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KILLING VECTORS ON CONTACT RIEMANNIAN MANIFOLDS AND FIBERINGS RELATED TO THE HOPF FIBRATIONS

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1. Introduction. Let (M, g) be a Riemannian manifold. Then K-contact Riemannian structures and Sasakian structures (= normal contact Riemannian structures) on M are defined by Killing vectors ξ of unit length satisfying some conditions (cf. §2). Hence we denote by (M, ξ, g) a K-contact Riemannian manifold or a Sasakian manifold.

Every (M, ξ, g) is odd dimensional.

In this paper, after preliminaries in §2 and §3, we first try to give conditions for Killing vectors to be infinitesimal automorphisms of (M, ξ, g) in terms of curvature of (M, ξ, g) in §4~§8.

THEOREM A. *Let (M, ξ, g) be a 3-dimensional K-contact Riemannian manifold which is not of constant curvature. Then every Killing vector is an infinitesimal automorphism of (M, ξ, g).*

By $\phi = -\nabla\xi$, we have a (1,1)-tensor field on M. ϕ satisfies $\phi\phi X = -X + g(\xi, X)\xi$ for each vector field X on M.

THEOREM B. *Let (M, ξ, g) be a 7-dimensional compact Sasakian manifold which is not of constant curvature. Assume that ϕ-holomorphic sectional curvature $H(X) < 3$. Then every Killing vector is an infinitesimal automorphism of (M, ξ, g).*

For general $(4r+3)$-dimensional cases, we need stronger conditions on curvature than those in Theorem B, r being an integer $\geqq 1$.

THEOREM C. *Let (M, ξ, g) be a $(4r+3)$-dimensional compact Sasakian manifold which is not of constant curvature. Assume that curvature is positive (more generally, ϕ-holomorphic special bisectional curvature is positive). Then*

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every Killing vector is an infinitesimal automorphism of (M, ξ, g).

The remaining cases are $(4r+1)$-dimensional, r being an integer $\geqq 1$.

THEOREM D. *Let* (M, ξ, g) *be a* $(4r+1)$*-dimensional complete Sasakian manifold which is not of constant curvature. Then every Killing vector is an infinitesimal automorphism of* (M, ξ, g).

As we have seen in [22], discussions on these problems concern Sasakian 3-structures on (M, g).

In §9, we give slightly general statements of the above theorems.

Analogously to the Hopf fibrations of spheres and the Boothby-Wang's fiberings of regular contact manifolds, we consider fibrations of (M, g) admitting a K-contact 3-structure in §11 and §12.

THEOREM E. *Let* (M, g) *be a complete Riemannian manifold admitting a Sasakian* 3*-structure* $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$. *If one of the Sasakian structures, for example* $\xi_{(1)}$, *is regular, then* $(M, \xi_{(1)}, \xi_{(2)}, \xi_{(3)}, g)$ *is a* $S^3[1]$*- or* $RP^3[1]$*-principal bundle over an Einstein manifold* (B, h).

In §13 we show that in many cases results on K-contact 3-structures are generalized to results on 3-K-contact structures.

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2. Preliminaries. Let (M, g) be a Riemannian manifold. By ∇ and R we denote the Riemannian connection and the Riemannian curvature tensor $(R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y])$, respectively. Let ξ be a unit Killing vector on (M, g), which satisfies

$$R(X, \xi)\xi = g(X, \xi)\xi - X \tag{2.1}$$

for any vector field X on M. Define a $(1,1)$-tensor field ϕ by $\phi = -\nabla\xi$ and a 1-form ($=$ contact form) η by $\eta = g(\xi, \ )$. Then (ϕ, ξ, η, g) is a K-contact Riemannian structure (cf. [5], etc.). We denote this K-contact Riemannian manifold by (M, ξ, g). On (M, ξ, g) we have

$$\phi\xi = -\nabla_\xi \xi = 0, \tag{2.2}$$

$$\phi\phi X = -X + g(\xi, X)\xi, \tag{2.3}$$

$$(2.4) \qquad g(\phi X, \phi Y) = g(X, Y) - g(\xi, X)g(\xi, Y).$$

If a unit Killing vector ξ satisfies

$$(2.5) \qquad R(X, \xi)Y = g(X, Y)\xi - g(\xi, Y)X, \quad \text{or}$$

$$(2.5)' \qquad -\nabla_X(\nabla\xi)Y = g(X, Y)\xi - g(\xi, Y)X$$

for any vector fields X and Y on M, then (M, ξ, g) is called a Sasakian manifold (=normal contact Riemannian manifold) (cf. [12], [13], etc.). A Sasakian manifold is a K-contact Riemannian manifold.

On a Sasakian manifold (M, ξ, g), by the Ricci identity, we have the following relation (cf. for example, Lemma 3.2 in [21]):

$$(2.6) \qquad \begin{aligned} \phi R(X, Y)(\phi Z) = &-R(X, Y)Z - g(Y, Z)X + g(X, Z)Y \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y. \end{aligned}$$

We define the distribution D by $D_p = \{X_p\,;\ g(\xi, X_p) = 0, X_p \in M_p\}$, where M_p denotes the tangent space to M at p. By $X \in D$ we understand that X is a vector field on M such that $X_p \in D_p$ for every p of M. By $X \in D_p$, we understand that X is a tangent vector belonging to D_p. By $K(X, Y)$ we denote the sectional curvature for a 2-plane determined by X and Y. By $H(X)$, $X \in D_p$(or $X \in D$) we denote the sectional curvature $K(X, \phi X)$, called ϕ-holomorphic sectional curvature.

Let X and Y be an orthonormal pair in D_p and put $g(X, \phi Y) = \cos\alpha$. Then by a direct calculation we have (cf. E. M. Moskal [8])

$$(2.7) \qquad \begin{aligned} K(X, Y) = (1/8)[&3(1+\cos\alpha)^2 H(X+\phi Y) + 3(1-\cos\alpha)^2 H(X-\phi Y) \\ &- H(X+Y) - H(X-Y) - H(X) - H(Y) + 6\sin^2\alpha]. \end{aligned}$$

Furthermore we have (for (2.7) and (2.8), see also [18])

$$(2.8) \qquad \begin{aligned} K(X, Y) + \sin^2\alpha\, K(X, \phi Y) = (1/4)[&(1+\cos\alpha)^2 H(X+\phi Y) \\ &+ (1-\cos\alpha)^2 H(X-\phi Y) + H(X+Y) + H(X-Y) - H(X) - H(Y) + 6\sin^2\alpha]. \end{aligned}$$

3. *K*-contact 3-structures and Sasakian 3-structures. Let $\xi_{(1)}, \xi_{(2)}, \xi_{(3)}$ be three K-contact structures on (M, g). Define $\phi_{(i)}(i = 1, 2, 3)$ by $\phi_{(i)} = -\nabla\xi_{(i)}$. Assume that

$$(3.1) \qquad g(\xi_{(i)}, \xi_{(j)}) = \delta_{ij}, \quad i, j = 1, 2, 3,$$

$$\xi_{(k)} = \phi_{(i)}\xi_{(j)} = -\phi_{(j)}\xi_{(i)}\,, \tag{3.2}$$

$$\phi_{(k)}X = \phi_{(i)}\phi_{(j)}X - g(\xi_{(j)}, X)\xi_{(i)} = -\phi_{(j)}\phi_{(i)}X + g(\xi_{(i)}, X)\xi_{(j)}\,, \tag{3.3}$$

where (i, j, k) is an even permutation of $(1, 2, 3)$. Then we say that $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ is a K-contact 3-structure on (M, g). Similarly, if $\xi_{(1)}, \xi_{(2)}, \xi_{(3)}$ are Sasakian structures and satisfy (3.1)$\sim$(3.3), then $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ is called a Sasakian 3-structure on (M, g).

(i) If (M, g) admits a K-contact 3-structure, then $\dim M = 4r+3$ for some integer $r \geqq 0$ (Y. Y. Kuo [7]).

(ii) (M, g) admitting a Sasakian 3-structure is an Einstein manifold (T. Kashiwada [6]).

(iii) Let $\xi_{(1)}$ and $\xi_{(2)}$ be two Sasakian structures on (M, g) such that $g(\xi_{(1)}, \xi_{(2)}) = 0$. Then $\xi_{(3)} = (1/2)[\xi_{(1)}, \xi_{(2)}]$ is also a Sasakian structure and orthogonal to $\xi_{(1)}$ and $\xi_{(2)}$. Hence $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ is a Sasakian 3-structure (Y. Y. Kuo [7]).

If the inner product $g(\xi, \xi')$ of two Sasakian structures ξ and ξ' on (M, g) is constant $(\neq 1, \neq -1)$, we can find Sasakian structure $\xi_{(2)}$ so that $\xi_{(1)} = \xi$ and $\xi_{(2)}$ are orthogonal. Hence (M, g) admits a Sasakian 3-structure.

In the case where $g(\xi, \xi')$ is not constant, we have

LEMMA 3.1. (S. Tachibana and W. N. Yu [15]) *Let (M, g) be a complete Riemannian manifold of m-dimension. If (M, g) admits two Sasakian structures ξ and ξ' with $g(\xi, \xi')$=non-constant, then (M, g) is of constant curvature* 1.

Originally, Lemma 3.1 was proved for complete and simply connected (M, g) with conclusion that (M, g) is isometric to a unit sphere S^m.

Let $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ be a K-contact 3-structure on (M, g). By E we denote the distribution defined by (putting $\xi_{(1)} = \xi$)

$$E_p = \{X_p \in D_p\,;\; g(X_p, \xi_{(2)}) = g(X_p, \xi_{(3)}) = 0\}\,. \tag{3.4}$$

Since $\dim M = 4r+3$, we have $\dim E_p = 4r$. If $X \in E_p$, we have

$$\phi_{(k)}X = \phi_{(i)}\phi_{(j)}X = -\phi_{(j)}\phi_{(i)}X\,, \tag{3.5}$$

where (k, i, j) is an even permutation of $(1, 2, 3)$.

We define $\phi_{(i)}$-holomorphic sectional curvature for $X \in E_p$ by

$$H(X) = H_{(1)}(X) = K(X, \phi_{(1)}X)\,,$$

$$H_{(2)}(X) = K(X, \phi_{(2)}X)\,, \quad H_{(3)}(X) = K(X, \phi_{(3)}X)\,.$$

In the remainder of this section we assume that $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ is a Sasakian 3-structure.

PROPOSITION 3. 2. *For $X \in E_p$, we have*

$$H_{(1)}(X) + H_{(2)}(X) + H_{(3)}(X) = 3\,. \tag{3.6}$$

PROOF. In (2. 6) we put $\phi = \phi_{(i)}$ and take X, Y, Z (of unit length) $\in E_p$ and consider the inner product with $W \in E_p$. Then we get

$$\begin{aligned} g(R(X,Y)\phi_{(i)}Z, \phi_{(i)}W) = {} & g(R(X,Y)Z,W) + g(Y,Z)g(X,W) \\ & - g(X,Z)g(Y,W) - g(\phi_{(i)}Y,Z)g(\phi_{(i)}X,W) \\ & + g(\phi_{(i)}X,Z)g(\phi_{(i)}Y,W)\,, \end{aligned} \tag{3.7}$$

where we have used (2. 3) and (2. 4), and $i = 1, 2, 3$. If we put $i=1$, $Z=X$, and $Y=W=\phi_{(3)}X$ in (3. 7), we get

$$g(R(X,\phi_{(3)}X)\phi_{(1)}X, \phi_{(1)}\phi_{(3)}X) = g(R(X,\phi_{(3)}X)X, \phi_{(3)}X) - 1\,, \tag{3.8}$$

that is,

$$-\,g(R(X,\phi_{(3)}X)\phi_{(1)}X, \phi_{(2)}X) = H_{(3)}(X) - 1\,. \tag{3.9}$$

Then we have two relations by even permutations of $(1, 2, 3)$ from (3. 9). Hence, (3. 6) follows from the Bianchi identity.

PROPOSITION 3. 3. *For $X \in E_p$ and for real numbers a, b $(a^2+b^2=1)$, we have*

$$H_{(1)}(X) = H_{(1)}(\phi_{(2)}X) = H_{(1)}(a\phi_{(2)}X + b\phi_{(3)}X)\,. \tag{3.10}$$

PROOF. By a permutation $(1\to 2\to 3\to 1)$ in (3.9), we have

$$\begin{aligned} H_{(1)}(X) - 1 &= -\,g(R(X,\phi_{(1)}X)\phi_{(2)}X, \phi_{(3)}X) \\ &= -\,g(R(\phi_{(2)}X,\phi_{(3)}X)X, \phi_{(1)}X) \\ &= g(R(\phi_{(2)}X,\phi_{(3)}X)\phi_{(3)}\phi_{(2)}X, \phi_{(3)}\phi_{(3)}X) \quad \text{by (3. 5).} \end{aligned} \tag{3.11}$$

On the other hand, in (3. 7) we put $i=3$ and replace X,Y,Z,W by $\phi_{(2)}X, \phi_{(3)}X, \phi_{(2)}X$, $\phi_{(3)}X$. Then we have

$$(3.12)\qquad g(R(\phi_{(2)}X, \phi_{(3)}X)\phi_{(3)}\phi_{(2)}X, \phi_{(3)}\phi_{(3)}X) = g(R(\phi_{(2)}X,\phi_{(3)}X)\phi_{(2)}X,\phi_{(3)}X) - 1\,.$$

By (3.11) and (3.12), we have

$$H_{(1)}(X) = g(R(\phi_{(2)}X, \phi_{(1)}\phi_{(2)}X)\phi_{(2)}X, \phi_{(1)}\phi_{(2)}X) = H_{(1)}(\phi_{(2)}X)\,.$$

Since $a\phi_{(2)}X + b\phi_{(3)}X = a\phi_{(2)}X + b\phi_{(1)}\phi_{(2)}X$, we have (3.10).

LEMMA 3.4. *Let $X \in E_p$. For real numbers a, b $(a^2+b^2=1)$ we have* $(i=2, 3)$

$$(3.13)\qquad H_{(1)}(a\xi_{(i)} + bX) = a^4 + 2a^2b^2 + b^4H_{(1)}(X)\,.$$

PROOF. By a straightforward calculation using (2.5) for $\xi_{(2)}$ and $\xi_{(3)} = \phi\xi_{(2)}$, we have

$$\begin{aligned} &g(R(a\xi_{(2)} + bX, a\phi\xi_{(2)} + b\phi X)(a\xi_{(2)} + bX), a\phi\xi_{(2)} + b\phi X) \\ &= a^4g(R(\xi_{(2)}, \phi\xi_{(2)})\xi_{(2)}, \phi\xi_{(2)}) + b^4g(R(X, \phi X)X, \phi X) \\ &\quad + a^2b^2g(R(\xi_{(2)}, \phi X)\xi_{(2)}, \phi X) + a^2b^2g(R(X, \phi\xi_{(2)})X, \xi\phi_{(2)})\,, \end{aligned}$$

from which we have (3.13) for $i=2$, and the case of $i=3$ is similar.

REMARK. Since $c\xi_{(2)} + d\xi_{(3)}$ for constant c, d $(c^2+d^2=1)$ is also Sasakian, Lemma 3.4 shows that

$$(3.13)'\qquad H_{(1)}(a(c\xi_{(2)} + d\xi_{(3)}) + bX) = a^4 + 2a^2b^2 + b^4H_{(1)}(X)\,.$$

4. Theorem A. A 3-dimensional K-contact Riemannian manifold (M, ξ, g) is necessarily Sasakian and it is a D-Einstein manifold, i.e.,

$$(4.1)\qquad R_1(X, Y) = ag(X, Y) + bg(\xi, X)g(\xi, Y)\,,$$

where a and b are functions on M and R_1 denotes the Ricci curvature tensor (cf. [16], [17]). Consequently the scalar curvature S is given by $S = 3a+b$.

THEOREM A. *Let (M, ξ, g) be a 3-dimensional K-contact Riemannian manifold which is not of constant curvature. Then every Killing vector is an infinitesimal automorphism.*

To prove Theorem A, it suffices to show the following.

PROPOSITION 4.1. *Let (M, ξ, g) and (M', ξ', g') be two 3-dimensional K-contact Riemannian manifolds. If they admits an isometry $\varphi(\varphi^* g' = g)$ such that $\varphi\xi \neq \xi'$ and $\varphi\xi \neq -\xi'$, then (M, g) is of constant curvature.*

PROOF. Let x be an arbitrary point of M and put $y = \varphi x$. Since φ is an isometry, we have $S_x = S_y'$ and

$$(4.2) \qquad R_{1x}(X, Y) = (\varphi^* R_1')_x(X, Y) = R'_{1y}(\varphi X, \varphi Y).$$

By (4.1) we get

$$(4.3) \qquad 3a_x + b_x = 3a_y' + b_y',$$

$$(4.4) \quad a_x g_x(X, Y) + b_x g_x(\xi, X) g_x(\xi, Y) = a_y' g_y'(\varphi X, \varphi Y) + b_y' g_y'(\xi', \varphi X) g_y'(\xi', \varphi Y).$$

Since $\dim M = 3$, we have $Z \in D_x$ such that $g_y'(\xi', \varphi Z) = 0$. Putting $X = Y = Z$ in (4.4), we get $a_x = a_y'$. Then (4.3) implies $b_x = b_y'$. If we put $X = Y = \xi$ in (4.4), we have $b_x = b_y'[g_y'(\xi', \varphi\xi)]^2$. Hence, if $b_x \neq 0$, we have $[g_y'(\xi', \varphi\xi)]^2 = 1$. If (M, g) is not of constant curvature, we have a non-empty open set U where b is non-vanishing. Then we have $\varphi\xi = \xi'$ on U or $\varphi\xi = -\xi'$ on U. Since $\varphi\xi, \xi'$(or $-\xi'$) are Killing vectors on (M', g'), and since they coincide on U, they coincide on M'. This contradicts the assumption of φ, and hence, $b = 0$ on M. Consequently, (M, g), (M', g') are of constant curvature 1.

By $I(M, g)$ and $A(M, \xi, g)$, we denote the isometry group and the automorphism group of (M, ξ, g), respectively.

COROLLARY 4.2. *Let (M, ξ, g) be a 3-dimensional K-contact Riemannian manifold. Then we have either*

(i) *(M, g) is of constant curvature, or*

(ii-1) *$I(M, g) = A(M, \xi, g)$ or*

-2) *$I(M, g) = A(M, \xi, g) \cup A'(M, \xi, g)$,*

where $A'(M, \xi, g) = \{\varphi f; f \in A(M, \xi, g), \varphi \in I(M, g) : \varphi\xi = -\xi\}$.

5. Einstein-Kählerian manifolds. Let (N, J, G) be a $2n$-dimensional Kählerian manifold with (almost) complex structure tensor J and Kählerian metric tensor G. Holomorphic sectional curvature is defined by $'H(\sigma) = {}'H(u) = {}'K(u, Ju)$, where σ denotes the holomorphic section determined by u. For two holomorphic sections σ and σ', holomorphic bisectional curvature $'H(\sigma, \sigma')$ is defined in [4]. In this paper we consider holomorphic special bisectional curvature $'H(\sigma, \sigma')$, where the word "special" means $\sigma \perp \sigma'$. In this case

$$'H(\sigma, \sigma') = {}'K(u, v) + {}'K(u, Jv),$$

where $u \in \sigma$ and $v \in \sigma'$. Generalizing a result of M. Berger [1], S. I. Goldberg and S. Kobayashi [4] proved the followings: On an Einstein-Kählerian manifold (N, J, G) assume that the maximum value $'H_1$ of holomorphic sectional curvature is attained at x of N. Let u be a unit tangent vector at x such that $'H_1 = 'H(u)$.

(i) For an orthonormal basis $(u_1, \cdots, u_n, u_{1^*} = Ju_1, \cdots, u_{n^*} = Ju_n)$ at x such that

$$u_1 = u, \quad \text{and} \tag{5.1}$$

$$'R_{11^*i\alpha} = G('R(u_1, Ju_1)u_i, u_\alpha) = 0 \tag{5.2}$$

for all i and α such that $[\alpha \neq i^*;\ 2 \leqq i \leqq n, 2 \leqq \alpha \leqq n$ or $n+2 \leqq \alpha \leqq 2n]$, if $'R_{11^*ii^*}$ (holomorphic special bisectional curvature) is positive, then (N, J, G) has constant holomorphic sectional curvature $'H_1$.

Especially,

(ii) If (N, J, G) is of positive holomorphic bisectional curvature, then it is of constant holomorphic sectional curvature.

6. Local fiberings. Let p be a point of a K-contact Riemannian manifold (M, ξ, g). We have a sufficiently small coordinate neighborhood U of p, which is cubical and flat with respect to ξ (cf. [10]). Then U is a regular K-contact Riemannian manifold with the induced structure and we have a fibering

$$\pi : U \rightarrow U/\xi = N. \tag{6.1}$$

Since U is a K-contact Riemannian manifold, N is an almost Kählerian manifold. We denote the almost Kählerian structure tensors by J and G. Then we have

$$\phi u^* = (Ju)^*, \tag{6.2}$$

$$g = \pi^* G + \eta \otimes \eta, \tag{6.3}$$

where u^* on U is the horizontal lift of a vector field u on N with respect to the contact form η. Further

$$d\eta(u^*, v^*) = 2g(u^*, \phi v^*) = 2G(u, Jv) \cdot \pi. \tag{6.4}$$

Denoting by $'R$ the Riemannian curvature tensor on N, we have

$$\begin{aligned} R(u^*, v^*)z^* = {} & ('R(u, v)z)^* + 2g(u^*, \phi v^*)\phi z^* \\ & + g(u^*, \phi z^*)\phi v^* - g(v^*, \phi z^*)\phi u^* + \langle u, v, z \rangle \xi, \end{aligned} \tag{6.5}$$

where $<u,v,z>$ denotes some function depending on u, v, z and u, v, z are vector fields on N (cf. [9], [17], [18], etc.). The relation between holomorphic sectional curvature $'H(u)$ on N and ϕ-holomorphic sectional curvature $H(u^*)$ on U is

$$H(u^*) = 'H(u)\cdot\pi - 3\,. \tag{6.6}$$

The relation between ϕ-holomorphic special bisectional curvature $H(\rho,\rho')=K(X,Y)+K(X,\phi Y)$ $(X\in\rho\subset D, Y\in\rho'\subset D)$ on U and holomorphic special bisectional curvature $'H(\pi\rho.\ \pi\rho')$ on N is

$$H(\rho,\rho') = 'H(\pi\rho,\pi\rho')\cdot\pi\,. \tag{6,7}$$

U is a D-Einstein space if and only if N is an Einstein space ([17]). If (M,ξ,g) is Sasakian, then (N,J,G) is Kählerian.

7. Theorem B. Now we prove the following Proposition.

PROPOSITION 7.1. *Let $(\xi=\xi_{(1)},\xi_{(2)},\xi_{(3)})$ be a Sasakian 3-structure on a compact Riemannian manifold (M,g) of dimension 7. If*

$$H(X) = H_{(1)}(X) = K(X,\phi X) < 3$$

for any non-zero vector $X\in E$, then (M,g) is of constant curvature.

PROOF. Let x be a point of M. Put

$$H_x{}^* = \max\{H(X) = H_{(1)}(X), X\in E_x\}.$$

Case I, where $H_x{}^*\leqq 1$ for any x of M. Let $X\in E_x$ be any unit vector. Take a ϕ-basis $(\xi=\xi_{(1)},\xi_{(2)},\xi_{(3)}=\phi\xi_{(2)}, X,\phi X, Y=\phi_{(2)}X, \phi Y=\phi_{(3)}X)$. Since $\cos\alpha=g(X,\phi Y)=0$, by (2.8) we have

$$\begin{aligned}4(K(X,Y)+K(X,\phi Y)) &= H(X+\phi Y)+H(X-\phi Y)\\ &\quad + H(X+Y)+H(X-Y)-H(X)-H(Y)+6.\end{aligned}$$

Noticing $K(X,Y)=H_{(2)}(X)$ and $K(X,\phi Y)=H_{(3)}(X)$, and applying (3.6) and (3,10), we have

$$6 = H(X+\phi Y)+H(X-\phi Y)+H(X+Y)+H(X-Y)+2H(X)\,.$$

Since $H_x{}^*\leqq 1$, we have $H(X+\phi Y)=H(X-\phi Y)=H(X+Y)=H(X-Y)=H(X)$

$=1$. By (3.13)$'$, (M, ξ, g) has constant ϕ-holomorphic sectional curvature 1. Therefore (M, g) is of constant curvature 1 (cf. [18]).

Case II, where $1 < H_p^*$ for some p. Since M is compact, we can assume that H_p^* is the maximum value on M. Let $V \in E_p$ such that $H_p^* = H(V)$. Let U be a regular neighborhood of p and let $\pi: U \to U/\xi = N$ be a (local) fibering. Let $u_1 = \pi_p V$. Then, by (6.6), we see that $'H(u_1)_q = H_p^* + 3$ is the maximum on N, where $q = \pi p$. We define a vector u_3 by $u_3 = \pi_p \xi_{(2)}$. Then $Ju_3 = \pi_p \phi \xi_{(2)} = \pi_p \xi_{(3)}$. In (6.5), if we replace u, v, z by u_1, Ju_1, u_3, we have

$$R(u_1^*, \phi u_1^*)\xi_{(2)} = ('R(u_1, Ju_1)u_3)^* + 2g(u_1^*, \phi\phi u_1^*)\phi\xi_{(2)}$$
$$+ 0 - 0 + <u_1, Ju_1, u_3> \xi_{(1)}$$

at p. Projecting this, we have

$$'R(u_1, Ju_1)u_3 = 2Ju_3 .$$

This shows that u_3 and Ju_3 are characteristic vectors of a symmetric bilinear form α_{u_1} defined by $\alpha_{u_1}(y, z) = G('R(u_1, Ju_1)y, Jz)$. Hence, a J-basis:

$$u_1, Ju_1, u_2 = \pi_p \phi_{(2)} u_1^*, Ju_2 = \pi_p \phi_{(3)} u_1^*, u_3, Ju_3$$

satisfies the conditions (5.1) and (5.2). We define three holomorphic sections by $\sigma = (u_1, Ju_1)$, $\sigma' = (u_2, Ju_2)$ and $\sigma'' = (u_3, Ju_3)$. Then, by (6.7), we have

$$\begin{aligned} 'H(\sigma, \sigma') \cdot \pi &= H((u_1^*, \phi u_1^*), (\phi_{(2)} u_1^*, \phi_{(3)} u_1^*)) \\ &= K(u_1^*, \phi_{(2)} u_1^*) + K(u_1^*, \phi_{(3)} u_1^*) \\ &= H_{(2)}(u_1^*) + H_{(3)}(u_1^*) \\ &= 3 - H_{(1)}(u_1^*) \quad \text{by (3.6)}. \end{aligned}$$

Therefore, $'H(\sigma, \sigma') > 0$, which implies $'R_{11^*22^*} > 0$ in §5. Next, by (2.1), we have

$$'H(\sigma, \sigma'') \cdot \pi = K(u_1^*, \xi_{(2)}) + K(u_1^*, \xi_{(3)}) = 2,$$

which implies $'R_{11^*33^*} = 2 > 0$ in §5. Since (U, g) admits a Sasakian 3-structure, it is an Einstein manifold and (N, J, G) is an Einstein-Kählerian manifold. By (i) of §5, (N, J, G) is of constant holomorphic sectional curvature $H_p^* + 3$. Therefore (U, ξ, g) is of constant ϕ-holomorphic sectional curvature H_p^*. In particular we have $H_p^* = K(\xi_{(2)}, \phi\xi_{(2)}) = 1$, which is a contradiction.

Hence, only case I is possible, and (M, g) is of constant curvature.

LEMMA 7.2. (Theorem 4.4, [22]) *Let (M, ξ, g) be a complete Sasakian manifold which is not of constant curvature. Then we have either*

(i) $\dim I(M, g) = \dim A(M, \xi, g)$

$\rightleftarrows$ *(M, g) admitting no Sasakian 3-structure, or*

(ii) $\dim I(M, g) = \dim A(M, \xi, g) + 2$

$\rightleftarrows$ *(M, g) admitting a Sasakian 3-structure.*

THEOREM B. *Let (M, ξ, g) be a 7-dimensional compact Sasakian manifold which is not of constant curvature. Assume that ϕ-holomorphic sectional curvature $H(X) < 3$. Then every Killing vector is an infinitesimal automorphism of (M, ξ, g), i.e.,*

$$\dim I(M, g) = \dim A(M, \xi, g) .$$

PROOF. By Lemma 7.2, if $\dim I(M, g) \neq \dim A(M, \xi, g)$, we have a Sasakian 3-structure such that $\xi_{(1)} = \xi$. By the assumption $H(X) < 3$, Theorem B follows from Proposition 7.1.

8. Theorems C and D. By a theorem of E. M. Moskal [8] (for proof, also see [23], §7) we see that every compact Einstein-Sasakian manifold with positive curvature (or positive ϕ-holomorphic special bisectional curvature) is of constant curvature 1. Therefore, Lemma 7.2 and the fact that (M, g) admitting a Sasakian 3-structure is an Einstein manifold imply the following theorem.

THEOREM C. *Let (M, ξ, g) be a $(4r+3)$-dimensional compact Sasakian manifold which is not of constant curvature. Assume that every sectional curvature is positive (more generally, every ϕ-holomorphic special bisectional curvature is positive). Then we have*

$$\dim I(M, g) = \dim A(M, \xi, g) .$$

For $\dim M = 4r+1$ (r : an integer $\geqq 1$), there is no Sasakian 3-structure on (M, g). Hence,

THEOREM D. *Let (M, ξ, g) be a $(4r+1)$-dimensional complete Sasakian manifold which is not of constant curvature. Then*

$$\dim I(M, g) = \dim A(M, \xi, g) .$$

9. Infinitesimal translations. In this section, we give more general statements of Theorems B and C. The Riemannian curvature tensor of (M, g) of constant curvature k satisfies

$$R(X, Z)Y = k[g(X, Y)Z - g(Z, Y)X]. \tag{9.1}$$

A Killing vector of constant length is called an infinitesimal translation (cf. for example, K. Yano [24]).

THEOREM 9.1. *Let (M, g) be a compact Riemannian manifold. Assume that on (M, g) there are two (non-proportional) infinitesimal translations ξ and ξ', satisfying*

$$R(X, \xi)Y = k[g(X, Y)\xi - g(\xi, Y)X], \tag{9.2}$$

$$R(X, \xi')Y = k[g(X, Y)\xi' - g(\xi', Y)X] \tag{9.3}$$

for a positive constant k.

(i) *If $\dim M = 7$ and sectional curvature is smaller than $3k$, then (M, g) is of constant curvature k.*

(ii) *If $\dim M = 4r+3$ and sectional curvature is positive, then (M, g) is of constant curuature k.*

(iii) *If $\dim M = 3$ or $\dim M = 4r+1$, then (M, g) is of constant curvature k.*

Proof. By a homothetic deformation, we can assume that $k = 1$. Since (9.2) and (9.3) are linear homogeneous in ξ and ξ', we can assume that they are of unit length. Then, if $g(\xi, \xi')$ is constant, (M, g) admits a Sasakian 3-structure, and (i), (ii), (iii) hold by Theorem 7.1, etc. If $g(\xi, \xi')$ is not constant, (M, g) is of constant curvature by Lemma 3.1.

10. The Hopf-fibrations. Let $S^{2n+1}[1]$ be a unit sphere with the natural Sasakian structure of constant (ϕ-holomorphic sectional) curvature 1. Since ξ on $S^{2n+1}[1]$ is regular, we have the fibering:

$$\pi : S^{2n+1}[1] \longrightarrow S^{2n+1}[1]/\xi = CP^n[4], \tag{10.1}$$

where $CP^n[4]$ denotes a complex n-dimensional projective space with Fubini-Study metric of constant holomorphic sectional curvature 4. The map $\pi : S^3 \to S^2 = CP^1$ is the classical Hopf map.

For $S^{4r+3}[1]$, we have a Sasakian 3-structure $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$. The 3-dimensional distribution defined by $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ is completely integrable. Each maximal integral submanifold is isomorphic to $S^3[1]$. In this case, the Hopf fibration is:

$$\pi: S^{4r+3}[1] \longrightarrow S^{4r+3}[1]/(\xi_{(1)}, \xi_{(2)}, \xi_{(3)}) = QP^r, \tag{10.2}$$

where QP^r denotes the quaternionic projective space (cf. N. Steenrod [14], p. 105-).

(10.1) and (10.2) are principal bundles with group S^1 and S^3, respectively. A generalization of (10.1) for regular contact manifolds is the Boothby-wang's fiberings [2].

In the next section, we give a generalization of (10.2).

11. Fiberings of (M, g) admitting a K-contact 3-structure. Let $(\xi_{(1)},\xi_{(2)},\xi_{(3)})$ be a K-contact 3-structure on (M, g) (cf. §13). We define the 3-dimensional distribution by $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$. Since we have

$$[\xi_{(1)}, \xi_{(2)}] = \nabla_{\xi_{(1)}}\xi_{(2)} - \nabla_{\xi_{(2)}}\xi_{(1)} = 2\phi_{(1)}\xi_{(2)} = 2\xi_{(3)},$$

etc. by (3.2), etc., it is completely integrable. Each maximal integral submanifold (leaf) L is totally geodesic and of constant curvature 1. By the restriction, L admits a K-contact 3-structure (and hence, a Sasakian 3-structure, since $\dim L=3$). Now we assume that $\xi_{(1)}$ is regular and that (M, g) is complete. Then we show that all leaves are isomorphic. To begin with,

LEMMA 11.1. *In the classification of 3-dimensional space forms (M, g) admitting a Sasakian 3-structure $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ (cf. S. Sasaki [11]), only $S^3[1]$ and $RP^3[1]$ are regular with respect to $\xi_{(1)}$.*

PROOF. Each (M, g) of the classification is of the form $S^3[1]/\Gamma$, where Γ is a finite subgroup of the automorphism group of the Sasakian 3-structure. By I and $-I$ ($I^\triangle$ and $-I^\triangle$, resp.) we denote the identity and the anti-podal map of $S^3[1]$ (of S^2, resp.). Assume that (M, g) is neither $S^3[1]$ nor a real projective space $RP^3[1]=S^3[1]/\{I, -I\}$. Then, Γ contains φ such that $\varphi\neq I$ and $\varphi\neq -I$. Since φ is an automorphism of $(S^3[1], \xi, g)$, it induces an automorphism $\varphi^\triangle$ of the Kählerian manifold $S^3[1]/\xi = CP^1[4] = S^2$, where $\xi = \xi_{(1)}$.

(i) If $\varphi^\triangle = I^\triangle$, we have $\varphi = \exp r\xi$ for some r. Since $[\xi_{(1)}, \xi_{(2)}] = 2\xi_{(3)}$ and $[\xi_{(1)}, \xi_{(3)}] = -2\xi_{(2)}$, we have

$$(\exp r\xi)\xi_{(2)} = (\cos 2r)\xi_{(2)} - (\sin 2r)\xi_{(3)}.$$

$\varphi\xi_{(2)} = \xi_{(2)}$ implies $r=\pi$ and $\varphi = \exp \pi\xi = -I$ on $S^3[1]$, which is a contradiction to the assumption of φ.

(ii) If $\varphi^\triangle = -I^\triangle$, and if $S^3[1]/\Gamma$ is regular with respect to ξ, then $(S^3[1]/\Gamma)/\xi$ is Kählerian and orientable. However, since ξ is invariant by Γ, we have

$$(S^3[1]/\Gamma)/\xi = (S^3[1]/\xi)/\Gamma^\triangle = (S^3[1]/\xi)/(\varphi^\triangle, **) = RP^2/(**).$$

Because every complete Riemannian manifold of even dimension with constant curvature (>0) is S^m or RP^m, $(**) = $ (identity). Since RP^2 is not orientable, this is a contradiction.

(iii) If $\varphi^\triangle \neq I^\triangle$ and $\varphi^\triangle \neq -I^\triangle$, then $\varphi^\triangle$ has fixed points. Since Γ is a finite group, the set of all such points is composed of finite number of points. Therefore, on $S^3[1]/\Gamma$, ξ is not regular (cf. also, S. Tanno [20]).

LEMMA 11. 2. *Assume that a complete Riemannian manifold (M, g) admits a K-contact 3-structure $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$. If $\xi_{(1)}$ is regular, then $\xi_{(2)}$ and $\xi_{(3)}$ are regular, and all leaves L are isomorphic to $S^3[1]$ or $RP^3[1]$.*

PROOF. This follows from Lemma 11. 1.

REMARK. $S^3[1]$ and $RP^3[1]$ are Lie groups (cf. [14], p. 37, p. 115). In fact, let Q be the space of quaternions $(\boldsymbol{q}=x_1+x_2\boldsymbol{i}+x_3\boldsymbol{j}+x_4\boldsymbol{k})$ and let $S^3=\{\boldsymbol{q}\in Q; |\boldsymbol{q}|=1\}$. Then the right translation R_q and the left translation L_q by $\boldsymbol{q}\in S^3$ are defined by $R_q\boldsymbol{q}'=\boldsymbol{q}'\cdot\boldsymbol{q}$ and $L_q\boldsymbol{q}'=\boldsymbol{q}\cdot\boldsymbol{q}'$, respectively. We define a Sasakian 3-structure $(\xi^0_{(1)}, \xi^0_{(2)}, \xi^0_{(3)})$ such that

$$(\exp t\xi_{(1)})\boldsymbol{q}' = (\cos t)\boldsymbol{q}' + (\sin t)\boldsymbol{q}'\cdot\boldsymbol{i}, \qquad \boldsymbol{q}'\in S^3$$

etc. ($\xi^0_{(2)}$ for $\boldsymbol{j}$, $\xi^0_{(3)}$ for $\boldsymbol{k}$). Then $\xi^0_{(1)}, \xi^0_{(2)}, \xi^0_{(3)}$, are left invariant vector fields. We denote by $\mathfrak{g}$ the Lie algebra of $S^3[1]$ or $RP^3[1]$.

THEOREM 11. 3. *Let (M, g) be a complete Riemannian manifold admitting a K-contact 3-structure $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$. Assume that $\xi_{(1)}$ is regular. Then $(M, \xi_{(1)}, \xi_{(2)}, \xi_{(3)}, g)$ is a $S^3[1]$- or $RP^3[1]$-principal bundle over a Riemannian manifold (B, h). h and g are related by*

$$g(X, Y) = h(\pi X, \pi Y)\cdot\pi + \textstyle\sum_{i=1}^3 g(\xi_{(i)}, X)g(\xi_{(i)}, Y). \tag{11.2}$$

A $\mathfrak{g}$-valued 1-form w defined by

$$w(X) = \textstyle\sum_{i=1}^3 g(\xi_{(i)}, X)\xi^0_{(i)} \tag{11.3}$$

is an infinitesimal connection form.

PROOF. By Lemmas 11. 1 and 11. 2, we see that $(M, \xi_{(1)}, \xi_{(2)}, \xi_{(3)}, g)$ is a $S^3[1]$- or $RP^3[1]$-principal bundle over a manifold B. First we show that w defined by (11. 3) is an infinitesimal connection form. Since $S^3[1]$ or $RP^3[1]$ acts to the right, $\xi_{(i)}$ are considered as the fundamental vector fields corresponding to $\xi^0_{(i)}$, respectively. Clearly, $w(\xi_{(i)})=\xi^0_{(i)}$. To prove $R_a{}^*w=ad(a^{-1})w$, it suffices to show it for $a=\exp r\xi_{(1)}$. For this a we have $R_a{}^{-1}\xi_{(1)}=\xi_{(1)}$, and

$$R_a{}^{-1}\xi_{(2)} = \lambda\xi_{(2)} + \mu\xi_{(3)}, \quad R_a{}^{-1}\xi_{(3)} = -\mu\xi_{(2)} + \lambda\xi_{(3)},$$
$$R_a\xi^0_{(2)} = \lambda\xi^0_{(2)} - \mu\xi^0_{(3)}, \quad R_a\xi^0_{(3)} = \mu\xi^0_{(2)} + \lambda\xi^0_{(3)},$$

where λ and μ are constants depending on a $(\lambda^2+\mu^2=1)$. Then we have

$$\begin{aligned}(R_a{}^*w)_p(X) &= w_{pa}(R_aX) = \textstyle\sum_{i=1}^3 g_{pa}(\xi_{(i)}, R_aX)\xi^0_{(i)}\\ &= \textstyle\sum g_p(R_a{}^{-1}\xi_{(i)}, X)\xi^0_{(i)}\\ &= g_p(\xi_{(1)}, X)\xi^0_{(1)} + g_p(\lambda\xi_{(2)} + \mu\xi_{(3)}, X)\xi^0_{(2)} + g_p(-\mu\xi_{(2)} + \lambda\xi_{(3)}, X)\xi^0_{(3)}\\ &= \textstyle\sum g_p(\xi_{(i)}, X)R_a\xi^0_{(i)} = ad(a^{-1})w_p(X).\end{aligned}$$

Hence, w is an infinitesimal connection form on the principal bundle. Let x and y be vector fields on B and let x^* and y^* be their horizontal lifts with respect to w. We define a $(0,2)$-tensor h on B by $h(x,y)=g(x^*,y^*)$. Since $\xi_{(i)}$ are Killing vectors, h is well defined and satisfies (11.2).

REMARK. The map $\pi:(M,\xi_{(1)},\xi_{(2)},\xi_{(3)},g)\to(B,h)$ is harmonic in the sense of Eells-Sampson [3] (cf. Proposition, p. 127). This is the same for the Boothby-Wang's fiberings.

12. The Riemannian curvature tensors. We consider the fibering of Theorem 11.3. By $'\nabla$ we denote the Riemannian connection of (B,h). Let x,y,z be vector fields on B, and let x^*,y^*,z^* be their horizontal lifts. First we note that

$$[\xi_{(i)}, x^*] = L_{\xi_i}x^* = 0, \tag{12.1}$$

because the horizontal distribution is invariant and x^* is the horizontal lift of x. Now we have

$$\begin{aligned}2g(\nabla_{x^*}y^*, Z) &= x^*\cdot g(y^*,Z) + y^*\cdot g(x^*,Z) - Z\cdot g(x^*,y^*)\\ &\quad + g([x^*,y^*],Z) + g([Z,x^*],y^*) - g(x^*,[y^*,Z]).\end{aligned} \tag{12.2}$$

Putting $Z=z^*$, projecting this identity on B, and noticing $\pi[x^*,y^*]=[x,y]$, we have

$$(12.3)\qquad 2h(\pi(\nabla_{x^*}y^*), z) = x\cdot h(y, z) + y\cdot h(x, z) - z\cdot h(x, y) + h([x, y], z) + h([z, x], y) - h(x, [y, z]) = 2h('\nabla_x y, z)\,.$$

Therefore, we have

$$(12.4)\qquad \nabla_{x^*}y^* = ('\nabla_x y)^* + \sum a_i\xi_{(i)}\,,$$

where $a_i = g(\xi_{(i)}, \nabla_{x^*}y^*)$. Putting $Z = \xi_{(i)}$ in (12.2), we have

$$(12.5)\qquad 2a_i = -\xi_{(i)}\cdot g(x^*, y^*) + g([x^*, y^*], \xi_{(i)}) = g([x^*, y^*], \xi_{(i)}) = \eta_{(i)}([x^*, y^*]) = -d\eta_{(i)}(x^*, y^*)$$

$$(12.6)\qquad = -2g(x^*, \phi_{(i)}y^*)\,.$$

By (12.4), (12.5) and (12.6), we have

$$(12.7)\qquad [x^*, y^*] = [x, y]^* - 2\sum_{i=1}^{3} g(x^*, \phi_{(i)}y^*)\xi_{(i)}\,.$$

By $'R$ we denote the Riemannian curvature tensor of (B, h).

$$('\nabla_x{}'\nabla_y z)^* = \nabla_{x^*}('\nabla_y z)^* + \sum g(x^*, \phi_{(i)}('\nabla_y z)^*)\xi_{(i)}\,.$$

By (12.4), etc., we get

$$('\nabla_x{}'\nabla_y z)^* = \nabla_{x^*}\nabla_{y^*}z^* + \sum g(y^*, \phi_{(i)}z^*)\nabla_{x^*}\xi_{(i)} + \sum [g(\nabla_{x^*}y^*, \phi_{(i)}z^*) + g(y^*, \nabla_{x^*}\phi_{(i)}\cdot z^*) + g(y^*, \phi_{(i)}\nabla_{x^*}z^*) + g(x^*, \phi_{(i)}\nabla_{y^*}z^*)]\xi_{(i)}\,.$$

On the other hand, we get

$$('\nabla_{[x,y]}z)^* = \nabla_{[x,y]^*}z^* + \sum g([x, y]^*, \phi_{(i)}z^*)\xi_{(i)} = \nabla_{[x^*,y^*]}z^* + 2\sum g(x^*, \phi_{(i)}y^*)\nabla_{\xi_i}z^* + \sum g([x^*, y^*], \phi_{(i)}z^*)\xi_{(i)}\,.$$

Therefore, using $\nabla_{\xi_{(i)}}z^* = \nabla_{z^*}\xi_{(i)} = -\phi_{(i)}z^*$, we have

$$(12.8)\qquad ('R(x, y)z)^* = R(x^*, y^*)z^* + \sum [g(y^*, \phi_{(i)}z^*)\phi_{(i)}x^* - g(x^*, \phi_{(i)}z^*)\phi_{(i)}y^*$$

$$- 2g(x^*, \phi_{(i)}y^*)\phi_{(i)}z^*] + \sum[g(x^*, \nabla_{y^*}\phi_{(i)}\cdot z^*) - g(y^*, \nabla_{x^*}\phi_{(i)}\cdot z^*)]\xi_{(i)} .$$

PROPOSITION 12.1. *In the fibering of Theorem* 11.3, *let x, y be an orthonormal (local) vector fields on B (or tangent vectors at a point of B). Then we have*

$$'K(x,y)\cdot\pi = K(x^*,y^*) + 3\sum_{i=1}^{3}[g(y^*, \phi_{(i)}x^*)]^2 . \tag{12.9}$$

PROOF. Putting $z=x$ in (12.8) and taking the inner products of y^* and the both sides of (12.8), we get

$$h('R(x,y)x,y)\cdot\pi = g(R(x^*,y^*)x^*,y^*) + 3\sum[g(y^*, \phi_{(i)}x^*)]^2 ,$$

from which we have (12.9).

THEOREM 12.2. *In the fibering of Theorem* 11.3, *assume that* $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ *is a Sasakian* 3*-structure and* $\dim M=7$. *Then (M,g) is of constant curvature* 1 *if and only if (B,h) is of constant curvature* 4.

PROOF. Let x, y be any orthonormal pair in B_q, $q\in B$. Then x^*, y^* are orthonormal and y^* is expressed by

$$y^* = \sum_{i=1}^{3} b_i\phi_{(i)}x^* , \qquad b_i = g(y^*, \phi_{(i)}x^*) .$$

Since $\sum b_i^2=1$, (12.9) implies $'K(x,y)\cdot\pi = K(x^*,y^*)+3$. Hence, if (M,g) is of constant curvature 1, (B,h) is of constant curvature 4. Conversely, if (B,h) is of constant curvature 4, we have $H_{(1)}(X)=1$ for any non-zero $X\in E_p$. This implies that (M,g) has constant $\phi_{(1)}$-holomorphic sectional curvature 1 by (3.13)$'$. Thus, (M,g) is of constant curvature 1.

EXAMPLE. The Hopf fibration of S^7 is; $\pi: S^7\to QP^1=S^4$.

THEORFM 12.3. *In the fibering of Theorem* 11.3, *(M,g) is an Einstein manifold if and only if (B,h) is an Einstein manifold such that*

$$'R_1(x,y) = (4r+8)h(x,y) , \qquad 4r = \dim B .$$

PROOF. Let p be an arbitrary point of M and put $q=\pi p$. Let $(\xi_{(i)}, X_u, \phi_{(i)}X_u;\ i=1,2,3, u=1,\cdots,r)$ be an orthonormal basis at p. If we denote $\pi_p X_u$ by πX_u, etc., $(\pi X_u, \pi\phi_{(i)}X_u)$ is an orthonormal basis at q. By (12.8), we have

$$h_q('R(x, \pi X_u)y, \pi X_u) = g_p(R(x^*, X_u)y^*, X_u) + 3\sum g_p(\phi_{(i)}x^*, X_u)g_p(\phi_{(i)}y^*, X_u)\,, \tag{12.10}$$

$$h_q('R(x, \pi\phi_{(j)}X_u)y, \pi\phi_{(j)}X_u) = g_p(R(x^*, \phi_{(j)}X_u)y^*, \phi_{(j)}X_u) + 3\sum_i g_p(\phi_{(i)}x^*, \phi_{(j)}X_u)g_p(\phi_{(i)}y^*, \phi_{(j)}X_u) \tag{12.11}$$

for $j=1,2,3$. On the other hand, by (2.1). we have

$$0 = \sum g_p(R(x^*, \xi_{(i)})y^*, \xi_{(i)}) - 3g_p(x^*, y^*)\,. \tag{12.12}$$

First we notice that

$$\sum_u g(x^*, X_u)g(y^*, X_u) + \sum_{j,u} g(x^*, \phi_{(j)}X_u)g(y^*, \phi_{(j)}X_u) = g(x^*, y^*)\,.$$

Then by $(12.10)\sim(12.12)$, we have

$$'R_{1q}(x, y) = R_{1p}(x^*, y^*) + 6g_p(x^*, y^*)\,. \tag{12.13}$$

If (M, g) is an Einstein manifold, we have $R_1=(m-1)g=(4r+2)g$ (cf. (2.1)). Therefore, we have $'R_1(x, y)=(4r+8)\,h(x, y)$. Conversely, if (B, h) is an Einstein manifold such that $'R_1=(4r+8)h$, then $R_1(x^*, y^*)=(m-1)\,g(x^*, y^*)$ holds. Since $R_1(X, \xi_{(i)})=(m-1)\eta_{(i)}(X)$ (cf. (1.6) of [21]), (M, g) is an Einstein manifold.

In the fibering of Theorem 11.3, if $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ is a Sasakian 3-structure, then (B, h) is an Einstein manifold. Hence, we have

THEOREM E. *Let (M, g) be a complete Riemannian manifold admitting a Sasakian 3-structure $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$. If one of the Sasakian structures is regular, then $(M, \xi_{(1)}, \xi_{(2)}, \xi_{(3)}, g)$ is a $S^3[1]$- or $RP^3[1]$-principal bundle over an Einstein manifold (B, h) such that $'R_1=(4r+8)h$, $4r=\dim B$.*

13. 3-*K*-contact structures. We define a 3-*K-contact structure* on (M, g) by three K-contact structures $\xi_{(1)}, \xi_{(2)}, \xi_{(3)}$ satisfying (3.1) and (3.2). Some results on K-contact 3-structures are generalized to results on 3-K-contact structures.

LEMMA 13.1. *Let $\xi_{(1)}$ and $\xi_{(2)}$ be two K-contact structures on (M, g) such that $g(\xi_{(1)}, \xi_{(2)})=0$.* Then *$(\xi_{(1)}, \xi_{(2)}, \xi_{(3)}=(1/2)[\xi_{(1)}, \xi_{(2)}])$ is a 3-K-contact structure.*

PROOF. Since

$$[\xi_{(1)}, \xi_{(2)}] = \nabla_{\xi_{(1)}}\xi_{(2)} - \nabla_{\xi_{(2)}}\xi_{(1)} = 2\nabla_{\xi_{(1)}}\xi_{(2)} = -2\phi_{(2)}\xi_{(1)}$$
$$= -2\nabla_{\xi_{(2)}}\xi_{(1)} = 2\phi_{(1)}\xi_{(2)},$$

$\xi_{(3)} = \phi_{(1)}\xi_{(2)}$ is also a unit Killing vector. Then we have

$$(13.1)\qquad [\xi_{(1)}, \xi_{(3)}] = L_{\xi_{(1)}}\xi_{(3)} = L_{\xi_{(1)}}(\phi_{(1)}\xi_{(2)}) = \phi_{(1)}[\xi_{(1)}, \xi_{(2)}]$$
$$= 2\phi_{(1)}\xi_{(3)} = 2\phi_{(1)}\phi_{(1)}\xi_{(2)} = -2\xi_{(2)},$$

$$(13.2)\qquad [\xi_{(2)}, \xi_{(3)}] = L_{\xi_{(2)}}(-\phi_{(2)}\xi_{(1)}) = 2\xi_{(1)}.$$

Hence, $\xi_{(i)}$, $i=1,2,3$, satisfy (3.1), (3.2) where $\phi_{(3)} = -\nabla\xi_{(3)}$. We show that $\xi_{(3)}$ is a K-contact structure. Since $\xi_{(1)}$ satisfies

$$(13.3)\qquad R(X, \xi_{(1)})\xi_{(1)} = g(X, \xi_{(1)})\xi_{(1)} - X,$$

operating the Lie derivation $L_{\xi_{(2)}}$ to (13.3), we have

$$(13.4)\qquad R(X, \xi_{(3)})\xi_{(1)} + R(X, \xi_{(1)})\xi_{(3)} = g(X, \xi_{(3)})\xi_{(1)} + g(X, \xi_{(1)})\xi_{(3)}.$$

Oprating $L_{\xi_{(2)}}$ again to (13.4), and using (13.3), we have

$$(13.5)\qquad R(X, \xi_{(3)})\xi_{(3)} = g(X, \xi_{(3)})\xi_{(3)} - X$$

Therefore, $\xi_{(3)}$ is a K-contact structure.

PROPOSITION 13.2. *A 3-K-contact structure on (M, g) is a K-contact 3-structure if and only if*

$$(13.6)\qquad R(X, \xi_{(1)})\xi_{(2)} = g(X, \xi_{(2)})\xi_{(1)}.$$

PROOF. Operating ∇_X to $\phi_{(1)}\xi_{(2)} = \xi_{(3)}$, we have

$$\nabla_X\phi_{(1)} \cdot \xi_{(2)} - \phi_{(1)}\phi_{(2)}X = -\phi_{(3)}X.$$

Since $\nabla_X\phi_{(1)} = -\nabla_X(\nabla\xi_{(1)})$ and $\nabla_X(\nabla\xi_{(1)}) + R(X, \xi_{(1)}) = 0$, we have

$$(13.7)\qquad R(X, \xi_{(1)})\xi_{(2)} - \phi_{(1)}\phi_{(2)}X = -\phi_{(3)}X.$$

Hence, if (13.6) holds, we have $(3.3)_{k=3}$. If we oprate $L_{\xi_{(1)}}$ to (13.6), we have $R(X, \xi_{(1)})\xi_{(3)} = g(X, \xi_{(3)})\xi_{(1)}$, and then we get $(3.3)_{k=2}$. Similarly, we get $(3.3)_{k=1}$.

REMARK. In the above discussion, if $\xi_{(1)}$ and $\xi_{(2)}$ are Sasakian, then replacing (13.3) by (2.5) for $\xi_{(1)}$ we see that $\xi_{(3)}$ is Sasakian. Since we have (13.6) for Sasakian $\xi_{(1)}$, we have (iii) in §3.

PROPOSITION 13.3. *Theorem* 11.3, *Proposition* 12.1 *and Theorem* 12.3 *are true for a* 3-*K-contact structure.*

In fact, in proofs of Propositions listed above, (3.3) are not used. Only two points we must notice here are:

(i) we have a basis of the form $(\xi_{(i)}, X_j, \phi_{(i)}X_j)$ at each point. If $\dim M=3$, this is clear. If $\dim M>3$, we have a unit $X_1 \in M_p$, which is orthogonal to $\xi_{(i)}, i=1,2,3$. If we put $X=X_1$ in (13.4), we get $R(X_1, \xi_{(3)})\xi_{(1)}+R(X_1, \xi_{(1)})\xi_{(3)}=0$. Similarly, we have

$$R(X_1, \xi_{(1)})\xi_{(2)} + R(X_1, \xi_{(2)})\xi_{(1)} = 0 \tag{13.8}$$

By (13.7) and (13.7)$'$($\leftarrow \phi_{(2)}\xi_{(1)} = -\xi_{(3)}$):

$$R(X, \xi_{(2)})\xi_{(1)} - \phi_{(2)}\phi_{(1)}X = \phi_{(3)}X\,, \tag{13.7$'$}$$

(13.8) is written as

$$\phi_{(1)}\phi_{(2)}X_1 + \phi_{(2)}\phi_{(1)}X_1 = 0\,. \tag{13.9}$$

By (13.9), (13.9)$'$, (13.9)$''$, we see that $(\xi_{(1)}, X_1, \phi_{(i)}X_1)$ is orthonormal. These steps complete a basis stated above.

(ii) With respect to (12.11)$\to$(12.13), it is required that $(\xi_{(i)}, X_j, \phi_{(1)}\phi_{(2)}X_j, \phi_{(1)}\phi_{(3)}X_j, \phi_{(2)}\phi_{(3)}X_j)$ is also an orthonormal basis. This is also assured by (13.9).

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