(f_*Y)$ by $\widetilde{\nabla}_X Y$ and identify f_*X with X. With these identifications in mind we write

$$\widetilde{\nabla}_X Y = \nabla_X Y+g(AX, Y)\xi$$

$$\widetilde{\nabla}_X\xi = -AX.$$

A direct computation applied to these equations yields the Gauss and Codazzi equations :

$$\tilde{R}(X, Y)Z = R(X, Y)Z + g(AX, Z)AY - g(AY, Z)AX$$
$$\tilde{R}(X, Y)\xi = (\nabla_X A)Y - (\nabla_Y A)X.$$

In any inner product space V, we may identify $V \wedge V$ with the space of skew symmetric endomorphisms of V by setting

$$(a \wedge b)(c) = <b, c>a - <a, c>b.$$

With this notation the Gauss equation may be written

$$R(X, Y) = \tilde{R}(X,Y) + AX \wedge AY.$$

A complete, simply connected, connected Riemannian manifold of constant curvature is called a *real space form*. For each real number $\tilde{c}$ and each integer $n>1$ there is (up to isometry) exactly one n-dimensional real space form of constant curvature $\tilde{c}$.

All real space forms are *frame homogeneous*, i.e., for any pair of points x and y and any orthonormal frames u at x and v at y there is an isometry ϕ such that $\phi(x) = y$ and ϕ_* maps u onto v.

The real space forms are

(i) Euclidean space E^n ($\boldsymbol{R}^n$ with the usual inner product $X \cdot Y = \sum_{i=1}^{n} X^i Y^i$), $\tilde{c} = 0$.

(ii) Real hyperbolic space $H^n(\tilde{c})$ (the interior of the disk of radius $2a$ in $\boldsymbol{R}^n$ with $g(X, Y) = X \cdot Y/(1-r^2/4a^2)^2$, $\tilde{c} = -1/a^2 < 0$.

(iii) The sphere $S^n(\tilde{c})$ of radius a in Euclidean space with the metric induced from E^{n+1}, $\tilde{c} = 1/a^2 > 0$.

In a space of constant curvature $\tilde{c}$, the curvature tensor is expressed by $\tilde{R}(X, Y) = \tilde{c} X \wedge Y$. Thus for hypersurfaces of spaces of constant curvature, the equations of Gauss and Codazzi reduce to

$$R(X, Y) = \tilde{c} X \wedge Y + AX \wedge AY$$

and

$$(\nabla_X A)Y - (\nabla_Y A)X = 0.$$

The following is a list of the hypersurfaces which will appear in our classification theorems. Unless specified otherwise each is of the form (M, f) where f is an inclusion mapping. For $\widetilde{M}=E^{n+1}$ we have

1) Hyperplanes : $M=\{x \mid x_{n+1}=0\}$, $A=0$, $M \cong E^n$.

2) Spheres : $M=\{x \mid \|x\|^2=1/c\}$, $A=\sqrt{c}\, I$, $M \cong S^n(c)$.

3) Cylinders over spheres : $M=\{x \mid x_1^2+\cdots x_p^2=1/c\}$,

$$A=\sqrt{c}\, I_p \oplus 0,\ n>p>1.$$

4) Cylinders over complete plane curves : Let $K: \boldsymbol{R} \to E^2$ be a complete plane curve. $K(\boldsymbol{R})$ is isometric either to E^1 or S^1 depending on whether or not it is simply connected. If i denotes the identity map in E^{n-1} then $(E^1 \times E^{n-1}, K \times i\}$ is a complete hypersurface in E^{n+1}. $A=\lambda I_1 \oplus 0$ for some scalar function λ on the curve.

We also consider the following hypersurfaces in $S^{n+1}(\tilde{c})$

1) Great spheres : $M=\{x \mid \|x\|^2=1/\tilde{c},\ x_{n+2}=0\}$; $A=0$, $M \cong S^n(\tilde{c})$.

2) Small spheres : $M=\{x \mid \|x\|^2=1/\tilde{c},\ x_{n+2}=\sqrt{1/\tilde{c}-1/c}$,

$$A=\sqrt{c-\tilde{c}}\ I,\ M \cong S^n(c).$$

3) Product of spheres : $M=\{(x,y) \mid \|x\|^2=1/c_1,\ \|y\|^2=1/c_2,\ x \in E^{p+1}$,

$$y \in E^{n-p+1},\ 1/c_1+1/c_2=1/\tilde{c},\ 1<p<n-1\}.$$

$$M \cong S^p(c_1) \times S^{n-p}(c_2)\,;\ A=(1/\sqrt{c_1+c_2})\,(c_1 I_p \oplus(-c_2) I_{n-p}).$$

4) $M=E^1 \times S^{n-1}(c_2)$. $f(t,x)=((1/\sqrt{c_1}) \cos t,\ (1/\sqrt{c_1}) \sin t,\ x) \in S^{n+1}(\tilde{c}) \subseteq E^{n+2}$ where $1/c_1=1/\tilde{c}-1/c_2$. $A=(1/\sqrt{c_1+c_2})\,(c_1 I_1 \oplus(-c_2) I_{n-1})$.

For $H^{n+1}(\tilde{c})$ we will need only one kind of hypersurface

1) Hyperplanes: $M=\{x \mid x_{n+1}=0\}$; $A=0$, $M \cong H^n(\tilde{c})$.

Later, when classifying hypersurfaces up to congruence, the statement "(M,f) is a great sphere" (for instance) means that (M, f) is congruent to the hypersurface described above as a great sphere.

Let (M^n, f) be a hypersurface in $\widetilde{M}^{n+1}$. We recall that the rank of A_x is called the type number at x and is written $t(x)$. The type number depends on f but not on the choice of ξ. We will use $M^n(c)$ to denote a Riemannian manifold of constant curvature c.

PROPOSITION 1.1. *Let* (M^n, f) *be a hypersurface in* $\widetilde{M}^{n+1}(\tilde{c})$. *Then if* $t(x) \geqq 2$, $\ker A_x = \{X \in T_x(M) | R(X, Y) = \tilde{c} X \wedge Y$ *for all* $Y \in T_x(M)\}$.

PROOF. Denote the latter space by $T_0(x)$. If $AX = 0$, then $0 = AX \wedge AY = R(X, Y) - \tilde{c} X \wedge Y$ by the Gauss equation. Thus $\ker A_x \subseteq T_0(x)$. If we now choose X arbitrary in $T_0(x)$, there is a $Y \in T_x(M)$ such that $AY \neq 0$ and $g(AX, AY) = 0$ at x. Since $AX \wedge AY = 0$ at x, $g(AY, AX)AX = g(AX, AX)AY$ at x. Thus $g(AX, AX) = 0$, and hence $AX = 0$, namely, $X \in \ker A_x$.

PROPOSITION 1.2. *Let* f *and* $\bar{f}$ *be isometric immersions of* M^n *as a hypersurface in* $\widetilde{M}^{n+1}(\tilde{c})$. *If* $t(x)$ *for* f *is* $\geqq 3$ *for all* x, *then* $A = \pm \bar{A}$, *where* $\bar{A}$ *is the second fundamental form corresponding to* $\bar{f}$.

PROOF. We first observe that $t(x) = \bar{t}(x)$. For if $\bar{t}(x) \leqq 1$, then $\bar{A}X \wedge \bar{A}Y = 0$ for all X and Y and hence $\dim T_0(x) = n$ contrary to the fact that $t(x) \geqq 3$. Thus $\bar{t}(x) \geqq 2$ for all x. Hence $\ker \bar{A}_x = T_0(x) = \ker A_x$. Since A and $\bar{A}$ are symmetric, $\mathrm{Im}\, A_x = \mathrm{Im}\, \bar{A}_x = T_0(x)^{\perp}$. In particular $t(x) = \bar{t}(x)$.

Furthermore, for arbitrary $X \in T_x(M)$, $AX \wedge \bar{A}X = 0$. For if not, we may choose Y so that $AX \wedge \bar{A}X \wedge \bar{A}Y \neq 0$. But $AX \wedge AY = \bar{A}X \wedge \bar{A}Y$ since $R(X, Y) - \tilde{c} X \wedge Y$ is independent of the immersion. Thus $AX \wedge AX \wedge AY \neq 0$ which is a contradiction.

Thus, for each X there is a scalar c, possibly depending on X such that $AX = c\, \bar{A}X$. Choose X_1 and X_2 linearly independent in $T_0(x)^{\perp}$. Then $AX_1 = c_1 \bar{A} X_1$, $AX_2 = c_2 \bar{A} X_2$ and $A(X_1 + X_2) = c_3 \bar{A}(X_1 + X_2)$. But A and $\bar{A}$ are one to one on $T_0(x)^{\perp}$ and so $c_1 = c_2 = c_3$. Thus $AX = c\bar{A}X$ for some c independent of X. This equation also holds of course for $X \in \ker A$. Now

$$AX \wedge AY = c^2 \bar{A}X \wedge \bar{A}Y \quad \text{so} \quad c = \pm 1.$$

We conclude that $A_x = \pm \bar{A}_x$.

It is a standard result (see for example [5], p.207) that (M, f) and $(M, \bar{f})$ are congruent whenever they have the same second fundamental form. Under the hypothesis of Proposition 1.2, we can choose ξ in such a way that A and $\bar{A}$ coincide on M. Thus we have proved

THEOREM 1.3. *Let* f *and* $\bar{f}$ *be immersions of* M^n *as a hypersurface in a real space form* $\widetilde{M}^{n+1}$. *If* $t(x)$ *is* $\geqq 3$ *at each* x, *then* (M^n, f) *and* $(M^n, \bar{f})$ *are congruent.*

We recall that a hypersurface is said to be umbilical if A_x is a multiple of the identity for every x. We will show in §2 that in this case A is in fact a constant multiple of the identity. If A is identically zero, the hypersurface is totally geodesic. This means that in Dombrowski's formulation $\bar{\nabla}_X(f_*Y)=f_*(\nabla_X Y)$ for all X and $Y\in\mathfrak{X}(M)$. This definition of a totally geodesic submanifold makes sense for higher codimension as well.

We now investigate the possibility of realizing certain spaces of constant curvature c as hypersurfaces in a space of constant curvature $\tilde{c}$.

PROPOSITION 1.4. *Let $M^n(c)$ be a hypersurface in $\widetilde{M}^{n+1}(\tilde{c})$ where $c\neq\tilde{c}$ and $n>2$. Then $c>\tilde{c}$, rank $A=n$ and M^n is umbilical.*

PROOF. Let x be an arbitrary point of M. Choose mutually orthogonal unit eigenvectors $\{e_i\}_{i=1}^n$ for A_x and let $\{\lambda_i\}_{i=1}^n$ be the corresponding eigenvalues. Then by the Gauss equation

$$R(e_i, e_j) = (\lambda_i\lambda_j+\tilde{c})e_i\wedge e_j.$$

Since M has constant curvature c, $\lambda_i\lambda_j+\tilde{c}=c$ for distinct i and j. We conclude that all the λ_i are equal and non-zero. In fact $\lambda_i^2=c-\tilde{c}$. Thus $c>\tilde{c}$, rank $A=n$ and $A^2=(c-\tilde{c})I$.

COROLLARY 1.5. *Let f and $\bar{f}$ be immersions of $M^n(c)$ as a hypersurface in a real space form $\widetilde{M}^{n+1}(\tilde{c})$. Then if $n>2$ and $c\neq\tilde{c}$, (M^n, f) and $(M^n, \bar{f})$ are congruent.*

PROOF. Since $t(x)=n\geqq 3$ for all $x\in M$ we may apply Theorem 1.3 to obtain the result.

The case $c=\tilde{c}$ is more difficult and we use a theorem of O'Neill and Stiel [6] which may be stated as follows.

THEOREM 1.6. *A complete hypersurface $M^n(\tilde{c})$ of $\widetilde{M}^{n+1}(\tilde{c})$ is totally geodesic provided that $\tilde{c}>0$.*

No such result is true in the case $\tilde{c}=0$ since cylinders over complete plane curves need not be totally geodesic in E^{n+1}.

Theorem 1.6 provides us with a rigidity theorem in light of the following proposition.

PROPOSITION 1.7. *Let f and $\bar{f}$ be totally geodesic immersions of M^n as a hypersurface in a real space form $\widetilde{M}^{n+1}$. Then (M^n,f) and $(M^n,\bar{f})$ are congruent.*

PROOF. Since the second fundamental forms A and $\bar{A}$ are both zero, the standard result used in Theorem 1.3 gives the desired conclusion.

PROPOSITION 1.8. *Let (M^n, f) be a hypersurface of a real space form $\widetilde{M}^{n+1}$ such that either $t(x) \geqq 3$ for all x or $t(x)=0$ for all x. Then for every isometry ϕ of M, there is an isometry τ of $\widetilde{M}^{n+1}$ such that $f \circ \phi = \tau \circ f$.*

PROOF. We apply Theorem 1.3 or Proposition 1.7 to the immersions f and $f \circ \phi$.

For a given Riemannian manifold M, we denote by $\widehat{M}$ its simply connected Riemannian covering. The associated covering map π is a local isometry. If (M^n, f) is a hypersurface of $\widetilde{M}^{n+1}$, so is $(\widehat{M}^n, f \circ \pi)$. Let $\widehat{A}$ denote the second fundamental form for the latter hypersurface corresponding to some normal field ξ. Then A and $\widehat{A}$ are closely related. In fact, we have

PROPOSITION 1.9. *At each point x of $\widehat{M}$, $\widehat{A}_x = \pm \pi_*^{-1} A_{\pi(x)} \pi_*$.*

PROOF. Let X be a vector field on $\widehat{M}$.

$$\bar{\nabla}_X \hat{\xi} = -(f \circ \pi)_* \widehat{A} X = -f_* \pi_* \widehat{A} X$$

here we are regarding $\hat{\xi}$ as a vector field in $\widetilde{M}$ along $f \circ \pi$. On the other hand every vector field on M is locally of the form $\pi_* X$ for some $X \in \mathfrak{X}(\widehat{M})$. Regarding ξ as a vector field in $\widetilde{M}$ along f we write $\bar{\nabla}_{\pi_* X} \xi = -f_* A \pi_* X$.

Since $(f \circ \pi)_* (T_x(\widehat{M})) = f_*(\pi_* T_x(\widehat{M})) = f_*(T_{\pi(x)}(M))$, we see that $\hat{\xi}_x = \pm \xi_{\pi(x)}$. Thus for each $x \in M$

$$(\bar{\nabla}_X \hat{\xi})_x = \pm (\bar{\nabla}_{\pi_* X} \xi)_{\pi(x)}.$$

Since f_* is one to one, this implies $\pi_* \widehat{A} X = \pm A \pi_* X$ for all $X \in \mathfrak{X}(\widehat{M})$. Since π_* is one to one, the conclusion follows.

We remark that $A_{\pi(x)}$ and $\widehat{A}_x$ have the same rank and except perhaps for a difference in sign, the same eigenvalues. Also, since π is a local isometry, any local intrinsic property of M is also possessed by $\widehat{M}$. We shall make use of these facts freely in subsequent work.

2. Decomposition of hypersurfaces. Let (M^n, f) be a hypersurface in $\widetilde{M}^{n+1}$. The eigenvalues of A_x are called principal curvatures at x and the corresponding eigenvectors are called principal vectors. The following lemma assures us that the prncipal curvatures vary continuously on M.

LEMMA 2.1. *Let A be a symmetric tensor field of type* $(1,1)$ *on a connected Riemannian manifold M^n. Then there exist n continuous functions $\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \cdots \geqq \lambda_n$ such that for each x, $\{\lambda_i(x)\}_{i=1}^n$ are the eigenvalues of A_x.*

PROOF. Let $f(t, x) = t^n + a_1 t^{n-1} + a_2 t^{n-2} + \cdots + a_n$ be the characteristic polynomial of A_x. Each a_i is a differentiable function of x. Suppose $\{\xi_i\}_{i=1}^r$ are the distinct eigenvalues of A_{x_0} and let m_i be their respective multiplicities. Assume $\xi_1 > \xi_2 > \cdots > \xi_r$. Let $\varepsilon_0 > 0$ be arbitrary and let $\varepsilon = \min\{\varepsilon_0, 1/2, (1/2)\max|\xi_i - \xi_j|\}$. Let $C_i = \{z \in \boldsymbol{C} \mid |z - \xi_i| = \varepsilon\}$. Clearly $f(z,x_0) \neq 0$ on C_i. Choose $\delta_0 > 0$ so that if $d(x, x_0) < \delta_0$ (where d is the distance function on M^n arising from the Riemannian metric) then $f(z, x) \neq 0$ on C_i. Then $m_i = \dfrac{1}{2\pi i}\displaystyle\int_{C_i} \frac{f'(z,x_0)}{f(z,x_0)}\,dz$. However

$$\left|\frac{1}{2\pi i}\int_{C_i}\left(\frac{f'(z,x_0)}{f(z,x_0)} - \frac{f'(z,x)}{f(z,x)}\right)dz\right| \leqq \varepsilon \sup_{z \in C_i}\left|\frac{f'(z,x_0)}{f(z,x_0)} - \frac{f'(z,x)}{f(z,x)}\right|.$$

But this expression converges uniformly to 0 on C_i as $x \to x_0$. Thus there is $\delta < \delta_0$ such that

$$d(x,x_0) < \delta \text{ implies } \sup_{z \in C_i}\left|\frac{f'(z,x_0)}{f(z,x_0)} - \frac{f'(z,x)}{f(z,x)}\right| < 1.$$

Hence

$$\left|m_i - \frac{1}{2\pi i}\int_{C_i}\frac{f'(z,x)}{f(z,x)}dz\right| < \frac{1}{2} \quad \text{if} \quad d(x,x_0) < \delta.$$

Since the integral is integer-valued and is in fact the number of zeros of $f(z,x)$ inside C_i, we see that $f(z,x)$ has m_i zeros inside C_i.

In our case each λ_i is real and so if $d(x, x_0) < \delta$ then $|\lambda_i(x) - \lambda_i(x_0)| < \varepsilon \leqq \varepsilon_0$.

PROPOSITION 2.2. *Let A be as in Lemma 2.1 and suppose that exactly two eigenvalues $\lambda > \mu$ are distinct at each point. Then λ and μ have constant multiplicities and are differentiable.*

PROOF. Let $x_0 \in M$. Then

$$\lambda_1(x_0) = \lambda_2(x_0) = \cdots = \lambda_p(x_0) = \lambda_0 > \mu_0 = \lambda_{p+1}(x_0) = \cdots = \lambda_n(x_0).$$

By continuity of each λ_i, x_0 has a neighborhood where

$$\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_p > \frac{\lambda_0 + \mu_0}{2} > \lambda_{p+1} \geqq \cdots \geqq \lambda_n.$$

Since only two eigenvalues are distinct, $\lambda_1 = \lambda_2 = \cdots \lambda_p = \lambda > \mu = \lambda_{p+1} = \cdots = \lambda_n$ within this neighborhood. We have shown that for each integer p, $U_p = \{x \mid \text{multiplicity } \lambda(x) = p\}$ is open. But M is a finite union of such sets and M is connected, so λ has constant multiplicity.

Let p be the multiplicity of the larger eigenvalue λ. Define two functions $f = p\lambda + q\mu$ $(q = n-p)$ and $g = p(p-1)\lambda^2 + 2pq\lambda\mu + q(q-1)\mu^2$. Both f and g are differentiable since they are coefficients of the characteristic polynomial of A. Now $f^2 = p^2\lambda^2 + 2pq\lambda\mu + q^2\mu^2$. Thus $p\lambda^2 + q\mu^2 = f^2 - g$ and since $(q\mu)^2 = (f - p\lambda)^2$, we have $pn\lambda^2 - 2pf\lambda + f^2 - q(f^2 - g) = 0$. Set $h(t, x) = pnt^2 - 2pft + (1-q)f^2 + qg$. Note $h(\lambda, x) = 0$. $\partial h/\partial t = 2pnt - 2pf = 2p(nt - p\lambda - q\mu)$. Thus $\partial h/\partial t(\lambda_0, x_0) \neq 0$. By the quadratic formula, λ is the unique root of $h(t, x) = 0$ near x_0 which coincides with λ_0 at x_0. Since λ_0 is not a repeated root, λ is differentiable. Since $\mu = (\text{trace } A - p\lambda)/q$, μ is also differentiable.

PROPOSITION 2.3. *Let (M^n, f) be a hypersurface in $\tilde{M}^{n+1}(\tilde{c})$. If at each point of M, exactly two principal curvatures $\lambda \neq \mu$ are distinct, then the distribution $T_\lambda = \{X \mid AX = \lambda X\}$ is differentiable and involutive. If* dim *$T_\lambda > 1$, then $X\lambda = 0$ for $X \in T_\lambda$.*

PROOF. Let us restrict ourselves to a neighborhood of a point x_0 where ξ is defined and $\lambda > \mu$. By 2.2, λ and μ are differentiable and have constant multiplicities. Choose differentiable vector fields $X_1, X_2, \cdots, X_n$ near x_0 in such a way that $\{X_1, \cdots, X_p\}$ and $\{X_{p+1}, \cdots, X_n\}$ are bases at x_0 for T_λ and T_μ respectively. Let $Y_i = (A - \mu)X_i$, $1 \leqq i \leqq p$ and $Y_i = (A - \lambda)X_i$ for $p+1 \leqq i \leqq n$. Then the Y_i are differentiable and linearly independent near x_0. Furthermore $\{Y_1, \cdots, Y_p\}$ is a basis for T_λ and $\{Y_{p+1}, \cdots, Y_n\}$ for T_μ near x_0. This follows from the fact that $(A - \lambda)Y_i = (A - \lambda)(A - \mu)X_i = (A^2 - (\lambda + \mu)A + \lambda\mu)X_i = 0$ for $1 \leqq i \leqq p$ since $t^2 - (\lambda + \mu)t + \lambda\mu$ is the minimal polynomial of A.

Finally if X and Y are vectors in T_λ, $A[X, Y] = A(\nabla_X Y) - A(\nabla_Y X) = \nabla_X(AY) - \nabla_Y(AX)$ by the Codazzi equation. However, $AY = \lambda Y$ and $AX = \lambda X$ so

$$A[X, Y] = (X\lambda)Y - (Y\lambda)X + \lambda[X, Y]$$

$$(A - \lambda)[X, Y] = (X\lambda)Y - (Y\lambda)X.$$

The left side of this equation lies in T_μ and the right side is in T_λ. But $T_\lambda \cap T_\mu = (0)$ so $(A-\lambda)[X, Y] = 0$ and $[X, Y] \in T_\lambda$.

Also $(X\lambda)Y-(Y\lambda)X = 0$. If $\dim T_\lambda > 1$ we may choose X and Y to be linearly independent. Thus $X\lambda = 0$.

PROPOSITION 2.4. *Let (M^n, f) be an umbilical hypersurface in $\widetilde{M}^{n+1}(\tilde{c})$. Then M^n has constant curvature $c \geqq \tilde{c}$. The equality holds if and only if (M^n, f) is totally geodesic.*

PROOF. As above $A = \lambda I$ where λ is constant. The Gauss equation gives for any pair of tangent vectors X and Y

$$R(X, Y) = (\lambda^2 + \tilde{c})X \wedge Y$$

so the sectional curvature of all planes is $\lambda^2 + \tilde{c}$. Thus M^n has constant curvature $c = \lambda^2 + \tilde{c}$ and $c = \tilde{c}$ if and only if $\lambda = 0$.

THEOREM 2.5. *Let M^n be a hypersurface in $\widetilde{M}^{n+1}(\tilde{c})$ whose principal curvatures are constant. If exactly two are distinct, then M is locally isometric to the product of two spaces of constant curvature.*

PROOF. Let $\lambda > \mu$ be the two distinct principal curvatures. By Proposition 2.3, T_λ and T_μ are differentiable and involutive distributions.

If $X \in T_\lambda$, $Y \in T_\mu$, the Codazzi equation gives $\nabla_X(\mu Y) - \nabla_Y(\lambda X) = A\nabla_X Y - A\nabla_Y X$. Since λ and μ are constant, we get $(A-\lambda)\nabla_Y X = (A-\mu)\nabla_X Y$. The left side is in T_λ while the right side is in T_μ. Hence both sides are zero, $\nabla_Y X \in T_\lambda$, $\nabla_X Y \in T_\mu$. Now if $Z \in T_\lambda$

$$g(\nabla_Z X, Y) + g(X, \nabla_Z Y) = \nabla_Z(g(X, Y)) = 0.$$

On the other hand $\nabla_Z Y \in T_\mu$ so $g(X, \nabla_Z Y) = 0$. Thus we have shown $\nabla_Z X \in T_\mu^\perp$ for all Z and $X \in T_\lambda$. But $T_\lambda = T_\mu^\perp$. We may write our results $\nabla_{T_\lambda} T_\mu \subseteq T_\mu$ and $\nabla_{T_\lambda} T_\lambda \subseteq T_\lambda$. This means that T_λ (and similarly T_μ) is a parallel distribution.

It now follows from [4], p. 182, I, that M is locally isometric to the Riemannian product of the maximal integral manifolds M_λ and M_μ. Furthermore each integral manifold is totally geodesic in the sense that the Riemannian connection on M_λ (resp M_μ) is just the restriction of ∇ to M_λ(resp M_μ). The curvature tensors of M_λ and M_μ are therefore just the restrictions of R. If $X, Y \in T_\lambda$, then $R(X, Y) = (\lambda^2 + \tilde{c})X \wedge Y$. Thus if $\dim M_\lambda > 1$, M_λ is a space of constant curvature $\lambda^2 + \tilde{c}$. Similarly, M_μ. If $\dim M_\lambda = 1$ the product decomposition still holds but the sectional curvature is not defined. Since all one dimensional

Riemannian manifolds are locally isometric, there is nothing further to say.

In either case, however, we may now find the relationship between λ and μ. Take $X \in T_\lambda$, $Y \in T_\mu$. Since each distribution is parallel, $R(X, Y)Y \in T_\mu$. Thus, assuming X and Y are orthonormal,

$$0 = g(X, R(X, Y)Y) = g(X,(\lambda\mu+\tilde{c})(X \wedge Y)Y) = (\lambda\mu+\tilde{c})(g(Y,Y)g(X,X)-0).$$

Thus $\lambda\mu+\tilde{c} = 0$.

The assumption that exactly two principal curvatures are distinct is not as restrictive as would appear at first glance. In fact, for hypersurfaces of $\tilde{M}^{n+1}(\tilde{c})$, $\tilde{c} \leqq 0$, it offers no restriction in light of the following theorem of E. Cartan [7]. For a proof and generalization of this theorem we refer to Gray [8].

THEOREM 2. 6. *Let M^n be a hypersurface in $\tilde{M}^{n+1}(\tilde{c})$, $\tilde{c} \leqq 0$, whose principal curvatures are constant. Then at most two of them are distinct.*

If $\tilde{c}$ were allowed to be positive in Theorem 2. 6, the statement would be false. In §6 we will give an example of a hypersurface in $S^{n+1}(\tilde{c})$ with three distinct constant principal curvatures.

3. Einstein hypersurfaces.

We recall that a Riemannian manifold is said to be Einstein if the Ricci tensor is a constant multiple of the metric tensor, that is $S = \rho g$. We shall prove a local classification theorem, first proved by Fialkow [3] for Einstein hypersurfaces in spaces of constant curvature.

THEOREM 3. 1. *Let M^n, $n > 2$, be an Einstein hypersurface in $\tilde{M}^{n+1}(\tilde{c})$. If $\rho > (n-1)\tilde{c}$ then M is umbilical with $\lambda^2 = \rho/(n-1)-\tilde{c}$ and thus M is of constant curvature $\rho/(n-1)$. If $\rho = (n-1)\tilde{c}$ then $t(x) \leqq 1$ for all x and M is of constant curvature $\tilde{c}$. If $\rho < (n-1)\tilde{c}$ then $\tilde{c} > 0$, $\rho = (n-2)\tilde{c}$ and M is locally isometric to $M_1^p\left(\frac{n-2}{p-1}\tilde{c}\right) \times M_2^{n-p}\left(\frac{n-2}{n-p-1}\tilde{c}\right)$ where $1 < p < n-1$.*

PROOF. The Ricci tensor for a hypersurface in a space of constant curvature is given by the formula

$$S(X, Y) = (n-1)\tilde{c}g(X, Y) + g(AX, Y)\mathrm{trace}A - g(AX, AY).$$

Let $x_0 \in M$ *and let* $\{e_i\}_{i=1}^n$ be the unit principal vectors at x_0 corresponding to the principal curvatures $\{\lambda_i\}$. Since $S = \rho g$, we have for each i, $S(e_i, e_i) = \rho = (n-1)\tilde{c} + \lambda_i(\text{trace } A) - \lambda_i^2$. Thus at each point, the principal curvatures satisfy the quadratic equation $t^2 - st + \rho - (n-1)\tilde{c} = 0$ where $s = $ trace A. Thus at most two principal curvatures can be distinct at each point. Let us denote them by $\lambda \geqq \mu$.

Suppose $\rho > (n-1)\tilde{c}$. If $\lambda \neq \mu$ at some point, then λ and μ have the same sign. Also $p\lambda + (n-p)\mu = \operatorname{trace} A = \lambda + \mu$ where p is the multiplicity of λ. Hence $(p-1)\lambda + (n-p-1)\mu = 0$. This is a contradiction since $p=1$ and $n-p=1$ imply $n=2$. Hence M is umbilic and $\lambda^2 - (n\lambda)\lambda + \rho - (n-1)\tilde{c} = 0$. This implies $(n-1)\lambda^2 = \rho - (n-1)c$ and by the Gauss equation M has constant curvature $\rho/(n-1)$.

If $\rho < (n-1)\tilde{c}$ and $\lambda = \mu$ at some point, then the same formula $(n-1)\lambda^2 = \rho - (n-1)\tilde{c}$ holds. This is a contradiction since $\rho - (n-1)\tilde{c} < 0$. Hence M has exactly two distinct principal curvatures at each point. As above, $(p-1)\lambda + (n-p-1)\mu = 0$. Since neither λ nor μ can be zero (their product being $\rho - (n-1)\tilde{c}$), we must have $1 < p < n-1$. Thus

$$\lambda^2 = -\frac{n-p-1}{p-1}(\rho - (n-1)\tilde{c}) \quad \text{and} \quad \mu^2 = -\frac{p-1}{n-p-1}(\rho - (n-1)\tilde{c}).$$

Applying 2.5 we see that M is locally isometric to the product of spaces of constant curvature $\lambda^2 + \tilde{c}$ and $\mu^2 + \tilde{c}$ respectively. It also follows that $\lambda\mu + \tilde{c} = 0$. This implies $\rho - (n-1)\tilde{c} = -\tilde{c}$, that is $\rho = (n-2)\tilde{c}$ and $\tilde{c} > 0$. Hence

$$\lambda^2 + \tilde{c} = \frac{n-p-1}{p-1}\,\tilde{c} + \tilde{c} = \frac{n-2}{p-1}\,\tilde{c}.$$

Similarly

$$\mu^2 + \tilde{c} = \frac{n-2}{n-p-1}\,\tilde{c}.$$

This completes the proof when

$$\rho < (n-1)\tilde{c}.$$

If $\rho = (n-1)\tilde{c}$, the product of two distinct eigenvalues (if such exist) is zero. Thus if $\lambda \neq 0$, trace $A = p\lambda$ so $\lambda^2 - p\lambda^2 = 0$ implies $p=1$. Thus $t(x) \leqq 1$ for all x and M has constant curvature $\tilde{c}$ by the Gauss equation.

COROLLARY 3.2. *Let M^n, $n > 2$ be an Einstein hypersurface in $\tilde{M}^{n+1}(\tilde{c})$. If $\tilde{c} \leqq 0$, then $\rho \geqq (n-1)\tilde{c}$ and M is a space of constant curvature $\rho/(n-1)$.*

For $\tilde{M} = E^{n+1}$ and $S^{n+1}(\tilde{c})$ we now give the global versions of Theorem 3.1.

THEOREM 3. 3. *The complete Einstein hypersurfaces in E^{n+1} are spheres, cylinders over complete plane curves and hyperplanes.*

PROOF. From 3. 1 we have two choices, $\rho > 0$ and $\rho = 0$. In the latter case, $t(x) \leqq 1$ for all x. Thus M is locally isometric to E^n and by a theorem of Hartman-Nirenberg [9] is a cylinder over a complete plane curve. If $t(x)=0$ for all x, then M is totally geodesic and is just a hyperplane.

If $\rho > 0$, M has constant curvature $\rho/(n-1)$ and by 1. 4 $t(x) = n$ for all x and M is umbilical. Let $\widehat{M}$ be the simply connected Riemannian covering of M and denote the covering map by π. Then $(\widehat{M}, f\circ\pi)$ is a complete simply connected space of constant curvature $\rho/(n-1) > 0$ and hence is isometric to $S^n(\rho/(n-1))$. Thus $\widehat{M}$ is orientable and ξ can be defined globally on $\widehat{M}$. By 1. 3 $(\widehat{M}, f\circ\pi)$ is congruent to the standard sphere in E^{n+1}. In particular $f\circ\pi$ is one to one, π is one to one and M is simply connected. (M, f) is congruent to the standard sphere of curvature $\rho/(n-1)$.

REMARK. A proof of the Hartman-Nirenberg theorem, more in the spirit of this paper, may be found in the appendix of [2]. For the case $\rho > 0$ in 3. 3, a more elementary proof, based on the fact that an umbilical hypersurface in E^{n+1} is part of a hyperplane or a sphere, may be found in [4], Volume II, p. 36.

THEOREM 3. 4. *The complete Einstein hypersurfaces in $S^{n+1}(\tilde{c})$ are the small spheres, the great spheres and certain products of spheres.*

PROOF. Let (M^n, f) be a complete Einstein hypersurface in $S^{n+1}(\tilde{c})$. Then $(\widehat{M}^n, f\circ\pi)$ has the same properties. There are three possibilities : (i) $\rho > (n-1)\tilde{c}$ implies $\widehat{M}$ is a complete simply connected space of constant curvature $\rho/(n-1)$ and is hence isometric to $S^n(\rho/(n-1))$; (ii) $\rho=(n-1)\tilde{c}$ implies $\widehat{M}$ is isometric to $S^n(\tilde{c})$; (iii) $\rho=(n-2)\tilde{c}$. Since $\widehat{M}$ is simply connected, ξ and hence T_λ and T_μ can be defined on all of $\widehat{M}$. By [4] p. 187, I, $\widehat{M}$ is isometric to $S^p\left(\frac{n-2}{p-1}\tilde{c}\right) \times S^{n-p}\left(\frac{n-2}{n-p-1}\tilde{c}\right)$. By 1. 3, 1. 6 or 1. 7 $(M, f\circ\pi)$ is respectively a small sphere, great sphere or product of spheres. In each case $f\circ\pi$ is one to one and thus π is one to one. It follows that M is simply connected and (M^n, f) is congruent to the appropriate hypersurface.

We note that any space of constant curvature $\tilde{c}$ is Einstein with $\rho = (n-1)\tilde{c}$. Also the product of two spaces of constant curvature $M^p(c_1) \times M^{n-p}(c_2)$ is Einstein if $(p-1)c_1 = (n-p-1)c_2$. Thus the above hypersurfaces are indeed Einstein.

4. Hypersurfaces for which $R(X, Y)\cdot R=0$. For any pair X and Y of vector fields on a Riemannian manifold, $R(X, Y)$ is an endomorphism of the tangent space at each point. The mapping of the algebra of tensor fields into itself given by

$$T \rightarrow \nabla_X \nabla_Y T - \nabla_Y \nabla_X T - \nabla_{[X,Y]} T$$

is the unique derivation which extends $R(X, Y)$. Thus it is natural to write $R(X, Y)\cdot T$ for the image of an arbitrary tensor field T under this mapping.

In particular, we consider the (1, 3) tensor field $R(X, Y)\cdot R$. It acts on a pair of vector fields U and V as follows

$$(R(X,Y)\cdot R)(U,V)=[R(X,Y), R(U,V)]-R(R(X,Y)U,V)-R(U,R(X,Y)V).$$

We first observe that on a two dimensional Riemannian manifold $R(X,Y)\cdot R$ always vanishes. In order to verify this it is sufficient (because of the symmetry properties of R) to look at $(R(X,Y)\cdot R)(X,Y)$ for linearly independent X and Y.

We also note that if M is a locally symmetric Riemannian manifold (i. e., $\nabla_X R=0$ for all X), then $R(X,Y)\cdot R=0$ for all X and Y. The problem of finding an appropriate converse to this statement has been studied by Nomizu. In [2] he conjectures that the converse is true for complete, irreducible Riemannian manifolds of dimension at least three. The main theorem of his paper verifies this conjecture for hypersurfaces in Euclidean space as follows.

THEOREM 0. 2. *The complete hypersurfaces of Euclidean space such that $R(X,Y)\cdot R=0$ and $t(x)\geqq 3$ for some x are* (i) *spheres and* (ii) *cylinders over spheres of dimension at least three.*

For a compact Riemannian manifold M, the condition $R(X, Y)\cdot R=0$ together with $\nabla S=0$ (in particular the Einstein condition $S=\rho g$) implies that M is locally symmetric. We refer the reader to Lichnerowicz [10] pp. 9-11 and Yano [11] p. 222.

We now proceed to investigate the implications of the condition $R(X,Y)\cdot R=0$ for hypersurfaces in spaces of constant curvature.

PROPOSITION 4. 1. *Let M^n be a hypersurface in $\widetilde{M}^{n+1}(\tilde{c})$. Then $R(X,Y)\cdot R=0$ (for all tangent vectors X and Y) if and only if for distinct i,j,k the principal curvatures satisfy $(\lambda_i\lambda_j+\tilde{c})(\lambda_i-\lambda_j)\lambda_k=0$.*

PROOF. Let $\{e_i\}$ be an orthonormal basis of eigenvectors of A_{x_0}, corresponding to eigenvalues λ_i. If i,j,k are distinct we use the Gauss equation to get

$$\begin{aligned}
[R(e_i,e_j),R(e_i,e_k)] &= (\lambda_i\lambda_j+\tilde{c})(\lambda_i\lambda_k+\tilde{c})[e_i \wedge e_j,e_i \wedge e_k] \\
&= (\lambda_i\lambda_j+\tilde{c})(\lambda_i\lambda_k+\tilde{c})e_k \wedge e_j \\
R(R(e_i,e_j)e_i,e_k) &= -(\lambda_i\lambda_j+\tilde{c})R(e_j,e_k) \\
&= (\lambda_i\lambda_j+\tilde{c})(\lambda_j\lambda_k+\tilde{c})e_k \wedge e_j \\
R(e_i, R(e_i,e_j)e_k) &= 0\,.
\end{aligned}$$

Thus if $R(X, Y)\cdot R = 0$ for all X and Y, we must have

$$(\lambda_i\lambda_j+\tilde{c})(\lambda_i\lambda_k+\tilde{c}) = (\lambda_i\lambda_j+\tilde{c})(\lambda_j\lambda_k+\tilde{c})$$

$$(\lambda_i\lambda_j+\tilde{c})(\lambda_i-\lambda_j)\lambda_k = 0.$$

Conversely, suppose that this condition holds. It is sufficient to verify $R(e_i,e_j)\cdot R = 0$ for $i \neq j$. If i,j,k and l are all distinct every term in the expression for $(R(e_i,e_j)\cdot R)(e_k,e_l) = 0$. By the above condition $(R(e_i,e_j)\cdot R)(e_i,e_k) = 0$. Finally, the skew symmetry takes care of the rest.

COROLLARY 4.2. *Let M^n be a hypersurface in E^{n+1}. Then*

(1) *if $t(x) \leqq 2$, we have $R(X,Y)\cdot R = 0$ for all $X,Y \in T_x(M)$*

(2) *if $t(x) \geqq 3$ and $R(X,Y)\cdot R = 0$ for all $X, Y \in T_x(M)$,*

then the non-zero principal curvatures are equal.

PROOF. The formula above reduces to $\lambda_i\lambda_j\lambda_k(\lambda_i-\lambda_j) = 0$. Thus if $t(x) \leqq 2$ the formula is always satisfied. If $t(x) \geqq 3$ then if λ_i and λ_j are non-zero, we can find $\lambda_k \neq 0$ and by the formula $\lambda_i = \lambda_j$.

REMARK. Conclusion (2) is the starting point for Nomizu's proof of 0.2. Conclusion (1) shows that the condition $R(X,Y)\cdot R = 0$ offers no restriction on the class of hypersurfaces with $t(x) \leqq 2$ everywhere. The existence of an irreducible, complete hypersurface in $E^{n+1} (n \geqq 3)$ with $t(x) \leqq 2$ everywhere which is not locally symmetric is undecided. However, such a surface would have at least one x with $t(x) = 2$ by the Hartman-Nirenberg theorem.

Henceforth we will deal with the case $\tilde{c} \neq 0$.

PROPOSITION 4.3. *Let M^n be a hypersurface in $\widetilde{M}^{n+1}(\tilde{c})$, $\tilde{c} \neq 0$, $n > 2$ with $R(X,Y)\cdot R = 0$. Then for any $x \in M$ either $t(x) = n$ or $t(x) \leqq 1$.*

PROOF. Suppose $t(x) \neq n$ and thus $\lambda_i = 0$ for some i. Then for any $j \neq k$ distinct from i, $(\lambda_i\lambda_j + \tilde{c})(\lambda_i - \lambda_j)\lambda_k = 0$ so $\lambda_j\lambda_k\tilde{c} = 0$. Thus λ_j is non-zero for at most one j and $t(x) \leqq 1$. Furthermore if $t(x) = n$, then for any j, $\lambda_1\lambda_j + \tilde{c})(\lambda_1 - \lambda_j) = 0$ so $\lambda_j = \lambda_1$ or $\lambda_j = -\tilde{c}/\lambda_1$. Thus at most two principal curvatures are distinct at each point.

PROPOSITION 4.4. *Assume the hypothesis of Proposition* 4.3 *and in addition that at each point exactly two principal curvatures are distinct and they have multiplicities* > 1. *Then* M *is locally isometric to a product of two spaces of constant curvature.*

PROOF. Propositions 2.1 and 2.2 imply T_λ and T_μ have constant dimension, are differentiable and involutive. By 2.3 $X\lambda = 0$ if $X \in T_\lambda$. However $\mu = -\tilde{c}/\lambda$ so $X\mu = -(\tilde{c}/\lambda^2)X\lambda = 0$. Similarly $Y\lambda = Y\mu = 0$ for $Y \in T_\mu$. This λ and μ are constant. Theorem 2.5 gives the result.

If the assumption that both λ and μ have multiplicity greater than 1 is dropped (for instance if μ has multiplicity 1) then $X\lambda = 0$ and $X\mu = 0$ for $X \in T_\lambda$. But we know nothing about $Y\lambda$ or $Y\mu$ where $Y \in T_\mu$. This leads us to make the following definition.

DEFINITION. Let M^n be a hypersurface in $\tilde{M}^{n+1}(\tilde{c})$. A point $x \in M$ is called *bad* if (i) A_x is non-singular and (ii) A_x has a simple eigenvalue. All other points are called *good.*

REMARK. Given a hypersurface M^n in $\tilde{M}^{n+1}(\tilde{c})$, $\tilde{c} \neq 0$ with $R(X,Y)\cdot R = 0$ we have shown that it consists of points of the following types

I $A = \lambda I \neq 0$.

II $A = 0$.

III A has 2 unequal non-zero eigenvalues of multiplicity > 1.

IV A has 2 unequal non-zero eigenvalues of multiplicity 1 and $n-1$.

V A has 2 unequal eigenvalues, λ of multiplicity 1, 0 of multiplicity $n-1$.

PROPOSITION 4.5. *Assume the hypothesis of* 4.3. *Then the set of bad points is open.*

PROOF. Let x_0 be a bad point. Choose a neighborhood U of x_0 where (i)

A_x is non-singular and (ii) $\lambda_1(x) > \frac{\lambda(x_0)+\mu(x_0)}{2} > \lambda_2(x) \geqq \lambda_3(x) \cdots \geqq \lambda_n(x)$. By the last statement in the proof of 4. 3, $\lambda_2(x)=\lambda_3(x)=\cdots=\lambda_n(x)$ in U. Hence all points in U are bad.

REMARK. The same argument shows that the set of points of type III is open and the multiplicities remain constant in sufficiently small neighborhoods of points of this type.

PROPOSITION 4. 6. *Under the hypothesis of* 4. 3 *if* $c>0$ *then the set of bad points is closed.*

PROOF. Suppose a sequence of bad points x_i converges to $x \in M$. Since $\lambda(x_i) \neq \mu(x_i)$, $\lambda(x_i)\mu(x_i)+\tilde{c}=0$. By continuity, $\lambda(x)\mu(x)+\tilde{c}=0$. Since $\lambda^2(x)+\tilde{c}\neq 0$, x must be of type III or IV. Since the points of type III form an open set, x must be of type IV.

PROPOSITION 4. 7. *Assume the hypothesis of* 4. 3 *and all points are good. Then either* $t(x) \leqq 1$ *for all* x *or* $t(x)=n$ *for all* x.

PROOF. Let $F=\{x \mid t(x) \leqq 1\} = \{x \mid \det A_x = 0\}$. Clearly F is closed. Since M is connected, it will be sufficient to show that F is also open. First consider a sequence of points y_i of type III converging to some point y_0. Since the principal curvatures are continuous and the equation $\lambda\mu+\tilde{c}=0$ holds for each member of the sequence, it also holds at y_0. It follows that y_0 cannot lie in F.

Let x_0 be an arbitrary member of F. The above argument shows that x_0 has a connected neighborhood U which contains no points of type III. We will now show that U has no points of type I. Suppose there is a point y of type I in U. Let $W=\{x \in U \mid \det A_x = \det A_y\}$. Clearly W is closed in U. Choose an arbitrary x in W. Since A_x is non-singular, x has a (connected) neighborhood $U' \subseteq U$ where A is non-singular. U' consists entirely of umbilics. By Proposition 2.4, λ is constant on U' and is equal to $\lambda(x)$. This shows that W is open and hence $W=U$. This cannot happen since x_0 belongs to F. We conclude that there are no type I points in U. Thus $U \subseteq F$ and the proof is complete.

COROLLARY 4. 8 *Let* M^n, $n>2$, *be a hypersurface in* $\tilde{M}^{n+1}(\tilde{c})$, $\tilde{c}\neq 0$. *If all points are good,* $R(X,Y)\cdot R=0$ *and* $t(x)\leqq 1$ *for some* x, *then* M *is a space of constant curvature* $\tilde{c}$.

PROPOSITION 4. 9. *Let* M^n *be a hypersurface in* $\tilde{M}^{n+1}(\tilde{c})$ ($\tilde{c}>0$, $n>2$). *Assume* $t(x)\geqq 2$ *for some* x *and that* M *has at least one good point. If*

$R(X,Y)\cdot R=0$ *the umbilics form an open and closed set.*

PROOF. We first note that all umbilics are of type I. This follows from Propositions 4.5, 4.6 and 4.7. Let $x_i \to x$ where the x_i are non-umbilic points. Then the x_i are of type III and $\lambda(x_i)\mu(x_i)+\tilde{c}=0$. By continuity of λ and μ, we get $\lambda(x)\mu(x)+\tilde{c}=0$. Since $\tilde{c}>0$, x cannot be an umbilic. Thus the set of non-umbilics is closed. The other direction is trivial. $\{x \mid \lambda(x)=\mu(x)\}$ is closed.

THEOREM 4.10. *The complete hypersurfaces of* $S^{n+1}(\tilde{c})$ *satisfying* $R(X,Y)\cdot R=0$ *and having at least one good point are* (i) *all small spheres,* (ii) *all great spheres,* (iii) *all products of spheres* (*of dimensions* >1).

PROOF. We first note that spheres and products of spheres satisfy $\nabla R=0$ and hence satisfy $R(X,Y)\cdot R=0$. They are also, of course, complete.

If $t(x)\leqq 1$ for some x, M^n is a space of constant curvature by 4.8. In particular it is Einstein. If $t(x)\geqq 2$ for some x and M is umbilic, again M is Einstein. The arguments of Theorem 3.4 give us great spheres and small spheres respectively. By 4.9 the other possibility is that there are no umbilics and every point is of type III.

$(\widehat{M}^n, f\circ\pi)$ has a globally defined normal, constant principal curvatures and globally defined parallel distributions T_λ and T_μ of dimension >1. Thus $\widehat{M}^n$ is isometric to $S^p(\lambda^2+\tilde{c})\times S^{n-p}(\mu^2+\tilde{c})$. By 1.3, $f\circ\pi$ is one to one. Thus π is one to one, M is simply connected and (M, f) is a product of spheres as required.

5. Homogeneous hypersurfaces.

A Riemannian manifold is said to be *homogeneous* if its group of isometries is transitive. A homogeneous Riemannian manifold is always complete ([4] p. 176, I).

In this section we study those homogeneous Riemannian manifolds which occur as hypersurfaces in real space forms.

THEOREM 5.1. *Let* (M^n, f) *be a homogeneous hypersurface in* $\widetilde{M}^{n+1}(\tilde{c})$. *Then either* $t(x)\leqq 1$ *for all* x *or* $t(x)$ *is constant on* M.

PROOF. We recall (Proposition 1.1) that if $t(x)\geqq 2$ then $\ker A_x=T_0(x)$ $=\{X \mid R(X,Y)=\tilde{c}X\wedge Y$ for all $Y\}$. Now if y is another point of M, there is an isometry ϕ of M with $\phi(x)=y$ since M is homogeneous. ϕ_* is a linear isomorphism of $T_x(M)$ onto $T_y(M)$ preserving inner products. Thus $\phi_*(T_0(x))$ $=\{\phi_*X \mid R(X,Y)=\tilde{c}X\wedge Y$ for all $Y\}=\{\phi_*X \mid R(\phi_*X,\phi_*Y)=\tilde{c}\phi_*X\wedge\phi_*Y$ for all $Y\}$ $=T_0(y)$. Thus $\dim T_0(x)\leqq\dim T_0(y)$. By symmetry, $\dim T_0(y)\leqq\dim T_0(x)$ so $t(x)=t(y)$.

If $t(x)\leqq 1$ for some x, then $\dim T_0(x)=n$. Since by the above argument, $\dim T_0$ is constant, $t(x)\leqq 1$ for all x. This completes the proof.

THEOREM 5.2. *Let (M^n, f) be a homogeneous hypersurface of a real space form $M^{n+1}(\tilde{c})$. Then if $t(x) \geqq 3$ for all $x \in M$, the principal curvatures of M are constant.*

PROOF. Choose $x, y \in M$ and an isometry ϕ of M such that $\phi(x) = y$. By Proposition 1.8, ϕ may be extended to an isometry τ of $\widetilde{M}$ (i.e., $\tau \circ f = f \circ \phi$). If ξ is a field of unit normals near x, then $\tau_* \xi$ is a field of unit normals near y. The following sequence of equalities implies that $A_y = \pm \phi_* A_x \phi_*^{-1}$ and so $A_y^2 = \phi_* A_x^2 \phi_*^{-1}$.

$$\tau_* \widetilde{\nabla}_{f_* X} \xi = -\tau_* f_* AX = -f_* \phi_* AX.$$

On the other hand,

$$\widetilde{\nabla}_{\tau_* f_* X} \tau_* \xi = \widetilde{\nabla}_{f_* \phi_* X} \tau_* \xi = \pm f_* A \phi^* X.$$

Thus A_x^2 and A_y^2 have the same eigenvalues. Thus the squares of the principal curvature functions are constant. Since these functions are continuous (Lemma 2.1), they are themselves constant.

THEOREM 5.3. *Let M^n be a homogeneous hypersurface in a real space form $\widetilde{M}^{n+1}(\tilde{c})$ where $\tilde{c} \leqq 0$. Then if for some x, $t(x) \neq 2$ either*

(i) M^n is a space of constant curvature $\tilde{c}$ and $t(x) \leqq 1$ for all x

(ii) M^n is a space of constant curvature $c > \tilde{c}$, $t(x) = n$ for all x and the immersion is umbilical

(iii) M is locally isometric to $M_1^p(\lambda^2 + \tilde{c}) \times M_2^{n-p}(\mu^2 + \tilde{c})$ where $\lambda\mu + \tilde{c} = 0$ and $1 < p < n-1$.

(iv) M is locally isometric to $M_1^1 \times M_2^{n-1}(\mu^2 + \tilde{c})$ for some μ.

PROOF. By 5.1, either $t(x) \leqq 1$ for all x or $t(x) \geqq 3$ is constant on M. If the former holds then M is a space of constant curvature $\tilde{c}$. If the latter holds the principal curvatures are constant by 5.2. If M is umbilic then by 2.4 M has constant curvature $c > \tilde{c}$ and $t(x) = n$. If M is not umbilic, then there are exactly two distinct principal curvatures by 2.6. By 2.5, M is locally isometric to the product of two spaces of constant curvature in the manner described by (iii) and (iv).

We are now in a position to give a new proof of the theorem of Nagano and Takahashi for Euclidean space.

THEOREM 0.1. *Let (M^n, f) be a homogeneous hypersurface in E^{n+1} such that $t(x) \neq 2$ for some x. Then M^n is isometric to one of the following*

(i) E^n

(ii) $S^1 \times E^{n-1}$

(iii) $S^k(c) \times E^{n-k}$, $c>0$, $2<k \leqq n-1$

(iv) $S^n(c)$, $c>0$.

PROOF. If $t(x) \leqq 1$ for all x, then we invoke, as in Theorem 3.3, Hartman-Nirenberg Theorem [9] to show that M is isometric to E^n or $S^1 \times E^{n-1}$ depending upon whether or not M is simply connected.

Otherwise $t(x) \geqq 3$ for all x and $t(x)$ is constant. The simply connected covering hypersurface $(\widehat{M}, f \circ \pi)$ has the same properties by 1.9. If $t(x)=n$, $\widehat{M}$ is a complete, simply connected space of constant curvature $c>0$ so is isometric to $S^n(c)$. If $t(x)=k<n$, then ξ is defined globally on $\widehat{M}$ and thus T_λ and T_0 are defined on all of $\widehat{M}$. By [4] p. 187 $\widehat{M}$ is isometric to the Riemannian product of the maximal integral manifolds. Each of these is complete, simply connected and of constant curvature λ^2 and 0 respectively. Then $\widehat{M}$ is isometric to $S^k(\lambda^2) \times E^{n-k}$. Now 1.3 implies that in each of these cases $(\widehat{M}, f \circ \pi)$ is congruent to the corresponding model spaces. Thus in particular $f \circ \pi$ is one and hence so is π. We conclude that M was already simply connected and π is an isometry. We have also proved

COROLLARY 5.4. *The homogeneous hypersurfaces in E^{n+1} such that $t(x) \neq 2$ for some x are* (i) *all hyperplanes*; (ii) *all cylinders over spheres of dimension greater than* 2, (iii) *all spheres*; (iv) *all cylinders over complete plane curves.*

THEOREM 5.5. *Let (M^n, f) be a homogeneous hypersurface in $H^{n+1}(\tilde{c})$, $\tilde{c}<0$ such that for some x, $t(x) \neq 2$. Then one of the following is true*:

(i) *M is a space of constant curvature $c>\tilde{c}$ and is umbilical.*

(ii) *M is a space of constant curvature $\tilde{c}$.*

(iii) *M is locally isometric to $M_1^p(\lambda^2+\tilde{c}) \times M_2^{n-p}(\mu^2+\tilde{c})$ where $1<p<n-1$ and $\lambda\mu+c=0$.*

(iv) *M is locally isometric to $M_1^1 \times M_2^{n-1}(\mu^2+\tilde{c})$ for some constant $\mu \neq 0$.*

PROOF. By 5.1 and 5.2, either $t(x) \leqq 1$ for all x or $3 \leqq t(x)=$ constant. In the former case M is a space of constant curvature $\tilde{c}$. If $t(x) \geqq 3$ for all x, then the principal curvatures are constant and at most two can be distinct (5.2 and 2.6) say $\lambda \geqq \mu$. If $\lambda=\mu$. M is a space of constant curvature $\lambda^2+\tilde{c}$ and the immersion is umbilical. If $\lambda>\mu$, we apply 2.5 to get conclusions (iii) or (iv).

THEOREM 5.6. *Let (M^n, f) be a homogeneous hypersurface in $S^{n+1}(\tilde{c})$ such that $t(x) \neq 2$ for some x. Assume further that at some point at most two principal curvatures are distinct. Then one of the following conclusions holds*:

(i) *M is a space of constant curvature $c \geqq \tilde{c}$ and (M, f) is umbilical.*

(ii) *M is locally isometric to $M_1^p(\lambda^2+\tilde{c}) \times M_2^{n-p}(\mu^2+\tilde{c})$ where $1<p<n-1$ and $\lambda\mu+\tilde{c}=0$.*

(iii) *M is locally isometric to $M_1^1 \times M_2^{n-1}(\mu^2+\tilde{c})$ for some constant μ.*

PROOF. The argument is the same as 5.5 with the following two exceptions. Proposition 2.6 does not apply when $\tilde{c}>0$ so we assume at most two principal curvatures are distinct. The fact that (M, f) is umbilic when $c=\tilde{c}$ follows from the fact that M is complete and from Theorem 1.6.

REMARK. If $t(x)=2$ but at most two principal curvatures are distinct at each point, the conclusions of 5.6 still hold. For if λ is the nonzero principal curvature, T_λ has dimension 2. For arbitrary $x, y \in M$ choose an isometry ϕ of M with $\phi(x)=y$. Then $\phi_*(T_0(x))=T_0(y)$. Since $T_\lambda(x)=T_0(x)^\perp$, ϕ_* preserves T_λ as well. Thus if X and Y are orthonormal vectors in $T_x(M)$, $R(X, Y) =(\lambda^2+\tilde{c}) X \wedge Y$. Applying ϕ_* we get $R(\phi_* X, \phi_* Y)=(\lambda^2+\tilde{c}) \phi_* X \wedge \phi_* Y$. Hence $\lambda^2(x)=\lambda^2(y)$. Thus λ is constant. Applying 2.5, $\lambda \cdot 0+\tilde{c}=0$ a contradiction.

THEOREM 5.7. *Let (M^n, f) be as in 5.6. Then (M^n, f) is either* (i) *a small sphere,* (ii) *a great sphere,* (iii) *a product of spheres or M^n is locally isometric to $M_1^1 \times M_2^{n-1}(\mu^2+\tilde{c})$ for some constant μ. In the last case if M is simply connected it is congruent to the immersion of $E^1 \times S^{n-1}(c_2)$ onto $S^1(c_1) \times S^{n-1}(c_2)$ where $1/c_1+1/c_2=1/\tilde{c}$.*

PROOF. $(\widehat{M}^n, f \circ \pi)$ satisfies the same hypotheses and in addition is simply-connected. The arguments of 3.4. give the first three results. In the last case $\widehat{M}$ is isometric to $E^1 \times S^{n-1}(c_2)$ $(c_2=\mu^2+\tilde{c})$ and by 1.3 $(\widehat{M}, f \circ \pi)$ is congruent to the hypersurface described above.

6. Minimal hypersurfaces.

A hypersurface $(\widehat{M}^n, f)$ in $\widehat{M}^{n+1}$ is said to be minimal if trace $A_x=0$ for all $x \in M$. The class of minimal hypersurfaces in $M^{n+1}(\tilde{c})$ is very large so we shall discuss only those which have one of the additional properties we have studied earlier. However, the minimality assumption will sometimes permit us to weaken the other hypotheses. For example we shall discuss minimal homogeneous hypersurfaces with $t(x)=2$.

PROPOSITION 6.1. *Let (M^n, f) be a homogeneous hypersurface in $\widetilde{M}^{n+1}(\tilde{c})$*

and assume $t(x)=2$ *on* M. *Then the non-zero principal curvatures* λ *and* μ *satisfy* $\lambda\mu+\tilde{c}=$ *constant.*

PROOF. Following the notation of 5.1 we consider the two dimensional distribution T_0. Denote by $K(x)$ the sectional curvature of this distribution at x. Since ϕ_* maps $T_0(x)$ onto $T_0(y)$, the sectional curvatures $K(x)$ and $K(y)$ are equal. On the other hand, $K(x)=\lambda(x)\mu(x)+\tilde{c}$ by the Gauss equation for all x. Hence $\lambda\mu+\tilde{c}$ is constant.

COROLLARY 6.2. *If* $\tilde{c}\leqq 0$ *and* $n>2$, *then there are no minimal homogeneous hypersurfaces of* $\widetilde{M}^{n+1}(\tilde{c})$ *with* $t(x)=2$.

PROOF. By 6.1, $\lambda\mu+\tilde{c}=\text{constant}$. By minimality $\mu=-\lambda$. Thus λ^2 is constant. By continuity of λ, λ is constant. But 2.6 implies that $\lambda, -\lambda$ and 0 cannot be distinct, a contradiction.

THEOREM 6.3. *The minimal homogeneous hypersurfaces of* E^{n+1} *and* $H^{n+1}(\tilde{c})$ *are hyperplanes if* $n>2$.

PROOF. 6.2 allows us to assume that $t(x)\neq 2$. The only minimal hypersurfaces in the classification of 5.4 are hyperplanes. For $H^{n+1}(\tilde{c})$ we look at 5.5. If $\lambda\mu+\tilde{c}=0$ and $p\lambda+(n-p)\mu=0$ then $p\lambda^2+(n-p)(-\tilde{c})=0$. But λ^2 and $(-\tilde{c})$ are positive. Thus the only minimal homogeneous hypersurfaces are totally geodesic spaces of constant curvature, i,e., hyperplanes.

PROPOSITION 6.4. *Let* (M^n, f), $n>2$, *be a minimal homogeneous hypersurfaces in* $S^{n+1}(\tilde{c})$ *such that for some point, at most two principal curvatures are distinct. Then* (M^n, f) *is a great sphere, a product of spheres isometric to* $S^p\left(\frac{n\tilde{c}}{p}\right)\times S^{n-p}\left(\frac{n\tilde{c}}{n-p}\right)$ *for* $1<p<n-1$ *or* M *is locally isometric to* $M_1\times M_2^{n-1}\left(\frac{n\tilde{c}}{n-1}\right)$.

PROOF. If $t(x)\neq 2$, Theorem 5.7 together with the assumption of minimality gives the appropriate restrictions on c, λ and μ. If $t(x)=2$ for all x, then the non-zero principal curvatures must be equal. Thus trace $A=2\lambda\neq 0$ a contradiction to minimality.

The necessity of the assumption on the number of distinct principal curvatures is demonstrated by the following example due to Wu-Yi Hsiang [12].

Consider E^9 as the space of 3 by 3 real matrices with inner product $\langle A,B\rangle = \text{trace } AB^t$. Consider the subset $\{A \mid A \text{ is symmetric, trace } A=0,\ \|A\|=1\}$.

Clearly this subset is isometric to $S^4(1)$ since it is the intersection of $S^8(1)$ with the four independent hyperplanes $x_{12}=x_{21}$, $x_{13}=x_{31}$, $x_{23}=x_{32}$, $x_{11}+x_{22}+x_{33}=0$. Let $SO(3)$ act on E^9 by conjugation. Then $S^4(1)$ is invariant under this action. The orbit of the point $\begin{pmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a certain submanifold M of $S^4(1)$. The isotropy subgroup at this point is easily seen to be the finite subgroup K of $SO(3)$ consisting of the four matrices $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Hence $SO(3)/K$ is diffeomorphic to M and the Riemannian metric of M induced by $S^4(1)$ can be transferred to $SO(3)/K$. The simply connected Riemannian covering $\widehat{M}$ of M is diffeomorphic to S^3 though not isometric to a sphere. In any case we have $(\widehat{M}, i\circ\pi)$ is a simply connected homogeneous hypersurface in $S^4(1)$. We can show that it is minimal as well and that the principal curvatures are constant. However, the following proposition shows that $\widehat{M}$ cannot be any of the spaces of Proposition 6.4.

PROPOSITION 6.5. *Let (M^n, f) be a hypersurface in $\widetilde{M}^{n+1}(\tilde{c})$, $\tilde{c}\neq 0$ and suppose that for some x, $t(x)<n$. Then M is locally irreducible.*

PROOF. Let $T_0=\{X \mid R(X,Y)=\tilde{c}X\wedge Y \text{ for all } Y\}$. Clearly dim $T_0(x)\geqq 1$. Suppose T is a parallel distribution near x. If $X\in T_0$, $Y\in T$ (X and Y non-zero), then $R(X\ Y)\cdot Y\in T$ since T is parallel and $R(X,Y)$ is in the holonomy algebra. Thus $\tilde{c}(g(Y,Y)X-g(X,Y)Y)\in T$. But $Y\in T$ and $\tilde{c}g(Y,Y)\neq 0$ so $X\in T$. Thus $T_0\subseteq T$. If $T_x(M)=\bigoplus_{i=1}^{r} T_i$ is a decomposition into parallel distributions, $T_0\subseteq\bigcap_{i=1}^{r} T_i=(0)$. This is a contradiction if $r\neq 1$. Thus M is locally irreducible.

PROPOSITION 6.6.

(i) *The complete minimal Einstein hypersurfaces of E^{n+1} and $H^{n+1}(\tilde{c})$ are hyperplanes if $n>2$.*

(ii) *The complete minimal Einstein hypersurfaces of $S^{n+1}(\tilde{c})$ are great spheres and products of spheres $S^{n/2}(2\tilde{c})\times S^{n/2}(2\tilde{c})$. The latter case occurs only if n is even.*

PROOF. Adding the assumption to minimality to 3.1, 3.2 and 3.4, we see that either (i) $\rho=(n-1)\bar{c}$ and we get hyperplanes or great spheres or (ii) $\rho=(n-2)\bar{c}>0$. As in 3.1 we also have $\lambda+\mu=\operatorname{trace} A=0$ so $\lambda^2+\bar{c}$ $=\frac{n-2}{p-1}\bar{c}=\mu^2+\bar{c}=\frac{n-2}{n-p-1}\bar{c}$. Thus $n=2p$ and $\frac{n-2}{p-1}\bar{c}=2\bar{c}$. By 3.4 we have the appropriate product of spheres.

PROPOSITION 6.7. *The complete minimal hypersurfaces of $H^{n+1}(\bar{c})$, $n>2$ satisfying $R(X,Y)\cdot R=0$ are hyperplanes.*

PROOF. By 4.3 either $t(x)=n$ or $t(x)\leqq 1$ in which case minimality implies $t(x)=0$. If $t(x)=n$ and λ and μ are distinct principal curvatures at x, then $p\lambda+(n-p)\mu=0\rightarrow p\lambda^2+(n-p)(-\bar{c})=0$ which is impossible since $\lambda^2>0$ and $-\bar{c}>0$. Thus $t(x)=0$ for all x.

PROPOSITION 6.8. *Let (M^n, f) be a complete minimal hypersurface in $S^{n+1}(\bar{c})$, $n>2$, such that $R(X,Y)\cdot R=0$. Then the conclusion of 6.4 holds.*

PROOF. As above either $t(x)=0$ or $t(x)=n$ and $p\lambda^2=(n-p)\bar{c}$, $(n-p)\mu^2=p\bar{c}$. The set of bad points is open and closed by 4.5 and 4.6. Also by 4.7, $t(x)=0$ for all x or $t(x)=n$ for all x. If all points are good, (M^n, f) is a great sphere or product of spheres by 4.10. If all points are bad, λ and μ are still constant and M is locally isometric to $M_1^1\times M_2^{n-1}(n\bar{c}/(n-1))$.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NOTRE DAME
NOTRE DAME, INDIANA, U. S. A.