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# CLUSTER SETS OF ALGEBROID FUNCTIONS

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**1. Introduction.** In the present paper we shall deal with the cluster sets of algebroid functions defined in the disk $U: |z| < 1$ and propose two main theorems for these functions; one is Plessner's theorem ([9], Satz I, cf. [5], p. 147, Theorem 8.2) and the other is Meier's theorem, i.e., the so-called topological analogue of Plessner's theorem ([7], Satz 5, cf. [5], p. 154, Theorem 8.8). Here, an algebroid function $f(z)$ defined in $U$ is a multiple-valued function $w = f(z)$ from $U$ into the extended $w$-plane $\Omega: |w| \leqq \infty$, or the Riemann sphere, defined by an irreducible (with respect to meromorphic coefficients) algebraic equation:

$$f^n(z) + a_1(z)f^{n-1}(z) + \cdots + a_n(z) = 0\,, \tag{1.1}$$

where $a_1(z), \cdots, a_n(z)$ are single-valued meromorphic functions in $U$ (cf. [12], [11], p. 15).

It should be noted that in the vicinity of a compact set of logarithmic capacity zero Noshiro ([8], §4) treated these functions and enunciated some results on cluster sets; but there was no discussion in $U$.

We now refer to Plessner's theorem for algebroids in $U$. Although we shall assume the theorem for the single-valued case, it seems that our extension of the theorem is not an immediate consequence of the single-valued case. Indeed, there seem to be some obstacles in treating the point at infinity.

In studying the cluster sets of multiple-valued functions one is naturally led to the notion of "set-mappings" (cf. [14]). It must be emphasized that some theorems as Collingwood's maximality theorems ([3], [4], cf. [5], p. 75 ff.) and Bagemihl's ambiguous point theorem ([1], Theorem 2, cf. [5], p. 85, Theorem 4.12) are true of algebroid functions, or more generally, of set-mappings ([14]). We shall use Collingwood's maximality theorem in proving Meier's theorem (cf. Lemma 1).

We explain the contents of the present paper. First in §2 we shall give terminologies, notation and definitions of some cluster sets of algebroid functions following the standard lines as in the single-valued cases (cf. [5]). In §§3 and 4 we shall give the proof of Plessner's theorem (Theorem 1). In §5 we shall prove two lemmas for proving Meier's theorem (Theorem 2). The so-called Meier's topological analogue of Fatou's theorem ([7], Satz 6, cf. [5], p. 154, Theorem 8.9) will be expressed in the form of a theorem (§6, Theorem 3), while Theorems 1 and 2 are stated in §2. As is well-known, the proof of Meier's theorem depends mainly upon the Fatou-type theorem, i.e., Theorem 3 in our paper, so that we shall omit the detailed proof which contains only topological arguments. However, there are some non-trivialities which do not appear in the single-valued case and which we shall note in the last part of §6.

**2. Terminologies, notation and Theorems.** We denote $n$ branches of $f(z)$ defined by (1.1) at $z \in U$ by $f_1(z), \cdots, f_n(z)$. For any subset $T \neq \emptyset$ (non-empty) of $U$ we shall denote by $f(T)$ the subset of $\Omega$ defined by the following: $w \in f(T)$ if and only if $w = f_k(z)$ for some $z \in T$ and some $k$, $1 \leqq k \leqq n$. In particular, if $T = \{z\}$, a one-point set, then we use $f^*(z)$ instead of $f(\{z\})$. Evidently,

$$f(T) = \bigcup_{z \in T} f^*(z) .$$

We now let $G \neq \emptyset$ be a subset of $U$ and $t \in \overline{G}$, the closure with respect to the disk: $|z| \leqq 1$. Then the cluster set $C_G(f, t)$ of $f$ at $t$ relative to $G$ is defined by

$$C_G(f, t) = \bigcap_{\delta} \overline{f(\delta \cap G)} ,$$

where the intersection is taken over all open disks $\delta$ containing $t$ and the closure is taken in $\Omega$. We can easily show that $w \in C_G(f, t)$ if and only if there exists a point-sequence $\{z_\nu\} \subset G$ with $z_\nu \to t$ and $f_{(\nu)}(z_\nu) \to w$, for some $f_{(\nu)}(z_\nu) \in f^*(z_\nu) (\nu = 1, 2, \cdots)$.

Let $e^{i\theta}$ be a point of the circle $K$: $|z| = 1$. We define some cluster sets as follows:

*(Full) cluster set* $C(f, e^{i\theta}) \equiv C_U(f, e^{i\theta})$. This is the cluster set of $f$ at $e^{i\theta}$ relative to $U$.

*Angular cluster set* $C_\Delta(f, e^{i\theta})$. By an angle $\Delta$ at $e^{i\theta}$ we shall mean the interior of a triangle lying in $U$ except for one vertex $e^{i\theta}$. In this paper $\Delta$ will always stand for an angle at some point of $K$. The angular cluster set is the cluster set of $f$ at $e^{i\theta}$ relative to $\Delta$.

*Curvilinear cluster set* $C_\gamma(f, e^{i\theta})$ *and chordal cluster set* $C_{\rho(\varphi)}(f, e^{i\theta})$. Let $\gamma$ be a simple open arc in $U$ with the initial point in $U$ and the terminal point $e^{i\theta}$. Then the cluster set of $f$ at $e^{i\theta}$ relative to $\gamma$ will be called the curvilinear cluster set of $f$ at $e^{i\theta}$. Particularly if $\gamma$ is a chord $\rho(\varphi)$ of $K$ passing through $e^{i\theta}$ and making a directed angle $\varphi$, $|\varphi|<\pi/2$, with the radius to $e^{i\theta}$, then the corresponding cluster set $C_{\rho(\varphi)}(f, e^{i\theta})$ is called chordal.

We now classify the points of $K$.

*Plessner point $e^{i\theta}$ of $f$.* This is the point where $C_{\Delta}(f, e^{i\theta})$ is total, i.e., $C_{\Delta}(f, e^{i\theta})=\Omega$ for any $\Delta$ at $e^{i\theta}$.

*Fatou point $e^{i\theta}$ of $f$.* This is the point at which

$$\bigcup_{\Delta} C_{\Delta}(f, e^{i\theta})$$

consists of at most $n$ points in $\Omega$, where the summation is taken over all $\Delta$ at $e^{i\theta}$. This is equivalent to say that $C_{\Delta}(f, e^{i\theta})$ contains at most $n$ points for any $\Delta$ at $e^{i\theta}$.

*Meier point $e^{i\theta}$ of $f$.* This is the point such that (a) $C(f, e^{i\theta}) \neq \Omega$ and that (b) $C_{\rho(\varphi)}(f, e^{i\theta}) = C(f, e^{i\theta})$ for all $\varphi$, $|\varphi| < \pi/2$.

We shall denote the totality of Plessner (Fatou, Meier, resp.) points of $f$ by $I(f)$ ($F(f)$, $M(f)$, resp.).

In the case where $n=1$, the function $f(z)$ is single-valued and meromorphic in $U$ and all definitions and notation in the above are the same as the usual ones (cf. [5]). We shall be, of course, interested in the case $n \geqq 2$.

For two measurable sets $A$ and $B$ on $K$ we denote $A \cong B$ if and only if $A \backslash B$ and $B \backslash A$ both are of linear measure zero on $K$. As to the definition of the category in the sense of Baire we refer to [5], p. 75.

THEOREM 1. *Let $f(z)$ be an algebroid function in $|z|<1$ defined by (1.1). Then the sets $F(f)$ and $I(f)$ both are measurable on the unit circle $K$ and we have*

(i) $$K \cong F(f) \cup I(f),$$

(ii) $$F(f) \cong F(a_1) \cap \cdots \cap F(a_n)$$

*and*

(iii) $$I(f) \cong I(a_1) \cup \cdots \cup I(a_n).$$

THEOREM 2. *Let $f(z)$ be an algebroid function in $|z|<1$. Then all points of the circle $|z|=1$ except perhaps for a set of first Baire category belong to $M(f) \cup I(f)$.*

3. **Proof of Theorem 1.** We shall prove Theorem 1 in two steps; first in the present section we shall prove (ii) and next in §4 we shall show (i) using (ii). Then combining (i) with (ii) we obtain (iii) by the classical

Plessner's theorem which we shall assume in what follows.

Let $g(z)$ be a single-valued meromorphic function in $U$. Then Plessner's theorem asserts the following decomposition of $K$:

$$K = F(g) \cup I(g) \cup E(g), \tag{3.1}$$

where $E(g)$ is a set of linear measure zero. Next, for a fixed $w \in \Omega$, we let $F(g:w)$ be the set of points $e^{i\theta} \in F(g)$ at which $g(z)$ has the angular limit $w$. Then the set $F(g:w)$ is measurable (cf. [10], p. 219, foot-note) and of linear measure zero by Lusin-Privalov's theorem (cf. [10], p. 212), unless $g(z) \equiv w$ in $U$. The set $F(g:\infty)$ is always of linear measure zero.

Let $\Omega_0 = \{w_1, w_2, \cdots\}$ be a countable set of points dense in $\Omega$ such that $|w_j| < \infty$ $(1 \leqq j < \infty)$. We set, for every $w_j \in \Omega_0$,

$$\begin{aligned} A_j(z) &= \left\{ \prod_{\nu=1}^{n} (w_j - f_\nu(z)) \right\}^{-1} \\ &\equiv \{w_j^n + a_1(z)\, w_j^{n-1} + \cdots + a_n(z)\}^{-1} \end{aligned} \tag{3.2}$$

in $U$, where we use the well-known relation:

$$\begin{cases} -a_1(z) = f_1(z) + \cdots + f_n(z) \\ \quad a_2(z) = f_1(z) f_2(z) + \cdots + f_{n-1}(z) f_n(z) \\ \quad \cdots\cdots\cdots\cdots \\ (-1)^n a_n(z) = f_1(z) f_2(z) \cdots f_n(z). \end{cases} \tag{3.3}$$

Then $A_j(z)$ $(1 \leqq j < \infty)$ are single-valued meromorphic functions $\not\equiv 0$ in $U$, so that $F(A_j:0)$ are of linear measure zero.

We first prove:

$$F(f) \setminus \left\{ \bigcap_{j=1}^{n} F(a_j) \right\} \text{ is of linear measure zero}. \tag{3.4}$$

For the proof we consider two possible cases, i.e.,

Case (I): $e^{i\theta} \in F(f)$ and $\bigcap_{\Delta} C_\Delta(f, e^{i\theta}) \not\ni \infty$,

Case (II): $e^{i\theta} \in F(f)$ and $\bigcap_{\Delta} C_\Delta(f, e^{i\theta}) \ni \infty$.

Case (I). We may find a $\Delta_1$ at $e^{i\theta}$ bisected by the radius drawn to $e^{i\theta}$ such that $C_{\Delta_1}(f, e^{i\theta}) \not\ni \infty$, so that we obtain a $\Delta \subset \Delta_1$ at $e^{i\theta}$ such that

$$\overline{f(\Delta)} \not\ni \infty .$$

This means that we may find a constant $M > 0$ with $|f_k(z)| < M$ for any $z \in \Delta$ and any $k$, $1 \leqq k \leqq n$. Therefore, by (3.3), $a_1(z), \cdots, a_n(z)$ are all bounded in $\Delta$, so that $e^{i\theta} \notin I(a_j)$ for any $j$, $1 \leqq j \leqq n$, or

$$e^{i\theta} \in \bigcap_{j=1}^{n} \{F(a_j) \cup E(a_j)\} . \tag{3.5}$$

Case (II). We can find a $\Delta$ at $e^{i\theta}$ such that $\overline{f(\Delta)} \neq \Omega$, so that we may find a point

$$w_{j_0} \in \Omega_0 \cap \{\Omega \setminus \overline{f(\Delta)}\} .$$

The function $A_{j_0}(z)$ corresponding to $w_{j_0}$ is, therefore, bounded in $\Delta$; this means that $e^{i\theta} \notin I(A_{j_0})$. Assume that $e^{i\theta} \in F(A_{j_0})$. Since $C_{\Delta}(f, e^{i\theta}) \ni \infty$, there exists a sequence of points $\{z_\nu\} \subset \Delta$ such that $z_\nu \to e^{i\theta}$ and $f_{(\nu)}(z_\nu) \to \infty$ as $\nu \to \infty$, where $f_{(\nu)}(z_\nu) \in f^*(z_\nu)$. We may assume that $f_1(z_\nu) \to \infty$ as $\nu \to \infty$, by re-suffixing $f_1(z_\nu), \cdots, f_n(z_\nu)$ of $f^*(z_\nu)$. We can choose then a subsequence $\{z_{\nu_j}\} \subset \{z_\nu\}$ such that

$$f_1(z_{\nu_j}) \to \infty \quad \text{and} \quad f_k(z_{\nu_j}) \to \alpha_k \in C_\Delta(f, e^{i\theta})$$

as $j \to \infty$ for $2 \leqq k \leqq n$. Therefore we have

$$A_{j_0}(z_{\nu_j}) \to 0 \quad \text{as} \quad j \to \infty ,$$

since $\alpha_k \neq w_{j_0}$ for $2 \leqq k \leqq n$, which proves $e^{i\theta} \in F(A_{j_0}:0)$. We have thus obtained

$$e^{i\theta} \in \bigcup_{j=1}^{\infty} \{F(A_j:0) \cup E(A_j)\} , \tag{3.6}$$

the right-hand-side set being a set of linear measure zero which we denote by $Q$.

Combining (3.5) with (3.6) we have

$$F(f) \subset \left[\bigcap_{j=1}^{n} \{F(a_j) \cup E(a_j)\}\right] \cup Q,$$

so that we have (3.4).

To prove that

$$(3.7)\qquad \left\{\bigcap_{j=1}^{n} F(a_j)\right\} \setminus F(f) \text{ is of linear measure zero,}$$

we need a preparation.

The multiple-valued function $f$ can be realized as a single-valued meromorphic function $\mathcal{F}$ from a Riemann surface $\Phi$ into $\Omega$, $\Phi$ being a covering Riemann surface over $U$ with the projection map $z=\pi(p)$, having $n$ sheets and containing at most a countable number of branch points ($f(z) = \mathcal{F}\circ\pi^{-1}(z)$ with a slight ambiguity). Then the cluster set $C_\Delta(f, e^{i\theta})$ has the following equivalent definition:

$$C_\Delta(f, e^{i\theta}) = \bigcap_{r>0} \overline{\mathcal{F}(\mathcal{D}_r)},$$

where $\mathcal{D}_r = \pi^{-1}(\delta_r)$, $\delta_r$ being the intersection of $\Delta$ with the open disk with the centre $e^{i\theta}$ and the radius $r>0$.

We can prove easily that $C_\Delta(f, e^{i\theta})$ does not consist of just $k$ ($n+1 \leqq k < \infty$) components since $\mathcal{D}_r$ consists of at most $n$ components (cf., e.g., [13], especially Chap. II for the details).

We are now ready to prove (3.7); in fact, we can prove much more, i.e.,

$$(3.7)'\qquad \left\{\bigcap_{j=1}^{n} F(a_j)\right\} \setminus F(f) \subset \bigcup_{j=1}^{n} F(a_j : \infty).$$

Let $e^{i\theta}$ be a point of the left-hand-side set in the inclusion relation (3.7)$'$ and assume that $e^{i\theta} \notin \bigcup_{j=1}^{n} F(a_j : \infty)$. Then we can find some $\Delta$ at $e^{i\theta}$ such that $C_\Delta(f, e^{i\theta})$ contains $n+1$ distinct points $\alpha_1, \cdots, \alpha_{n+1}$ in $\Omega$. We may assume that all $\alpha_1, \cdots, \alpha_{n+1}$ are distinct from $\infty$, since $C_\Delta(f, e^{i\theta})$ contains $n+2$ distinct points as was stated. We can find, therefore, $n+1$ sequences of points: $\{z_j^{(\nu)}\}_{j=1}^{\infty} \subset \Delta$ ($1 \leqq \nu \leqq n+1$) such that

$$z_j^{(\nu)} \to e^{i\theta} \text{ and } f_{(\nu)}(z_j^{(\nu)}) \to \alpha_\nu \text{ as } j \to \infty$$

for $\nu$, $1 \leqq \nu \leqq n+1$, where $f_{(\nu)}(z_j^{(\nu)}) \in f^*(z_j^{(\nu)})$. Let $\beta_j$ be the angular limit of $a_j$, which is finite by our assumption ($1 \leqq j \leqq n$). Then from (1.1) of §1 we have

$$\{f_{(\nu)}(z_j^{(\nu)})\}^n + a_1(z_j^{(\nu)})\{f_{(\nu)}(z_j^{(\nu)})\}^{n-1} + \cdots + a_n(z_j^{(\nu)}) = 0$$

and letting $j \to \infty$, we have

$$\alpha_\nu^n + \beta_1 \alpha_\nu^{n-1} + \cdots + \beta_n = 0 \quad (\nu = 1, 2, \cdots, n+1).$$

This is a contradiction since the matrix

$$\begin{pmatrix} \alpha_1^n & \alpha_1^{n-1} \cdots 1 \\ \alpha_2^n & \alpha_2^{n-1} \cdots 1 \\ \cdots & \cdots \\ \alpha_{n+1}^n & \alpha_{n+1}^{n-1} \cdots 1 \end{pmatrix}$$

is invertible. Since $F(a_j : \infty)$ $(1 \leqq j \leqq n)$ are of linear measure zero we have (3. 7) as a consequence of (3. 7)$'$.

Now that (3. 4) and (3. 7) have been proved we have (ii) of Theorem 1.

4. **Proof of Theorem 1 (continued)**. There exist a countable number of angles $\Delta_1(0)$, $\Delta_2(0)$, $\cdots$ at $z=1$ such that for any angle $\Delta(0)$ at $z=1$ we may find a $\Delta_j(0)$ with $\Delta_j(0) \subset \Delta(0)$. We denote by $\Delta_j(\theta)$ the angle at $e^{i\theta}$ obtained by rotation of $\Delta_j(0)$ $(1 \leqq j < \infty)$. We denote by $k_1$, $k_2$, $\cdots$ the totality of closed spherical disks with the centres in $\Omega_0$, $\Omega_0$ being defined in §3, and the spherical radii of rational numbers.

We denote by $G_{j,\nu}$ the set of points $e^{i\theta} \in K$ such that

$$\overline{f(\Delta_j(\theta))} \cap \mathring{k}_\nu = \emptyset ,$$

where $\mathring{k}_\nu$ is the interior of $k_\nu$ $(1 \leqq j < \infty,\ 1 \leqq \nu < \infty)$. Then the set $G_{j,\nu}$ is a closed subset of $K$ for $1 \leqq j < \infty$ and $1 \leqq \nu < \infty$.

Set $E = K \backslash \{F(f) \cup I(f)\}$ and let $e^{i\theta} \in E$. Then we can find an angle $\Delta(\theta)$ at $e^{i\theta}$ such that $C_{\Delta(\theta)}(f, e^{i\theta}) \neq \Omega$ since $e^{i\theta} \notin I(f)$. Therefore, there exists a $k_\nu$ such that $C_{\Delta(\theta)}(f, e^{i\theta}) \cap k_\nu = \emptyset$, so that we can find a $\Delta_j(\theta) \subset \Delta(\theta)$ such that

$$\overline{f(\Delta_j(\theta))} \cap k_\nu = \emptyset .$$

This shows that $E \subset \bigcup_{j,\nu} G_{j,\nu}$, or

$$E = \bigcup_{j,\nu} E_{j,\nu} , \quad \text{where } E_{j,\nu} = E \cap G_{j,\nu} . \tag{4. 1}$$

The measurability of $E_{j,\nu}$ is obtained as an easy consequence of

$$E_{j,\nu}=G_{j,\nu}\backslash(G_{j,\nu}\backslash E_{j,\nu})$$

and

$$G_{j,\nu}\backslash E_{j,\nu} = G_{j,\nu} \cap F(f)\,,$$

since $G_{j,\nu}$ is closed and $F(f)$ is measurable by (ii) of Theorem 1, proved already in §3. Since measurability of $E_{j,\nu}$ $(1 \leqq j < \infty,\ 1 \leqq \nu < \infty)$ is proved, the rest we have to prove is, by (4.1), that the existence of a set $E_{j,\nu}$ of positive linear measure for some $j, \nu$ leads us to a contradiction, which proves that $E$ is of linear measure zero.

Let $P$ be a perfect set of positive linear measure contained in $E_{j,\nu}$. Let $r_j$ be the maximum of the distances from two vertices of the angle $\Delta_j(\theta)$, other than $e^{i\theta}$, to the origin $z=0$. We note that $r_j$ is, in fact, independent of the value $\theta$. Let $\alpha$ be the centre of the disk $k_\nu$. We say that a point $z \in U$ is an $\alpha$-point of $f$ if there is a point $p \in \Phi$ with $\pi(p) = z$ and $\mathcal{F}(p) = \alpha$. The $\alpha$-points of $f$ cannot accumulate at any point in $U$ since the surface $\Phi$ is $n$-sheeted. We choose $\rho$, $r_j < \rho < 1$ such that the circle $|z| = \rho$ contains no $\alpha$-point. Let $z_1, \cdots, z_m$ be the totality of $\alpha$-points of $f$ in the disk $R_\rho: |z| < \rho$. By deleting suitable $m$ closed disks in $R_\rho$ with centres $z_1, \cdots, z_m$ respectively, from $R_\rho$, we obtain a domain $R'_\rho$ such that

$$\overline{f(R'_\rho)} \cap \tau = \emptyset\,, \tag{4.2}$$

where $\tau$ is an open disk with the centre $\alpha$ contained in $\mathring{k}_\nu$.

We now set

$$D = R'_\rho \cup \left\{\bigcup_{e^{i\theta} \in P} \Delta_j(\theta)\right\}\,.$$

Then the boundary $\Gamma$ of $D$ consists of a finite number of rectifiable Jordan curves and $\Gamma \supset P$. We have, by the construction of $D$,

$$\overline{f(D)} \cap \tau = \emptyset\,.$$

Let $\beta_1, \cdots, \beta_{n+1}$ be $n+1$ distinct finite points in $\tau$ and consider the similar functions as in §3:

$$B_j(z) = \left\{\prod_{\nu=1}^{n} (\beta_j - f_\nu(z))\right\}^{-1}$$

$$\equiv \{\beta_j^n + a_1(z)\beta_j^{n-1} + \cdots + a_n(z)\}^{-1}\,, \quad (1 \leqq j \leqq n+1)\,.$$

Then all $B_j(z)$ are single-valued, bounded and analytic in $D$, so that they all have finite angular limits at a.e. point of $\Gamma$ by the generalized Fatou's theorem ([10], p. 129). We note that $B_j(z) \not\equiv 0$ for $1 \leqq j \leqq n+1$. By solving the following equation:

$$\beta_1^n + a_1(z)\beta_1^{n-1} + \cdots + a_n(z) = (B_1(z))^{-1}$$

$$\beta_2^n + a_1(z)\beta_2^{n-1} + \cdots + a_n(z) = (B_2(z))^{-1}$$

$$\cdots\cdots$$

$$\beta_{n+1}^n + a_1(z)\beta_{n+1}^{n-1} + \cdots + a_n(z) = (B_{n+1}(z))^{-1},$$

with respect to 1, $a_1(z), \cdots, a_n(z)$, we know that $a_1(z), \cdots, a_n(z)$ are linear combinations of $(B_1(z))^{-1}, \cdots, (B_{n+1}(z))^{-1}$ in $D$. Therefore, all $a_1(z), \cdots, a_n(z)$ have angular limits with respect to $D$ at a.e. point of $P$. We may say that these angular limits are also angular limits with respect to $U$ since the boundary of $D$ has a tangent at a.e. point of $P$ and this tangent must coincide with the tangent to $K$. We know, therefore, that

$$P \cap \left\{ \bigcap_{j=1}^{n} F(a_j) \right\}$$

is of positive linear measure. Combining this with (ii) of Theorem 1 we have

$$P \cap F(f) \neq \emptyset ,$$

which contradicts our hypothesis on $P$.

**5. Collingwood's maximality theorem and Schwarz's lemma.** We may consider $f^*(z)$ as a set-mapping from $U$ into $\Omega$ (cf. [14]). Then by Theorem 3 in [14], being an extension of Collingwood's maximality theorem (cf. [5], p. 79, Theorem 4.9) to set-mappings, and by the standard technique (cf. [4], §7, Theorem 4) we have the following

LEMMA 1. *For an algebroid function $f(z)$ in $U$ we have a set $J(f)$ on $K$ such that $K \backslash J(f)$ is of first category and that for every $e^{i\theta} \in J(f)$ we have*

$$C_\Delta(f, e^{i\theta}) = C(f, e^{i\theta})$$

*for any angle $\Delta$ at $e^{i\theta}$.*

PROOF. To make our paper complete we shall give the proof. As was stated above, we have only to prove the following: *Let $f(z)$ be algebroid in $U$. Let $\Delta(\theta)$ be the angle at $e^{i\theta}$ obtained by rotation of a fixed angle $\Delta(0)$ at $z=1$. Then we have*

$$C_{\Delta(\theta)}(f, e^{i\theta}) = C(f, e^{i\theta})$$

*except perhaps for a set of first Baire category on $K$.*

We show that the assumption of the set

$$E = \{e^{i\theta} \in K;\ C_{\Delta(\theta)}(f, e^{i\theta}) \neq C(f, e^{i\theta})\}$$

being of second category on $K$ leads us to a contradiction. For this end we shall show in four steps the following:

(5.1) There exist a subset $E_0 \subset E$ of category II on $K$, a non-empty closed set $T_0 \subset \Omega$, a positive constant $\alpha$ and a natural number $q$ such that for any $e^{i\theta} \in E_0$,

$$T_0 \cap C(f, e^{i\theta}) \neq \emptyset \tag{5.1.1}$$

and

$$\mathrm{dis}^*\{T_0, \overline{f(R_q \cap \Delta(\theta))}\} > \alpha, \tag{5.1.2}$$

where

$$R_q = \{1-2^{-q} < |z| < 1\}$$

and dis* means the spherical chordal distance on $\Omega$.

This is absurd. Indeed, let $\beta \subset K$ be an open arc where $E_0$ is dense. Choose $e^{i\theta} \in E_0 \cap \beta$ and let $A(\theta)$ be an open arc $(e^{i\theta_1}, e^{i\theta_2})$ of $K$, $\theta_1 < \theta < \theta_2$, such that $\overline{A(\theta)} \subset \beta$. Then for any

$$z \in R_q,\ \theta_1 < \arg z < \theta_2,$$

we may find $e^{i\varphi} \in E_0$ such that $z \in R_q \cap \Delta(\varphi)$ since $E_0$ is dense in $\beta$. Therefore we have by (5.1.2),

$$\mathrm{dis}^*(T_0, f^*(z)) \geqq \mathrm{dis}^*\{T_0, \overline{f(R_q \cap \Delta(\varphi))}\} > \alpha.$$

This contradicts (5.1.1) for $e^{i\theta} \in E_0$.

We shall take

$$E_0 = E_{NMq\nu};\ T_0 = s_M;\ \alpha = 1/\nu;\ q = q,$$

where all notation on the right-hand-sides will be defined in the sequel.

In the first step we set

$$C_{\Delta(\theta)}(f, e^{i\theta})_n = \{w \in \Omega\,;\ \text{dis}^*\{w, C_{\Delta(\theta)}(f, e^{i\theta})\} \leqq 1/n\}$$

and then we set

$$E_n = \{e^{i\theta} \in E\,;\ C(f, e^{i\theta}) \setminus C_{\Delta(\theta)}(f, e^{i\theta})_n \neq \emptyset\}$$

for $n = 1, 2, \cdots$. Then we have

$$E = \bigcup_n E_n\,.$$

Since $E$ is, by assumption, of category II, there exists an $E_N$ of category II on $K$. In the second step we first remark that $\Omega$ can be covered by a finite number of non-empty closed spherical disks $s_k$, $1 \leqq k \leqq m$ whose diameters are equal or less than $1/N$. We set

$$E_{Nk} = \{e^{i\theta} \in E_N\,;\ s_k \cap [C(f, e^{i\theta}) \setminus C_{\Delta(\theta)}(f, e^{i\theta})_N] \neq \emptyset\}\,,$$

$k=1, 2, \cdots, m$; so that we have

$$E_N = \bigcup_k E_{Nk}\,.$$

We can find $E_{NM}$ of second category. In the third step we prove that

$$C_{\Delta(\theta)}(f, e^{i\theta}) = \bigcap_Q \overline{f(R_Q \cap \Delta(\theta))}$$

does not intersect $s_M$ for any $e^{i\theta} \in E_{NM}$. In fact, we assume that we have a point $w \in s_M \cap C_{\Delta(\theta)}(f, e^{i\theta})$ for some $e^{i\theta} \in E_{NM}$. By definition we can take

$$\alpha \in s_M \cap [C(f, e^{i\theta}) \setminus C_{\Delta(\theta)}(f, e^{i\theta})_N]\,.$$

This is a contradiction. Now that for some $Q'$ we have

$$\overline{f(R_{Q'} \cap \Delta(\theta))} \cap s_M = \emptyset\,,$$

we obtain a decomposition:

$$E_{NM} = \bigcup_{Q} E_{NMQ},$$

where

$$E_{NMQ} = \{e^{i\theta} \in E_{NM} ; \overline{f(R_Q \cap \Delta(\theta))} \cap s_M = \emptyset\}$$

for $Q=1, 2, \cdots$. We therefore have a set $E_{NMq}$ of category II. Finally, in the fourth step we set

$$E_{NMq\eta} = \{e^{i\theta} \in E_{NMq} ; \operatorname{dis}^*\{s_M, \overline{f(R_q \cap \Delta(\theta))}\} > 1/\eta\}$$

for $\eta=1, 2, \cdots$. From the decomposition:

$$E_{NMq} = \bigcup_{\eta} E_{NMq\eta}$$

we can find $E_{NMq\nu}$ of second category. q.e.d.

For any set $S \neq \emptyset$ in the plane $\mathring{\Omega}$: $|z| < \infty$ and $\rho \geqq 0$ we shall denote $N(S, \rho) = \{w ; \operatorname{dis}(w, S) \leqq \rho\}$. We shall denote by $\delta(z_0, q)$ the open disk: $|z - z_0| < q$ in $\mathring{\Omega}$.

LEMMA 2. *Let $f(z)$ be an algebroid function in a disk $\delta(z_0, q)$ defined by*

$$f^n(z) + a_1(z)f^{n-1}(z) + \cdots + a_n(z) = 0. \tag{1.1}^{\text{bis}}$$

*Assume that we have a constant $M > 0$ with*

$$f^*(z) \subset \delta(0, M)$$

*for any $z \in \delta(z_0, q)$. Then we have*

$$f^*(z) \subset N(f^*(z_0), Hq^{-1/n}|z - z_0|^{1/n})$$

*for any $z \in \delta(z_0, q)$, where $H$ is a positive constant depending only upon $n$ and $M$.*

PROOF. We explain, first in the present paragraph, Henri Cartan's theorem ([2], p. 273, Théorème III) in a convenient form for our later use. Let $w_1, \cdots, w_n$ be not necessarily distinct $n$ complex numbers and let $h_1$ be an arbitrary positive constant. Then there exist at most $n$ open disks $\delta_1, \cdots, \delta_\mu$ in $\mathring{\Omega}$ such that

$$\sum_{j=1}^{\mu} (\text{the radius of } \delta_j) < 2eh_1, \tag{5.2}$$

$e$ being the base of natural logarithms, and that

$$\prod_{j=1}^{n} |w-w_j| > h_1^n \tag{5.3}$$

for any $w \in \mathring{\Omega} \setminus \bigcup_{j=1}^{\mu} \delta_j$.

We fix $z \in \delta(z_0, q)$ once and for all and we set

$$P(w) = \prod_{j=1}^{n} (w - f_j(z_0)) \equiv w^n + a_1(z_0)\, w^{n-1} + \cdots + a_n(z_0)$$

and

$$Q(w) = w^n + a_1(z)\, w^{n-1} + \cdots + a_n(z).$$

Furthermore we set

$$R(w) = Q(w) - P(w).$$

Then by the Schwarz inequality we have

$$|R(w)|^2 \leqq \sum_{j=1}^{n} |a_j(z) - a_j(z_0)|^2 \sum_{j=0}^{n-1} |w|^{2j},$$

so that we obtain

$$|R(w)| < M_1 \left\{ \sum_{j=1}^{n} |a_j(z) - a_j(z_0)|^2 \right\}^{1/2} \tag{5.4}$$

for any $w \in \delta(0, 2M)$, where

$$M_1 = \left\{ \sum_{j=0}^{n-1} (2M)^{2j} \right\}^{1/2}.$$

We set

$$(5.5) \qquad h = M_1^{1/n}\left\{\sum_{j=1}^{n} |a_j(z) - a_j(z_0)|^2\right\}^{1/2n}.$$

If $h=0$, we have nothing to prove; while if $h>0$, by Cartan's theorem, there are disks $\delta_1, \cdots, \delta_\mu$ satisfying (5.2) with $h_1=h$ such that

$$(5.6) \qquad |P(w)| = \prod_{j=1}^{n} |w-f_j(z_0)| > h^n$$

for any $w$ not in $\bigcup_{j=1}^{\mu} \delta_j$. Combining (5.6) with (5.4) and (5.5) we have

$$(5.7) \qquad |P(w)| > |R(w)|$$

on the boundary of $\overline{\delta(0, M)} \setminus \bigcup_{j=1}^{\mu} \delta_j$. Since $A = \delta(0, M) \cap \left(\bigcup_{j=1}^{\mu} \delta_j\right)$ contains $f_1(z_0)$, $\cdots, f_n(z_0)$, the roots of $P(w)=0$, we may find the roots $f_1(z), \cdots, f_n(z)$ of $Q(w)=0$ in $A$ by Rouché's theorem applied to every component of $A$. Hence by (5.2) with $h_1=h$ we have

$$(5.8) \qquad f^*(z) \subset N(f^*(z_0), 2eh).$$

On the other hand, $|a_j(z)| < M_2$ in $\delta(z_0, q)$, $1 \leqq j \leqq n$, where $M_2$ is a constant depending only on $n$ and $M$ by the relation (3.3) of §3. We therefore have by the Schwarz lemma and by (5.5) the following inequality:

$$h < M_1^{1/n} n^{1/2n} (2M_2)^{1/n} q^{-1/n} |z - z_0|^{1/n}.$$

Combined with (5.8) this gives our lemma.

REMARK. Our proof of the Schwarz lemma is suggested by the argument of Dufresnoy ([6], pp. 27-29). I am indebted to my colleague Professor N. Toda for the reference of this paper.

**6. Topological analogue of Fatou's theorem.** In this section we prove an extension of topological analogue of Fatou's theorem.

THEOREM 3. *Let $f(z)$ be an algebroid function defined in $U$ such that $f$ is bounded, i.e., there exists a constant $M>0$ with*

$$f^*(z) \subset \delta(0, M)$$

*for any* $z \in U$. *Then all points of* $K$ *except perhaps for a set of first Baire category belong to* $M(f)$.

PROOF. Let $e^{i\theta} \in K \backslash M(f)$. Then there exists a chord $\rho(\varphi)$ at $e^{i\theta}$ such that

$$C_{\rho(\varphi)}(f, e^{i\theta}) \neq C(f, e^{i\theta}).$$

We may suppose $0 \leqq \varphi < \pi/2$. Let

$$P \in C(f, e^{i\theta}) \setminus C_{\rho(\varphi)}(f, e^{i\theta})$$

and consider the closure $\bar{\delta}(P, 2\beta)$ of the disk $\delta(P, 2\beta)$, where

$$0 < 4\beta < \operatorname{dis}\{C_{\rho(\varphi)}(f, e^{i\theta}), P\} .$$

Then we may find a segment $\rho_1(\varphi) \subset \rho(\varphi)$, one end-point of which is $e^{i\theta}$, such that

$$\overline{f(\rho_1(\varphi))} \cap \bar{\delta}(P, 2\beta) = \emptyset . \tag{6.1}$$

We next set

$$R(\xi) = |\xi - e^{i\theta}| \quad \text{and} \quad \gamma(\xi) = R(\xi)\sin(\pi/4 - \varphi/2)$$

for $\xi \in \rho_1(\varphi)$. We take again a segment $\rho_2(\varphi) \subset \rho_1(\varphi)$ with one end-point $e^{i\theta}$ such that $\delta(\xi, \gamma(\xi)) \subset U$ for $\xi \in \rho_2(\varphi)$. We let

$$\gamma_0 = \min\{(\beta/H)^n, 1\} ,$$

where $H$ is a constant depending only on $n$ and $M$ as in Lemma 2. Then for any $z \in A_\xi \equiv \delta(\xi, \gamma_0\gamma(\xi)) \subset \delta(\xi, \gamma(\xi))$, $\xi \in \rho_2(\varphi)$, we have, by Lemma 2 with $q = \gamma(\xi)$, the following.

$$f^*(z) \subset N(f^*(\xi), \beta). \tag{6.2}$$

Combined with (6.1) this implies

$$\overline{f(A_\xi)} \cap \bar{\delta}(P, \beta/2) = \emptyset \tag{6.3}$$

for any $\xi \in \rho_2(\varphi)$. On the other hand, as $\rho_2(\varphi) \ni \xi \to e^{i\theta}$, the disks $A_\xi$ sweep an angle $\Delta$ at $e^{i\theta}$ bisected by the chord $\rho(\varphi)$, so that by (6.3) we have

$$\overline{f(\Delta)} \cap \bar{\zeta}(P, \beta/2) = \emptyset ,$$

and hence

$$C(f, e^{i\theta}) \neq C(f, e^{i\theta}).$$

The theorem now follows from Lemma 1. q.e.d.

Let $\gamma$ be an arbitrary simple arc in $U$ terminating at $e^{i\theta}$ and tangent at $e^{i\theta}$ to a chord $\rho(\varphi)$ at $e^{i\theta}$. Then we have

$$C_{\gamma}(f, e^{i\theta}) = C_{\rho(\varphi)}(f, e^{i\theta})$$

if $f$ is a bounded algebroid function in $U$. For the proof we use the same method as in the proof of Theorem 3, that is, we use similarly contracted disks as $A_{\xi}$ to obtain a swept $\Delta$ containing the parts of $\gamma$ and $\rho(\varphi)$ near $e^{i\theta}$.

Let $f(z)$ be algebroid in $U$. Then the multiple-valued function $h(z) = 1/\{f(z)-b\}$, $b$ being a complex constant, is again algebroid in $U$. This is an easy consequence of algebraic calculation applied to (1. 1) of §1. Next we remark that

$$M(f) = M(h).$$

This is an immediate consequence of the definition of cluster sets.

Now that our tools are ready we can prove Theorem 2 following the familiar lines ([7], Abschnitt D, cf. [5], p. 155).

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