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ON THE EMBEDDING AS A DOUBLE COMMUTATOR IN A TYPE 1 AW^*-ALGEBRA

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The purpose of this paper is to prove the following:

THEOREM. *Let M be a semi-finite AW^*-algebra with center Z. If M possesses a complete* set $\mathfrak{S}$ *of Z-valued bounded positive module homomorphisms which are completely additive on projections, then M can be embedded as a double commutator in an AW^*-algebra of type* 1 *with center which is isomorphic to Z.*

One of the problems concerning AW^*-algebras is: Whether or not there is a non-trivial AW^*-subalgebra of a W^*-algebra ([3], [16]) ? As an application of the above result, we shall show the following result which is a partial answer to this problem and is a generalization of [13, Theorem 5.2] on a problem of Feldman.

COROLLARY. *Let $\mathcal{B}$ be an AW^*-algebra of type* 1 *with center $\mathcal{Z}$ and let $\mathcal{A}$ be a semi-finite AW^*-subalgebra of $\mathcal{B}$ which contains $\mathcal{Z}$, then $\mathcal{A}=\mathcal{A}''$ (the double commutator of $\mathcal{A}$ in $\mathcal{B}$) in $\mathcal{B}$.*

Under the finiteness assumption on M and $\mathcal{A}$, H. Widom ([14]) showed the same result (see also [3], [4], [9] and [15]).

The main tool in this paper is a "non-commutative integration theory" with respect to a Z-valued trace Φ (a non-commutative vector measure) on the algebra of "locally measurable operators" affiliated with the given AW^*-algebra M.

This paper is devided into five sections. Section 1 is the preliminaries for the later sections and we will introduce the notion of "$\mathfrak{S}$-0-convergence" in M (Definition 1.1.2) such that for any orthogonal set $\{e_\alpha\}$ of projections in M with $e=\Sigma_\alpha e_\alpha$ and any element $a\in M$, $a^*ea=\Sigma_\alpha a^*e_\alpha a$ (unconditional sum of $a^*e_\alpha a$ with respect to $\mathfrak{S}$-0-convergence). In section 2, we shall prove the existence of a "$\mathfrak{S}$-0-continuous" natural application (Z-valued trace) Φ on M, using the Goldman's result ([4]). In section 3, along the same lines with [10], the extension theory of Φ to "locally measurable operators" affiliated with M ([11], [12]) are discussed. In particular, we shall show that the set $L^1(\Phi)$ of all Φ-integrable locally measurable operators is a complete normed module over Z. Section 4 concerns with the construction of AW^*-module $L^2(\Phi)$ (the collection of all Φ-square integrable locally measurable operators) over Z. The last section is devoted to prove our main theorem, more precisely to say, we shall show that the left regular representation π_1 of M on $L^2(\Phi)$ is a *-isomorphism of M into $\mathscr{B}(L^2(\Phi))$ (the set of all bounded module endomorphisms of $L^2(\Phi)$) such that $\pi_1(M)''=\pi_1(M)$ in $\mathscr{B}(L^2(\Phi))$ ($\pi_1(M)''$ is the double commutator of $\pi_1(M)$ in $\mathscr{B}(L^2(\Phi))$).

1. Definitions and preliminary results. An AW^*-algebra M means that it is both a C^*-algebra and a Baer*-ring ([7]).

The set of all self-adjoint elements, non-negative elements, projections, partial isometries and unitary elements in M is written with M_{sa}, M^+, M_p, M_{pi} and M_u, respectively.

We will say AW^*-algebra M to be semi-finite if every non-zero projection in M contains a non-zero finite projection in M.

For other informations about AW^*-algebras, in particular, the lattice structure theory of projections, and the algebra of "locally measurable operators", we refer to the papers [7], [8], [11], [12], [13], [14] and [16].

Denote the collection of all finite subset of a set A by $\mathscr{F}(A)$.

1.1. Order limits and center-valued c.a. states. Let Z be an abelian AW^*-algebra, then in virtue of the Gelfand representation, Z (resp. Z_{sa}) can be identified with the algebra $C(\Omega)$ (resp. $C_r(\Omega)$) of all complex (resp. real)-valued continuous functions on a stonian space Ω. Topologized the extended real line $[-\infty, +\infty]$ by the interval topology, let $C_r^*(\Omega)$ be the set of all $[-\infty, +\infty]$-valued continuous functions on Ω, then it is a complete lattice which is lattice isomorphic with the unit interval of the bounded complete lattice $C_r(\Omega)$ relative to the natural ordering for real functions and contains $C_r(\Omega)$ and $\boldsymbol{Z}$ (the set of all $[0, +\infty]$-valued continuous functions on Ω ([1])) as sublattices.

Let $\{a_\lambda\}$ be a net in $C_r^*(\Omega)$ and $a \in C_r^*(\Omega)$. By $a_\lambda \to a(0)$, we mean that $a=\limsup a_\lambda=\liminf a_\lambda$. In these circumstances, we say that the net $\{a_\lambda\}$ order converges to a. For any net $\{b_\lambda\}$ in $C(\Omega)$, $\{b_\lambda\}$ order-converges to b in $C(\Omega)$ if $(1/2)(b_\lambda+b_\lambda^*)\to(1/2)(b+b^*)(0)$ and $(1/2i)(b_\lambda-b_\lambda^*)\to(1/2i)(b-b^*)(0)$ where $i=\sqrt{-1}$. If Z is a von Neumann algebra, then $b_\lambda\to b(0)$ if and only if $\{b_\lambda\}$ converges strongly to b. In the case of an AW^*-algebra, the following criterion is useful for the later discussions.

LEMMA 1.1.1 ([14]). *Let $\{a_\lambda\}$ be a net in an abelian AW^* algebra Z and a be in Z, then $a_\lambda\to a(0)$ if and only if for any positive real number ε and a non-zero projection e in Z, there are a λ_0 and a non-zero projection f with $f\leqq e$ such that $\|(a_\lambda-a)f\|<\varepsilon$ for all $\lambda\geqq\lambda_0$.*

Next let N be an AW^*-algebra and $N^\natural$ be the center of N. A center-valued state ϕ on N is a non-negative module homomorphism ϕ from N to $N^\natural$. ϕ satisfies the following additional properties: (1) $\|\phi(a)\| \leqq k\|a\|$ for all $a \in N$ (k depends only on ϕ), (2) $|\phi(a^*b)|^2 \leqq \phi(a^*a)\phi(b^*b)$ for $a, b \in N$, (3) $\phi(b^*a^*ab) \leqq \|a^*a\| \phi(b^*b)$ for $a, b \in N$. By a center-valued c.a. state ϕ on N, we mean a center-valued state on N with the property that for any orthogonal family of projections $\{e_\alpha\}$ in N_p with $e = \Sigma_\alpha e_\alpha$ ($e \in N_p$), $\phi(e) = \Sigma_\alpha \phi(e_\alpha)$ in $N^\natural$, where $\Sigma_\alpha \phi(e_\alpha)$ is the unconditional sum of the $\phi(e_\alpha)$ in $N^\natural$.

LEMMA 1.1.2. *Let ϕ be a center-valued c.a. state on N, then for any $a \in N$ and any orthogonal family $\{e_\alpha\}$ of projections in N with $e = \Sigma_\alpha e_\alpha$, $\phi(a^*ea) = \Sigma_\alpha \phi(a^*e_\alpha a)$ in $N^\natural$.*

Since $N^{\natural +}$ is a bounded complete lattice, by Lemma 1.1.1, the proof is an obvious modification of that for a similar result in [3, Lemma 3].

In the followings, let M be a semi-finite AW^*-algebra with the center Z and suppose that there is a set $\mathfrak{S}$ of Z-valued c.a. states on M such that $\phi(a^*a) = 0$ for all $\phi \in \mathfrak{S}$ implies $a = 0$. Let $\mathcal{L}(\mathfrak{S})$ be the set of finite linear combinations of elements in $\{a^*\phi a,\ \phi \in \mathfrak{S},\ a \in M\}$, where $(a^*\phi a)(x) = \phi(axa^*)$ for $x \in M$.

DEFINITION 1.1.2. A net $\{a_\alpha\}$ in M $\mathfrak{S}$-0-converges to a in $M (a_\alpha \to a(\mathfrak{S}\text{-}0))$ if $\phi(a_\alpha - a) \to 0(0)$ in Z for all $\phi \in \mathcal{L}(\mathfrak{S})$.

REMARK. (1) Let $\{e_\alpha\}$ be an orthogonal family of projections in M with $\Sigma_\alpha e_\alpha = e (\in M_p)$, then $\Sigma_{\alpha \in J} e_\alpha \to e(\mathfrak{S}\text{-}0)(J \in \mathfrak{F}(\{\alpha\}))$ by Lemma 1.1.2. (2) Since $\mathfrak{S}$ is a separating set, an $\mathfrak{S}$-0-limit is unique.

1.2. Existence of a trace. Let N be a finite AW^*-algebra with the center $N^\natural$ which has a separating set $\mathfrak{S}'$ of center-valued c.a. states. Then, we have

PROPOSITION 1.2.1. *There is a unique central trace Φ having the additional property that for any increasing net $\{a_\gamma\}$ in N^+, with $a_\gamma \uparrow a(\mathfrak{S}'\text{-}0)$ for some $a \in N^+$, then $\Phi(a_\gamma) \uparrow \Phi(a)$ in $N^{\natural +}$.*

PROOF. Existence of a trace Φ on N is due to M. Goldman [4]. Therefore we have only to show that Φ satisfies the continuity described above. Since $\mathfrak{S}'$ is a separating set, by [4, Lemma 2.6], for any $p \in N^\natural_p$, there are a non-zero projection e in N ($e \leqq p$) and a non-negative mapping ϕ in $\mathcal{L}(\mathfrak{S}')$ with $\phi(e) \neq 0$ such that $\Phi(a) \leqq \phi(a)$ for all $a \in (eNe)^+$. Take a positive integer m and a non-zero central projection ($q \leqq p$) with $\Phi(e) \geqq (1/m)q$ such that there exists a projection $h \in N$ with $\Phi(h) = (1/m)q$. Hence we can choose a family $\{h_j\}_{j=1}^m$ of mutually orthogonal projections in N such that $h_1 \leqq e$, $h_i \sim h_j$ and $\sum_{j=1}^{m} h_j = q$. Let v_j be in N_{pi} such that $v_j^* v_j = h_1$, $v_j v_j^* = h_j$ and put $\psi(b) = \sum_{j=1}^{m} \phi(v_j^* b v_j)$ for $b \in N$, then $\psi \in \mathcal{L}(\mathfrak{S}')$ and $\psi(1-q)=0$. Now, noting that $v_i^* b v_j \in eNe$ for each pair of i and j, it follows that for each $b \in Nq$,

$$
\begin{aligned}
\psi(b^*b) &= \sum_{ij=1}^{m} \phi((v_i^* b^* v_j)(v_i^* b^* v_j)^*) \\
&\geqq \sum_{ij=1}^{m} \Phi((v_i^* b^* v_j)(v_i^* b^* v_j)^*) \\
&= \Phi(b^* b)\,.
\end{aligned}
$$

Hence by Zorm's lemma there are families $\{q_\alpha\} \subset N_p^\natural$ and $\{\phi_\alpha\} \subset \mathcal{L}(\mathfrak{S}')$ such that $q_\alpha q_\beta = 0$ $(\alpha \neq \beta)$, $\Sigma_\alpha q_\alpha = 1$, $\phi_\alpha(q_\alpha) \neq 0$, $\phi_\alpha(1-q_\alpha) = 0$ and $\phi_\alpha(b^*b) \geqq \Phi(b^*b)$ for all $b \in Nq_\alpha$ for each α. If $\{a_\gamma\}$ is an increasing net of N^+ such that $a_\gamma \uparrow a(\mathfrak{S}\text{-}0)$ for some $a \in N$, then $q_\alpha \Phi(a_\gamma) \uparrow q_\alpha \Phi(a)$ in $N^{\natural +}$ for each α. Therefore by Lemma 1.1.1, $\Phi(a_\gamma) \uparrow \Phi(a)(0)$. This completes the proof.

2. Existence of a natural application on M^+. Let Ω be the spectrum of the center Z of the given semi-finite AW^*-algebra M and $\boldsymbol{Z}$ be the collection of all $[0, +\infty]$-valued continuous functions on Ω.

To prove the existence of a natural application, we need the following, whose proof can be easily supplied by the reader.

LEMMA 2.1. *Let $\{a_\alpha\}$ be an increasing net in $\mathbf{Z}$ such that $a_\alpha \uparrow a(0)$ in $\mathbf{Z}$ for some $a \in \boldsymbol{Z}$, then for any $b \in \mathbf{Z}$, $ba_\alpha \uparrow ba(0)$ in $\mathbf{Z}$.*

Since M is semi-finite, there is a finite projection p in M such that $z(p)=1$. Let $\{p_\alpha\}_{\alpha \in \pi}$ be a maximal family of orthogonal equivalent projections in M such that $p \sim p_\alpha$ for each α and $p \in \{p_\alpha\}_{\alpha \in \pi}$. By the maximality of $\{p_\alpha\}_{\alpha \in \pi}$, there is a central projection z such that $p_0 = (1 - \Sigma_{\alpha \in \pi} p_\alpha) z \precsim pz \neq 0$. Therefore we can take families $\{z_\beta\} \subset Z_p$, $\{p_\beta\} \subset M_p$ and $\{p(\alpha_\beta, \beta)\}_{\alpha_\beta \in \pi_\beta \cup \{0\}}$ in M_p such that $z_\beta z_\gamma = 0$ $(\beta \neq \gamma)$, $p(\alpha_\beta, \beta) p(\gamma_\beta, \beta) = 0$ $(\alpha_\beta \neq \gamma_\beta)$, $z_\beta = p(0, \beta) + \Sigma_{\alpha_\beta \in \pi_\beta \cup \{0\}} p(\alpha_\beta, \beta) z_\beta$, $p(\alpha_\beta, \beta) z_\beta \sim p_\beta z_\beta$ for each $\alpha_\beta \in \pi_\beta$, $z(p_\beta) = z_\beta$, p_β is finite for each β, $p_\beta \in \{p(\alpha_\beta, \beta)\}_{\alpha_\beta \in \pi_\beta}$ for each β, $(1 - \Sigma_{\alpha_\beta \in \pi_\beta} p(\alpha_\beta, \beta) z_\beta = p(0, \beta) \precsim p_\beta z_\beta \neq 0$ and $\Sigma_\beta z_\beta = 1$. Noting that $z_\beta p_\beta M z_\beta p_\beta$ is a finite AW^*-algebra whose center is $Z z_\beta p_\beta$, if $\mathfrak{S}_\beta = \{(z_\beta p_\beta \phi z_\beta p_\beta) p_\beta, \phi \in \mathfrak{S}\}$ (where $(z_\beta p_\beta \phi z_\beta p_\beta) p_\beta(x) = p_\beta \phi(z_\beta p_\beta x z_\beta p_\beta)$, $x \in M$), then $\mathfrak{S}_\beta$ is a separating set of center-valued c.a. states on $z_\beta p_\beta M z_\beta p_\beta$. By Proposition 1.2.1, for each β, we can choose a $Z z_\beta p_\beta$-valued $\mathfrak{S}_\beta$-0-continuous trace Φ_β on $z_\beta p_\beta M z_\beta p_\beta$. Now let ψ_β be the

*-isomorphism of $Zz_\beta p_\beta$ onto Zz_β which is defined by $\psi_\beta^{-1}(x)=xp_\beta$ for each β and let $v(\alpha_\beta,\beta)$ be the partial isometry such that $v(\alpha_\beta,\beta)^*v(\alpha_\beta,\beta)=z_\beta p_\beta$, $v(\alpha_\beta,\beta)v(\alpha_\beta,\beta)^*=p(\alpha_\beta,\beta)$ for each $\alpha_\beta\in\pi_\beta$ and each β, $v(0,\beta)^*\,v(0,\beta)\leqq z_\beta p_\beta$ and $v(0,\beta)v(0,\beta)^*=p(0,\beta)$for each β. Define a new linear operation Φ on M^+ to $\boldsymbol{Z}$ as follows:

$$\Phi(h)=\Sigma_\beta\{\Sigma_{\alpha_\beta\in\pi_\beta\cup\{0\}}\psi_\beta(\Phi_\beta(v(\alpha_\beta,\beta)^*hz_\beta v(\alpha_\beta,\beta)))\},\qquad h\in M^+$$

where $\Sigma_{\alpha\in A}a_\alpha$ is the unconditional sum of the a_α in $\boldsymbol{Z}$, then Φ is a natural application on M^+, that is,

THEOREM 2.1. *The operation Φ on M^+ to $\boldsymbol{Z}$ satisfies the following properties:*

(1) *If $h_1,h_2\in M^+$ and λ is a non-negative number,* $\Phi(h_1+h_2)=\Phi(h_1)+\Phi(h_2)$ and $\Phi(\lambda h_1)=\lambda\Phi(h_1)$.

(2) *If $s\in M^+$ and $t\in Z^+$, then $\Phi(st)=t\Phi(s)$.*

(3) *If $a\in M^+$ and $u\in M_u$, $\Phi(uau^*)=\Phi(a)$.*

(4) *$\Phi(a)=0$ $(a\in M^+)$ implies $a=0$.*

(5) *For every increasing net $\{a_\mu\}$ in M^+ such that $a_\mu\uparrow a(\mathfrak{S}\text{-}0)$ for some $a\in M^+$, $\Phi(a_\mu)\uparrow\Phi(a)(0)$ in $\boldsymbol{Z}$.*

(6) *For* any *non-zero a in M^+, there is a non-zero b in M^+ majorized by a such that $\Phi(b)\in Z^+$.*

Using Lemma 2.1 and $\mathfrak{S}$-0-convergence instead of Lemma 2.12 and $\sigma(\mathfrak{S})$-topology in [13], the proof of this theorem proceeds in a manner entirely analogous to that of [13, Theorem 3.1], so we omit it.

Next let $\mathfrak{P}=\{s\in M^+,\ \Phi(s)\in Z^+\}$, then since $\mathfrak{P}$ satisfies the conditions of Lemma 1 in [2, Chapter 1 §1, 6], it follows that $\mathfrak{P}$ is the positive portion of a two-sided ideal $\mathfrak{N}$ and that there is a unique linear operation $\dot{\Phi}$ on $\mathfrak{N}$ to Z which coincides with Φ on $\mathfrak{P}$ with the properties; (a) $\dot{\Phi}(st)=\dot{\Phi}(ts)$ if $s\in M$, $t\in\mathfrak{N}$; (*b*) $\dot{\Phi}(st)=s\dot{\Phi}(t)$ if $s\in Z$ and $t\in\mathfrak{N}$.

Define $\mathrm{Rank}(x)=\Phi(LP(x))$ for every $x\in M$, where $LP(x)$ is the left projection of x in M, and $\mathrm{Rank}(x)$ has the following properties: (1) $\mathrm{Rank}(x)\geqq 0$, it is $=0$ only if $x=0$. (2) $\mathrm{Rank}(x)=\mathrm{Rank}(x^*)$, $\mathrm{Rank}(\alpha x)=\mathrm{Rank}(x)$ for every complex number $\alpha\neq 0$. (3) $\mathrm{Rank}(x+y)\leqq\mathrm{Rank}(x)+\mathrm{Rank}(y)$. (4) $\mathrm{Rank}(xy)\leqq\mathrm{Rank}(x),\mathrm{Rank}(y)$. In fact, (1) and the last half part of (2) are clear from definitions. By [7, Theorem 5.2], $LP(x)\sim LP(x^*)$, which implies by [13, Lemma 2.4] $\Phi(LP(x))=\Phi(LP(x^*))$. An easy calculation shows $LP(x+y)\leqq LP(x)\vee LP(y)$ and by the fact that $LP(x)\vee LP(y)-LP(x)\sim LP(y)-LP(x)\wedge LP(y)$, it follows that $\mathrm{Rank}(x+y)\leqq\mathrm{Rank}(x)+\mathrm{Rank}(y)$. $LP(xy)\leqq LP(x)$ shows that $\mathrm{Rank}(xy)\leqq\mathrm{Rank}(x)$ and $\mathrm{Rank}(xy)=\mathrm{Rank}((xy)^*)=\mathrm{Rank}(y^*x^*)\leqq\mathrm{Rank}(y^*)=\mathrm{Rank}(y)$. Thus (3) follows.

Therefore let $\mathcal{F}=\{a;\ a\in M,\ \mathrm{Rank}(a)\in Z^+\}$, then $\mathcal{F}$ is a two-sided ideal contained in $\mathfrak{N}$ such that $\mathcal{F}_p=\mathfrak{N}_p$. Moreover, by Theorem 2.1 (6) for any non-zero projecection e in M, we can choose a non-zero projection in $\mathcal{F}$ majorized by e.

3. An extension of Φ to "locally measurable operators". We shall now consider "locally measurable operators" affiliated with M ([12]). An essentially locally measurable operator (ELMO) is a family of ordered pairs $\{x_\alpha, e_\alpha\}$, where $\{x_\alpha\}\subset \mathcal{C}$ (the algebra of measurable operators affiliated with M) and $\{e_\alpha\}$ is an orthogonal family of central projections such that $\Sigma_\alpha e_\alpha=1$. Two ELMO's $\{x_\alpha, e_\alpha\}$ and $\{y_\beta, f_\beta\}$ are said to be equivalent if $e_\alpha f_\beta x_\alpha = e_\alpha f_\beta y_\beta$ for all α and β. The equivalence class of $\{x_\alpha, e_\alpha\}$ is denoted by (x_α, e_α) and it is called a locally measurable operator affiliated with M(LMO), and the collection of all LMO's affiliated with M is denoted by $\mathcal{M}$. Algebraic operations in $\mathcal{M}$ are componentwise, then it is a *-algebra in which $\mathcal{C}$ is naturally imbedded as a *-subalgebra. We use letters $x, y, z, \cdots$ for the elements in $\mathcal{M}$.

In [12], we showed the followings: (1) $\mathcal{M}$ is a Baer*-ring, and (2) every element x in $\mathcal{M}$ has a polar decomposition $x=w|x|(|x|=(x^*x)^{1/2})$ where $w^*w=RP(x)$ and $ww^*=LP(x)$. The self-adjoint part of $\mathcal{M}$ is partially ordered by defining $x\geqq y$ if $x-y=z^*z$ for some z. The subalgebra M is characterized as $\{x;\ x\in\mathcal{M},\ x^*x\leqq\alpha 1$ for some positive real number $\alpha\}$.

We want to extend Φ to $\mathcal{M}^+$ (the non-negative part of $\mathcal{M}$). The following definition is due to [10].

DEFINITION 3.1. For every $x\in\mathcal{M}^+$, we define

$$\Phi(x)=\mathrm{Sup}\{\Phi(a),\ a\in M^+,\ a\leqq x\},$$

where the supremum is taken in $\mathbf{Z}$.

It is clear that the new definition agrees with the old one in case $x\in M^+$. The following Lemma is helpful for the later discussions.

LEMMA 3.1. *For every* $x\in\mathcal{M}^+$, $\Phi(x)=\mathrm{Sup}\{\Phi(a);\ a\in\mathfrak{N}^+, a\leqq x\}=\mathrm{Sup}\{\Phi(a);\ a\in\mathcal{F}^+,\ a\leqq x\}$.

PROOF. Since $\Phi(x)\geqq\mathrm{Sup}\{\Phi(a),\ a\in\mathfrak{N}^+,\ a\leqq x\}\geqq\mathrm{Sup}\{\Phi(a),\ a\in\mathcal{F}^+,\ a\leqq x\}$, we have only to prove the converse. Let $b=\mathrm{Sup}\{\Phi(a);\ a\in\mathcal{F}^+, a\leqq x\}$ in $\mathbf{Z}$. By Theorem 2.1, there is an orthogonal family of projections $\{e_\alpha\}$ in $\mathcal{F}_p$ such that $\Sigma_\alpha e_\alpha=1$. For any $J\in\mathcal{F}(\{\alpha\})$ and $a\in M^+$, $a^{1/2}(\Sigma_{\alpha\in J}e_\alpha)a^{1/2}\leqq a$, $a^{1/2}(\Sigma_{\alpha\in J}e_\alpha)a^{1/2}\in\mathcal{F}^+$ and $a^{1/2}(\Sigma_{\alpha\in J}e_\alpha)a^{1/2}\uparrow a(\mathfrak{S}\text{-}0)$. Therefore again by Theorem 2.1, $\Phi(a)=\mathrm{Sup}\{\Phi(a^{1/2}(\Sigma_{\alpha\in J}e_\alpha)a^{1/2});\ J\in\mathcal{F}(\{\alpha\})\}$, that is, $\Phi(a)\leqq b$. Thus $b=\Phi(x)$ and the lemma follows.

REMARK. For any $x \in \mathcal{M}$, $\Phi(x^*x) = \Phi(xx^*)$. In fact, let $x = w|x|$ be the polar decomposition of x, then $xx^* = wx^*xw^*$ and $w^*xx^*w = x^*x$. If $x^*x \geqq a, a \in \mathcal{F}^+$, then $aw^*w = w^*wa = a$ and $xx^* = wx^*xw^* \geqq waw^* \in \mathcal{F}^+$. Thus, $\Phi(xx^*) \geqq \Phi(waw^*) = \Phi(w^*wa) = \Phi(a)$, which implies $\Phi(xx^*) \geqq \Phi(x^*x)$. By symmetry $\Phi(x^*x) = \Phi(xx^*)$.

Relations between the algebraic operations in $\mathcal{M}^+$ and our extended operation Φ are given in the following:

LEMMA 3.2. *Let s and t be in $\mathcal{M}^+$, then*

(1) $\Phi(s+t) = \Phi(s) + \Phi(t)$;

(2) $\Phi(\lambda t) = \lambda\Phi(t)$ *for any non-negative number* λ;

(3) $\Phi(usu^*) = \Phi(s)$ *for any* $u \in M_u$;

(4) $\Phi(as) = a\Phi(s)$ *for any* $a \in Z^+$.

PROOF. The statements (2) and (3) are clear from the definitions. For the assertion (1), since $\Phi(s)+\Phi(t) \leqq \Phi(s+t)$, we have only to show the converse. Let a be in $\mathcal{F}^+$ such that $a \leqq s+t$ and $c_n = a^{1/2}((1/n)1+s+t)^{-1}(s+t)^{1/2}$(note that since $s+t \geqq 0$, $s+t+(1/n)1$ is invertible in $\mathcal{M}$ and $(s+t+(1/n)1)^{-1} \in \{s+t\}''$ for each positive integer n), then c_n and $a^{1/2} - c_n(s+t)^{1/2}$ are bounded elements such that $\|a^{1/2} - c_n(s+t)^{1/2}\| \leqq 1/n$ and $\|c_n\| \leqq 1$ for each n. Observe that $a \in \mathcal{F}^+$, let $x = c_n s^{1/2}$ and $y = c_n t^{1/2}$, then $xx^* = c_n s c_n^* \leqq c_n(s+t)c_n^* \leqq a^{1/2}((1/n)1+s+t)^{-2}(s+t)^2 a^{1/2} \leqq a$ and by the same way, $yy^* \leqq a$, which implies x and y are in $\mathcal{F}$. Now put $a_1 = x^*x$ and $a_2 = y^*y$, then $a_1, a_2 \in \mathcal{F}^+$, $a_1 = s^{1/2}c_n^* c_n s^{1/2} \leqq s$ and $a_2 \leqq t$. Therefore we have

$$\begin{aligned}\Phi(s) + \Phi(t) &\geqq \Phi(a_1) + \Phi(a_2) = \Phi(x^*x) + \Phi(y^*y) \\ &= \Phi(xx^*) + \Phi(yy^*) = \Phi(c_n s c_n^*) + \Phi(c_n t c_n^*) \\ &= \Phi(c_n(s+t)c_n^*).\end{aligned}$$

Note that $LP(a)c_n = c_n$, it follows that $\{a^{1/2} - c_n(s+t)^{1/2}\}\{a^{1/2} - c_n(s+t)^{1/2}\}^* \leqq (1/n)LP(a)$. On the other hand, since $a^{1/2}(s+t)^{1/2}c_n^* = a^{1/2}(s+t)((1/n)1+s+t)^{-1}a^{1/2} \leqq a \in \mathcal{F}$, $a^{1/2}(s+t)^{1/2}c_n^* = c_n(s+t)^{1/2}a^{1/2}$, and $c_n(s+t)^{1/2} \in \mathcal{F}$, we get that

$$\Phi(a) - \Phi(c_n(s+t)c_n^*) = \Phi(\{a^{1/2} + c_n(s+t)^{1/2}\}\{a^{1/2} - c_n(s+t)^{1/2}\}^*).$$

Observe that $\|c_n(s+t)^{1/2}\| \leqq \|a^{1/2}\|$, it follows by the above arguments that

$$\begin{aligned}\|\Phi(a) - \Phi(c_n(s+t)c_n^*)\| &\leqq \|a^{1/2} + c_n(s+t)^{1/2}\| \, \|\Phi(|a^{1/2} - (s+t)^{1/2}c_n^*|)\| \\ &\leqq 2\|a\|^{1/2}(1/n)^{1/2}\|\Phi(LP(a))\|\end{aligned}$$

for each n, that is, $a \geqq c_n(s+t)c_n^*$ implies that

$$\Phi(s)+\Phi(t) \geqq \Phi(c_n(s+t)c_n^*)$$
$$\geqq \Phi(a) - 2(1/n)^{1/2}\|a\|^{1/2}\|\Phi(LP(a))\| \cdot 1$$

for all positive integer n, so that $\Phi(s)+\Phi(t) \geqq \Phi(a)$ for all $a \in \mathcal{F}^+$ with $a \leqq s+t$. Thus by Lemma 3.1, $\Phi(s)+\Phi(t) \geqq \Phi(s+t)$ and (1) follows.

To prove the assertion (4), since it is clear, by Lemma 2.1 and Lemma 3.1, that $a\Phi(t) \leqq \Phi(at)$ for any $t \in \mathcal{M}^+$ and $a \in Z^+$, it is sufficient to show the converse. Let c be in $\mathcal{F}^+$ with $c \leqq at$, then for each positive integer n, $c \leqq a+(1/n)t$, which implies $(a+(1/n)1)^{-1}a\Phi(c) \leqq a\ \Phi(t)$ by Theorem 2.1. Since $LP(a)c = c\ LP(a) = c$ and $(a+(1/n)1)^{-1}a \uparrow LP(a)$, we have $\Phi(c) \leqq a\Phi(t)$, so that $a\Phi(t) \geqq \Phi(at)$ by Lemma 3.1. This completes the proof.

Let $\mathcal{L}^+ = \{t; t \in \mathcal{M}^+, \Phi(t) \in Z^+\}$, then by the above lemma, $\mathcal{L}^+$ has the following properties:

(a) If $s \in \mathcal{L}^+$ and $u \in M_u$, then $usu^* \in \mathcal{L}^+$ and $\Phi(s) = \Phi(usu^*)$.

(b) Let $s \in \mathcal{L}^+$ and $t \in \mathcal{M}^+$ with $t \leqq s$, then $t \in \mathcal{L}^+$.

(c) For every s and $t \in \mathcal{L}^+$, $s+t \in \mathcal{L}^+$ and $\Phi(s+t) = \Phi(s)+\Phi(t)$.

Let $L^1(\Phi) = \left\{\sum_{i=1}^{n} t_i s_i^*, t_i^* t_i, s_i^* s_i \in \mathcal{L}^+\right\}$, then

THEOREM 3.1 ([10]). *$L^1(\Phi)$ is a unique invariant linear system (that is, $ML^1(\Phi)M \subset L^1(\Phi)$) such that $L^1(\Phi)^+ = \mathcal{L}^+$. Moreover, there is a unique non-negative linear operation $\dot{\Phi}$ on $L^1(\Phi)$ to Z, which coincides with Φ on $\mathcal{L}^+$, with the following properties:*

(1) *For $s \in L^1(\Phi)$ and $a \in M$, $\dot{\Phi}(at) = \dot{\Phi}(ta)$;*

(2) *for $a \in Z$ and $s \in L^1(\Phi)$, $\dot{\Phi}(at) = a\dot{\Phi}(t)$;*

(3) *for any $t \in L^1(\Phi)$, $\mathrm{Sup}\{|\dot{\Phi}(at)|;\ \|a\| \leqq 1,\ a \in M\} = \Phi(|t|)$;*

(4) *if s, $t \in L^1(\Phi)$, then $\Phi(|s+t|) \leqq \Phi(|s|)+\Phi(|t|)$.*

PROOF. The proof of the assertions except for (3) and (4) are obvious modifications of those for similar results in section 2 for the case $\mathfrak{N}$ and $\dot{\Phi}$. To prove the assertion (3), we argue as follows. Observe first that from the standard calculation, $|\dot{\Phi}(st)|^2 \leqq \Phi(s^*s)\Phi(t^*t)$ for any s and t with s^*s and $t^*t \in \mathcal{L}^+$. Let $t = u|t|$ be the polar decomposition of t in $L^1(\Phi)$, then for any $a \in M$ with $\|a\| \leqq 1$, it follows that

$$|\dot{\Phi}(at)|^2 = |\dot{\Phi}(au|t|)|^2 \leqq \Phi(|t|^{1/2}u^*a^*au|t|^{1/2})\Phi(|t|)$$
$$\leqq \Phi(|t|)^2,$$

So that $|\dot{\Phi}(at)| \leqq \Phi(|t|)$ and $\dot{\Phi}(u^*t) = \Phi(|t|)$ and $\|u\| \leqq 1$ implies the statement (3). Next let $s, t \in L^1(\Phi)$ and $s+t = w|s+t|$ be the polar decomposition of $s+t$, then by (3)

$$\Phi(|s+t|) = \Phi(w^*(s+t)) \leqq |\dot{\Phi}(w^*s)| + |\dot{\Phi}(w^*t)|$$
$$\leqq \Phi(|s|) + \Phi(|t|),$$

thus the proof is completed.

REMARK. (1) The linear map $\dot{\Phi}$ on $L^1(\Phi)$ is an extension of $\dot{\Phi}$ on $\mathfrak{N}$ which was defined in section 2. (2) If we set $|||s|||_1 = \|\Phi(|s|)\|$ for $s \in L^1(\Phi)$, then $L^1(\Phi)$ is a normed module over Z. (3) $L^1(\Phi) \subset \mathcal{C}$. In fact, since every element of $L^1(\Phi)$ is a finite linear combination of elemens in $\mathcal{L}^+$, we have only to show that $\mathcal{L}^+ \subset \mathcal{C}$. By the spectral theorem ([11, 12]), for any $t \in \mathcal{L}^+$ there exists an increasing sequence of projections $\{f_n\}$ in $\{t\}''$ (the double commutant of $\{t\}$ in $\mathcal{M}$) such that $tf_n \leqq (n+1)1$ and $(n+1)(1-f_n) \leqq t$ for each positive integer n, so that $\Phi(1-f_n) \leqq (1/(n+1))\Phi(t)$, this implies that $\{f_n\}$ is an SDD. Thus by [11, Theorem 5.1], $t \in \mathcal{C}$. This completes the proof.

THEOREM 3.2. *$L^1(\Phi)$ is a Banach space with respect to the norm* $|||\,,\,|||_1$.

PROOF. First of all, we shall show that for any monotone increasing sequence $\{t_n\}$ of elements in $\mathcal{L}^+$ which is $|||\,,\,|||_1$—Cauchy, there is $t \in \mathcal{L}^+$ such that $|||t_n - t|||_1 \to 0 (n \to \infty)$. By taking a subsequence, we can assume that $|||t_n - t_{n+1}|||_1 < 1/4^n$ for each positive integer n without loss of generality. Note that $t_{n+1} - t_n \geqq 0$ (resp. $t_n \geqq 0$), by the spectral theorem ([11]), we can choose a sequence $\{e_n\}$ in $\{t_{n+1} - t_n\}''$ (resp. $\{f_n\}$ in $\{t_n\}''$) of projections such that $0 \leqq (t_{n+1} - t_n)e_n \leqq 2^{-n}.1$ and $(t_{n+1} - t_n) \geqq 2^{-n}(1-e_n)$ (resp. $0 \leqq t_n f_n \leqq 2^n.1$ and $t_n \geqq 2^n(1-f_n)$) for each positive integer n. Now let $p_n = \bigwedge_{k \geqq n} e_k \wedge f_k$, then it follows that

$$\begin{aligned}\Phi(1-p_n) &\leqq \sum_{k=n}^{\infty} \Phi(1 - e_k \wedge f_k) \\ &\leqq \sum_{k=n}^{\infty} \{\Phi(1-e_k) + \Phi(1-f_k)\} \\ &\leqq \sum_{k=n}^{\infty} \{2^k \Phi(t_{k+1} - t_k) + (1/2^k)\Phi(t_k)\} \\ &\leqq (1 + \mathrm{Sup}\, |||t_k|||_1)2^{-n}.1\end{aligned}$$

for each n, so that $p_n \uparrow$ implies that $\Phi(1-p_n) \downarrow 0$ uniformly, $1 - p_n \in \mathcal{F}$ and $p_n \uparrow 1$, that is, $\{p_n\}$ is an SDD([11, Definition 3.1]). Since $p_n \leqq e_n \wedge f_n$, if $k \leqq n \leqq m$, then $(t_m-t_n)p_k \in M$ and $\|(t_m-t_n)p_k\| < 1/2^{n-1}$. Moreover, $t_kp_k = t_kf_kp_k$ and $t_kf_k \leqq 2^kf_k$, which implies $t_kp_k \in M$. By the mathematical induction, $(t_m-t_n)p_k \in M(m \geqq n \geqq k)$ implies $t_mp_k \in M$ for all $m \geqq k$. Now put $a(n, k) = p_kt_np_k + p_kt_n(1-p_k)+(1-p_k)t_np_k(n \geqq k)$, then $\{a(n, k)\} \subset M_{sa}$ for all $n \geqq k$. Since $\|a(n+1, k)-a(n, k)\| \leqq 3.\ 2^{-n}$ for all $n \geqq k$, it follows that $\{a(n,k)\}_{n\geqq k}$ is a uniformly Cauchy sequence in M_{sa}. Hence there exists an element $s(k) \in M_{sa}$ such that $a(n, k) \to s(k)(n\to\infty)$ uniformly. If $k_1 \geqq k_2$, then $p_{k_1} \geqq p_{k_2}$ implies $s(k_1)p_{k_2} = s(k_2)p_{k_2}$, so that $\{s(k), p_k\}$ is an EMO ([11, Dfinition 3.1]). Since $\|t_kp_k - t_mp_k\| \leqq 1/2^{k-1}$ for all $m \geqq k$, we get that $\|t_kp_k - s(k)p_k\| \leqq 1/2^{k-1}$ for each positiye integer k. Thus putting $t = [s(k),\ p_k](\in \mathcal{C}_{sa}$ ([11, Definition 3.4])), by [11, Theorem 3.1] $\|t_kp_k - tp_k\| = \|(t_k - s(k))p_k\| \leqq 1/2^{k-1}$ for all k, which implies that $t_k \to t(n.e.)$ $(k\to\infty)$ ([13, Definition 3.2]). Next we shall show that $t \geqq t_n$ for each n. Observe that $p_kt_mp_k \geqq p_kt_np_k \geqq 0$ $(m \geqq n \geqq k)$ and $p_kt_np_k \to p_ktp_k$ uniformly $(n\to\infty)$ and we have $p_ktp_k = p_ks(k)p_k \geqq p_kt_np_k \geqq 0$ for all $n \geqq k$. Thus by [11, Theorem 5.5], it follows that $t \geqq t_n$ for each n. Now we shall show that $\Phi(t) = \sup_n \Phi(t_n)$. Since $\Phi(t_n) \leqq \Phi(t)$ for all n, we have only to show the converse. Since $p_kt_np_k \uparrow p_ktp_k$ uniformly $(n\to\infty)$, for any $e \in \mathcal{F}_p$, $\|\Phi(ep_kt_kp_ke) - \Phi(ep_ktp_ke)\| \to 0(n\to\infty)$, which implies by Lemma 1.1.1, $\Phi(ep_kt_np_ke) \uparrow \Phi(ep_ktp_ke)\,(0)$ in Z^+. Since $\Phi(t_n) \geqq \Phi(t_n^{1/2}p_kep_kt_n^{1/2}) = \Phi(ep_kt_np_ke)$, it follows that

$$\Phi(t) \geqq \operatorname*{Sup}_n\ \Phi(t_n) \geqq \Phi(ep_ktp_ke) = \Phi(t^{1/2}p_kep_kt^{1/2}),$$

so that by the last paragraph of section 2 and Lemma 4.1, $\Phi(t^{1/2}p_kep_kt^{1/2}) \uparrow \Phi(t^{1/2}p_kt^{1/2})$ in $\mathbf{Z}$. Hence $\Phi(t) \geqq \operatorname*{Sup}_n \Phi(t_n) \geqq \Phi(t^{1/2}p_kt^{1/2})$. Again by Lemma 4.1, $\Phi(t) = \operatorname*{Sup}_n \Phi(t_n)$. $\operatorname*{Sup}_n |||\, t_n \,|||_1 < \infty$ implies $\Phi(t) \in Z$ and $t \in \mathcal{L}^+$. Since $\sum_{n=1}^{\infty} |||\, t_n - t_{n-1} \,|||_1 \leqq \sum_{n=1}^{\infty} 1/4^n < \infty$, for every positive number ε, there is a positive integer $k(\varepsilon)$ such that $\sum_{n=k}^{\infty} |||\, t_n - t_{n-1} \,|||_1 \leqq \varepsilon$ for all $k \geqq k(\varepsilon)$, that is, $\sum_{n=k+1}^{m} \Phi(t_n - t_{n-1}) = \Phi(t_m) - \Phi(t_k) \leqq \varepsilon \cdot 1$ for all $m \geqq k+1 \geqq k(\varepsilon)$. $\Phi(t_m) \uparrow \Phi(t)(0)$ implies $\Phi(t) - \Phi(t_k) \leqq \varepsilon \cdot 1$, that is, $|||\, t - t_k \,|||_1 \leqq \varepsilon$ for all $k \geqq k(\varepsilon)$. Thus the statement described above follows.

Using this fact, we can prove the completeness of $L^1(\Phi)$ by the similar way as that of [10, Theorem 14], so we omit the details. This completes the proof.

4. AW^*-module $L^2(\Phi)$ over Z. Let $L^2(\Phi) = \{s \in \mathcal{M},\ s^*s \in \mathcal{L}^+\}$, then for any s and t in $L^2(\Phi)$, $(s+t)^*(s+t) \leqq 2(s^*s + t^*t) \in \mathcal{L}^+$ shows by Lemma 3.2, $s+t \in L^2(\Phi)$. For any $a \in Z$ and $s \in L^2(\Phi)$, we have $\Phi(|a|^2s^*s) = |a|^2\Phi(s^*s) \in Z^+$, so that $as \in L^2(\Phi)$, that is, $L^2(\Phi)$ is a module over Z.

At first, we shall give the following lemma.

LEMMA 4.1. *Let $s \in \mathcal{M}$ and $\sigma_s(x) = \Phi(s^*xs)$ for any $x \in M^+$, then for any increasing net $\{a_\gamma\}$ in M^+ such that $a_\gamma \uparrow e(\mathfrak{S}\text{-}0)$ for some $e \in M_p$, $\sigma_s(a_\gamma) \uparrow \sigma_s(e)$ in $\boldsymbol{Z}$. In particular, σ_s is completely additive on projections.*

PROOF. Since $\sigma_s(e) \geqq \operatorname{Sup}_\gamma \sigma_s(a_\gamma)$, we have only to show the converse. Let $b \in \mathcal{F}^+$ with $b \leqq \mathrm{ess}^*e$, then $eb = be = b$ and $b^{1/2}(a_\gamma)b^{1/2} \uparrow b^{1/2}eb^{1/2}(\mathfrak{S}\text{-}0)$, so that by the continuity of Φ, $\Phi(b^{1/2}a_\gamma b^{1/2}) \uparrow \Phi(b^{1/2}eb^{1/2})$. On the other hand, since $\Phi(b^{1/2}a_\gamma b^{1/2}) = \Phi(a_\gamma^{1/2}ba_\gamma^{1/2}) \leqq \Phi(a_\gamma^{1/2}ss^*a_\gamma^{1/2}) = \Phi(s^*a_\gamma s)$, it follows that $\Phi(b) \leqq \operatorname{Sup}_\gamma \sigma_s(a_\gamma)$. Therefore by Lemma 3.1, $\sigma_s(e) \leqq \operatorname{Sup}_\gamma \sigma_s(a_\gamma)$ and the proof is now completed.

LEMMA 4.2 ([10]). *$L^2(\Phi)$ has the following properties:*

(1) *For s and t in $L^2(\Phi)^+$, $\Phi(st) \geqq 0$;*

(2) *if $s, t \in L^2(\Phi)$ with $|s| \leqq |t|$, then $\Phi(|s|^2) \leqq \dot{\Phi}(|s|\,|t|) \leqq \Phi(|t|^2)$;*

(3) *if s and t are self-adjoint elements in $L^2(\Phi)$ such that $\Phi(s^2) \leqq \Phi(t^2)$, then $\dot{\Phi}(st) \leqq \Phi(t^2)$;*

(4) *let t be in $L^2(\Phi)$ and $u \in M_u$, then $\Phi(|t|^2) = \Phi(|utu^*|^2)$;*

(5) *if $s, t \in L^2(\Phi)$, then $st \in L^1(\Phi)$, $|\dot{\Phi}(st)|^2 \leqq \Phi(|st|)^2 \leqq \Phi(s^*s)\Phi(t^*t)$* and

$$\Phi(s^*s)^{1/2} = \operatorname{Sup}\{|\Phi(st)|,\ \Phi(t^*t) \leqq 1\}\,.$$

PROOF. Let s and t be in $L^2(\Phi)^+$, then note that by the remark following Theorem 3.2, s and $t \in \mathcal{C}^+$, by [11, Theorem 5.1], we can write $t = [t_n, e_n]$, where $t_n, e_n \in \{t\}''$, $t_ne_n = t_n \geqq 0$ and $t_n \uparrow$. Let u be the Cayley transform of t, Γ is the spectrum of $\{u\}''$([1]) and $\Gamma_n = \{\gamma;\ |u(\gamma)+1| > 1/n\}^-$ where A^- is the closure of a set A. Denote the projection in $\{u\}''$ corresponding to the clopen subset Γ_n by f_n, then $f_n \uparrow LP(t)$ and $\gamma(\in \Gamma_n) \to (1+u(\gamma))^{-1}$ is a continuous function on Γ_n. Thus $e_nf_m \in L^2(\Phi)$ implies $e_nf_m \in \mathcal{F}_p$ for each pair of positive integers m and n. Since $te_nf_m \in \mathcal{F}$, $t^{1/2}e_nf_m \in \mathcal{F}$ and $st \in L^1(\Phi)$, it follows that

$$\begin{aligned}\dot{\Phi}(e_nf_mst) &= \dot{\Phi}(ste_nf_m) = \dot{\Phi}(s(te_nf_m)^{1/2}(te_nf_m)^{1/2}) \\ &= \dot{\Phi}(te_nf_m)^{1/2}s(te_nf_m)^{1/2}) \\ &= \Phi(s^{1/2}t^{1/2}e_nf_mt^{1/2}s^{1/2})\,.\end{aligned}$$

By Lemma 4.1, $\dot{\Phi}(e_nf_mst) \uparrow \Phi(s^{1/2}ts^{1/2})(0)$ in $\mathbf{Z}$. On the other hand, by Lemma 1.1.1, $\dot{\Phi}(e_nf_mst) \to \dot{\Phi}(st)\,(0)$ in Z, therefore $\dot{\Phi}(st) = \Phi(s^{1/2}ts^{1/2}) \geqq 0$, so that the statement (1) follows. To prove (2), we argue as follows. Let s, $t \in \mathrm{L}^2(\Phi)$ such that $|s| \leqq |t|$, then by (1), $|s|^{1/2}(|t| - |s|^{1/2}) \geqq 0$ implies that $\dot{\Phi}(|s|(|t| - |s|)) = \Phi(|s|^{1/2}(|t| - |s|)|s|^{1/2}) \geqq 0$, that is, $\dot{\Phi}(|s|\,|t|) \geqq \Phi(|s|^2)$. By the same way, $\Phi(|t|^2) \geqq \dot{\Phi}(|s|\,|t|)$. Next let $s,t \in L^2(\Phi)_{sa}$ such that $\Phi(s^2) \leqq \Phi(t^2)$, then $0 \leqq \Phi((t-s)^2)$

$= \Phi(t^2)-2\Phi(st)+\Phi(s^2)\leqq 2\,\Phi(t^2)-2\,\Phi(st)$ and this completes the proof of the statement (3). Let $t\in L^2(\Phi)$ and $u\in M_u$, then $|utu^*|^2u^*$, which implies by Lemma 3.2 (3) that the assertion (4) follows. Now we shall show the statement (5). Let s, t be in $L^2(\Phi)$ and $st=w|st|$ be the polar decomposition of st, then it follows, by the argument used in the proof of Theorem 3.1, that

$$\begin{aligned}|\dot{\Phi}(st)|^2 &= |\dot{\Phi}(w|st|)|^2 \leqq (\|w\|\Phi(|st|))^2 \leqq \Phi(|st|)^2\\ &= (\Phi(w^*st))^2 \leqq \Phi((w^*s)^*(w^*s))\Phi(t^*t)\\ &\leqq \Phi(s^*s)\Phi(t^*t)\,.\end{aligned}$$

Now let $a=\mathrm{Sup}\{|\dot{\Phi}(st)|\,;\,\Phi(t^*t)\leqq 1\}$ in Z, then by the above inequality $a\leqq\Phi(s^*s)^{1/2}$. Let $t_n=(\Phi(s^*s)+(1/n)1)^{-1/2}s^*(\in L^2(\Phi))$ for each positive integer n, then $\Phi(t_n^*t_n) = (\Phi(s^*s)+(1/n)1)^{-1}\Phi(s^*s) = (\Phi(s^*s)+(1/n)1)^{-1}\Phi(s^*s)\leqq 1$ and $\dot{\Phi}(st_n) =(\Phi(s^*s)+(1/n)1)^{-1/2}\Phi(s^*s)$, so that

$$(\Phi(s^*s)+(1/n)1)^{-1/2}\Phi(s^*s)^{1/2}\Phi(s^*s)^{1/2}\leqq a$$

for all n, that is, $a=\Phi(s^*s)^{1/2}$ and the statement (5) follows. This completes the proof.

Now for any pair a and b in $L^2(\Phi)$, we define $(a,b)_\Phi=\dot{\Phi}(b^*a)$, then $(\ ,\ )_\Phi$ satisfies the following properties:

(1) $(a,b)_\Phi = (b,a)_\Phi^*$,
(2) $(a,a)_\Phi\geqq 0, (a,a)_\Phi = 0$ only if $a=0$,
(3) $(sa+b,c)_\Phi = s(a,c)_\Phi + (b,c)_\Phi$,

for all a, b, $c\in L^2(\Phi)$ and $s\in Z$. If we define $|||a|||_2=\|(a,a)_\Phi\|^{1/2}$ for $a\in L^2(\Phi)$, then by ([9, §2]), $L^2(\Phi)$ is a normed module over Z with respect to $|||\,,\,|||_2$. Moreover, we have the following:

(1) Let $\{e_i\}$ be an orthogonal family of projections in Z such that $\sum_i e_i=e(\in Z_p)$ and if $a\in L^2(\Phi)$ such that $e_ia=0$ for all i, then $ea=0$.

(2) Let $\{e_i\}$ be an orthogonal family of projections in Z such that $\sum_i e_i=1$, and let $\{a_i\}$ be a bounded subset of $L^2(\Phi)$, then there exists in $L^2(\Phi)$ an element a such that $e_ia=e_ia_i$ for each i.

In fact, by the Baer*-ring property of $\mathcal{M}$ ([12, Theorem 3.1]), we can easily show the statement (1). On the other hand, since ([12, Theorem 4.1]), there exists a unique $a\in\mathcal{M}$ such that $e_ia=e_ia_i$, to prove the assertion (2), it suffices to show that $a\in L^2(\Phi)$. $e_ia^*a=e_ia_i^*a_i$ implies $e_ia^*a\in L^1(\Phi)$ for each i. Denote Sup $|||a_i|||_2$ by k and we have $\Phi(e_ia^*a)=e_i\Phi(a^*a)=e_i\Phi(a_ia_i)\leqq k^2e_i$ for all i, that is, $\Phi(a^*a)\leqq k^2\cdot 1, a\in L^2(\Phi)$ and $|||a|||_2\leqq k$. The statement (2) follows.

The rest of this section is devoted to prove that $L^2(\Phi)$ is complete with respect to the norm $|||\cdot|||_2$, that is, $L^2(\Phi)$ is an AW^*-module over Z. To prove this, we need the following lemma.

LEMMA 4.3. *Let $\{t_n\}$ be an increasing sequence in $L^2(\Phi)^+$ such that $|||t_n - t_m|||_2 \to 0 (m, n \to \infty)$, then there is an element $t \in L^2(\Phi)^+$ such that $|||t_n - t|||_2 \to 0 (n \to \infty)$.*

PROOF. By passing to a subsequence if necessary, we can suppose $|||t_{n+1} - t_n|||_2 < 1/16^n$ for each n. By the spectral theorem ([11]) we can choose sequences of projections $\{e_n\}$ in $\{t_{n+1} - t_n\}''$ and $\{f_n\}$ in $\{t_n\}''$ such that $0 \leqq (t_{n+1} - t_n)e_n \leqq (1/5^n)\cdot 1$, $(t_{n+1} - t_n) \geqq (1/5^n)(1 - e_n)$, $t_n f_n \leqq 2^n \cdot 1$ and $t_n \geqq 2^n(1 - f_n)$ for each n. Now put $p_n = \bigwedge_{k \geqq n} e_k \bigwedge f_k$, by the same arguments as in the proof of Theorem 3.2, $\{p_n\}$ is an SDD and there exists a sequence $\{s(k)\}$ in M_{sa} such that $t_n p_k \to s(k)p_k$ uniformly and $\{s(k), p_k\}$ is an EMO. Denote $[s(k), p_k]$ by t. Let $t_n^2 - t_n t_m = u_n |t_n^2 - t_n t_m|$ (resp. $t_n t_m - t_m^2 = v_n |t_n t_m - t_m^2|$) be the polar decomposition of $t_n^2 - t_n t_m$ (resp. $t_n t_m - t_m^2$), then by Theorem 3.1 (4) and Lemma 4.2, we get that

$$\begin{aligned}\Phi(|t_n^2 - t_m^2|) &\leqq \Phi(|t_n^2 - t_n t_m|) + \Phi(|t_n t_m - t_m^2|)\\ &= \Phi(u_n^* t_n (t_n - t_m)) + \Phi(v_n^* (t_n - t_m) t_m)\\ &\leqq (|||t_n|||_2 + |||t_m|||_2)|||t_n - t_m|||_2 \cdot 1\end{aligned}$$

for each pair of integers m and n. Thus $\{t_n^2\}$ is a $|||\cdot|||_1$-Cauchy sequence in $L^1(\Phi)$. By Theorem 3.2, there exists an $s \in L^1(\Phi)$ such that $|||t_n^2 - s|||_1 \to 0 (n \to \infty)$ and $t_n^2 \to s$ *n. e.*$(n \to \infty)$. Let $r_k = \bigwedge_{n \geqq k} ((t_{n+1} - t_n)^{-1}[p_n]) \bigwedge (t_n^{-1}[p_n])$ and $q_n = p_n \bigwedge r_n$, then by [11, Lemma 3.1], $\{q_n\}$ is an SDD. For any pair k and n with $n \geqq k$,

$$\begin{aligned}(t_{n+1}^2 - t_n^2)q_k &= t_{n+1}(t_{n+1} - t_n)q_k + (t_{n+1} - t_n)t_n q_k\\ &= t_{n+1}p_n(t_{n+1} - t_n)q_k + (t_{n+1} - t_n)p_n t_n q_k,\end{aligned}$$

therefore $(t_{n+1}^2 - t_n^2)q_k \in M$ and $\|(t_{n+1}^2 - t_n^2)q_k\| < 2 \cdot (2/5)^n$, so that by the similar reason to that of Theorem 3.2, there is a sequence of elements $\{s(k)'\}$ in M_{sa} such that $t_m^2 q_k \to s(k)' q_k$ uniformly $(m \to \infty)$ and $\{s(k)', q_k\}$ is an EMO. Let $t' = [s(k)', q_k] \in C$, then $t_n^2 \to t'$ *n. e.* $(n \to \infty)$. Thus $q_k s(k)^2 q_k = q_k s(k)' q_k$ for all k, so that by the Baer*-ring property of M, there is an SDD $\{q_k'\}$ such that $s(k)^2 q_k' = s(k)' q_k'$ for each k, while $t_n^2 \to s$(*n. e.*), by the unicity of n. e. limit, it follows that $t^2 = t' = s \in L^1(\Phi)$, that is, $t \in L^2(\Phi)$. On the other hand $t \geqq t_n$ implies by Lemma 4.2,

$$\begin{aligned}\Phi((t-t_n)^2) &= \Phi(t^2)-2\dot{\Phi}(tt_n)+\Phi(t_n^2)\\ &\leqq \Phi(t^2)-\Phi(t_n^2)\\ &= \dot{\Phi}(s-t_n^2)\leqq |||\, s-t_n^2\,|||_1\cdot 1.\end{aligned}$$

Thus $|||\, t-t_n\,|||_2 \to 0 (n\to\infty)$ and $t_n\to t(\text{n. e.})(n\to\infty)$. This completes the proof.

THEOREM 4.1. *$L^2(\Phi)$ is a faithful AW^*-module over Z([9]) with respect to the norm $|||\,,\,|||_2$.*

PROOF. The proof of that $L^2(\Phi)$ is an AW^*-module is an obvious modification of that for Theorem 3.2, thus it is sufficient to show that $L^2(\Phi)$ is faithful. In fact if $a\in Z$ with $at=0$ for all $t\in L^2(\Phi)$, then the semi-finiteness of Φ and the Baer*-ring property of C show the desired property that $a=0$. This completes the proof.

5. Proof of the main theorem. In the followings, we always denote $L^2(\Phi)$ by $\mathfrak{M}$. By [9, Theorem 7], the set $\mathcal{B}(\mathfrak{M})$ of all bounded module homomorphisms of $\mathfrak{M}$ into $\mathfrak{M}$ is an AW^*-algebra of type 1 with the center Z. The left (resp. right) regular representation π_1(resp. π_2) of M is a *-homomorphism (resp. *-antihomomorphism) of M into $\mathcal{B}(\mathfrak{M})$ which is defined by $\pi_1(x)t=xt$(resp. $\pi_2(x)t=tx$) for any $x\in M$ and $t\in\mathfrak{M}$. Since $\mathcal{F}\subset\mathfrak{M}$, $\pi_1(x)=0$(resp. $\pi_2(x)=0$) implies that there exists an orthogonal family $\{e_\alpha\}$ of projections in $\mathfrak{M}$ such that $xe_\alpha=0$(resp. $e_\alpha x=0$) for each α and $\sum_\alpha e_\alpha=1$. By [7, Lemma 2.2], $x=0$, that is, π_1(resp. π_2) is a *-isomorphism (resp. *-antiisomorphism).

LEMMA 5.1. *$\pi_1(M)$ and $\pi_2(M)$ are AW^*-subalgebras of $\mathcal{B}(\mathfrak{M})$.*

PROOF. We have only to prove the first of these statements, the second follows similarly. By [8, Definition], it suffices to show that for any orthogonal set $\{e_i\}_{i\in I}$ of projections in M with $e=\sum_{i\in I}e_i$, $\pi_1\left(\sum_{i\in J}e_i\right)\uparrow \pi_1(e)$ in $\mathcal{B}(\mathfrak{M})(J\in\mathcal{F}(I))$. In fact, since $\left(\pi_1(e)-\pi_1\left(\sum_{i\in J}e_i\right)x,\, x\right)_\Phi=\Phi\left(x^*\left(e-\sum_{i\in J}e_i\right)x\right)$, therefore from Lemma 4.1 and [14, Lemma 1.4] $\sum_{i\in J}\pi_1(e_i)\uparrow\pi_1(e)$ in $\mathcal{B}(\mathfrak{M})$. This completes the proof.

LEMMA 5.2. *For any $a\in\mathfrak{M}$, there is a sequence $\{a_n\}$ in $M\cap\mathfrak{M}$ such that $|||\,a_n\,|||_2\leqq|||\,a\,|||_2$ and $|a_n-a|_\Phi\to 0(0)$ in Z^+, whre $|x|_\Phi=(x,x)_\Phi^{1/2}$ for any $x\in\mathfrak{M}$.*

PROOF. Let $a=u|a|$ be the polar decomposition of a in C, then for any $b\in\mathcal{F}^+$, $|u(|a|-b)|_\Phi\leqq||a|-b|_\Phi$, so that we have only to prove the assertion for the case when $a \geqq 0$. Let v be the Cayley transform of a, then from the spectral theorem ([11]), there are an SDD $\{e_n\}$ in $\{v\}''$ and a sequence of projections $\{f_n\}$ in $\{v\}''$ such that $n(1-e_n) \leqq a$, ae_n and $(1+v)f_n$ is invertible in f_nMf_n for each n. Since $a_n = ae_nf_n \in \mathcal{F}^+$ and $a^2 \geqq a_n^2 \geqq a_m^2$ if $m<n$, then

$$0 \leqq \Phi(a^2) - \Phi(a_n^2) = \Phi(a^2(1-e_nf_n)) \leqq \Phi(a^2(1-e_nf_m)),$$

so that by Lemma 4.1, $0 \leqq 0-\lim(\Phi(a^2)-\Phi(a_n^2)) \leqq \Phi(a^2(1-f_m))$ for all m, which implies by Lemma 1.1.1, $\Phi(a_n^2) \uparrow \Phi(a^2)(0)$. While from Lemma 4.2, it follows that $\Phi((a-a_n)^2) \leqq \Phi(a^2)-\Phi(a_n^2)$. This shows that $|a-a_n|_\Phi \to 0(0)$ and the proof is completed.

LEMMA 5.3. *$\pi_1(M)'' = \pi_2(M)'$ and $\pi_2(M)'' = \pi_1(M)'$ in $\mathcal{B}(\mathfrak{M})$ where $\mathfrak{A}'$ is the commutant of $\mathfrak{A}$ in $\mathcal{B}(\mathfrak{M})$.*

PROOF. The methods which will be used here are patterned after those of [2, Chapter 1, Section 5]. Since $\pi_1(M)' \supset \pi_2(M)$ and $\pi_2(M)' \supset \pi_1(M)$, we have only to prove the converse inclusion. Let x be a left (resp. right) bounded element in $\mathfrak{M}$, that is, an element x such that there is $B_1(x)$ (resp. $B_2(x)$) in $\mathcal{B}(\mathfrak{M})$ such that $B_1(x)a = \pi_2(a)x$(resp. $B_2(x)a = \pi_1(a)x$) for all $M \cap \mathfrak{M}$. First of all, we shall show that the set $\mathfrak{M}_1 = \{B_1(x)\,;\, x$ is left bounded$\}$ is a left ideal of $\pi_2(M)'$. In fact, for any a and b in $M \cap \mathfrak{M}$, an easy calculation shows that $(B_1(x)\pi_2(a)b, y)_\Phi = (\pi_2(a)B_1(x)b, y)_\Phi$ for any $y \in L^2(\Phi)$. Therefore, by Lemma 1.1.1, Lemma 5.2 and the Schwarz' inequality, $(c, (B_1(x)\pi_2(a))^*y)_\Phi = (c, (\pi_2(a)B_1(x))^*y)_\Phi$ for any $c \in \mathfrak{M}$, that is, $B_1(x)\pi_2(a) = \pi_2(a)B_1(x)$ for any $a \in M \cap \mathfrak{M}$. The semi-finiteness of Φ implies that there is an increasing family of projections $\{e_\alpha\}$ in $M \cap \mathfrak{M}$ such that for any $a \in M$, $ae_\alpha \in M$ and $\pi_2(ae_\alpha) \to \pi_2(a)$ weakly ([14, p. 311]). Thus $B_1(x)\pi_2(a) = \pi_2(a)B_1(x)$ for all $a \in M$, that is, $\mathfrak{M}_1 \subset \pi_2(M)'$. Since for any $T \in \pi_2(M)'$, $TB_1(x)a = T \cdot \pi_2(a)x = \pi_2(a)Tx$ for all $a \in M \cap \mathfrak{M}$, Tx is left bounded and $B_1(Tx) = TB_1(x)$. Hence the assertion follows. From the same reason, $\mathfrak{M}_2 = \{B_2(x)\,;\, x$ is right bounded$\}$ is a left ideal of $\pi_1(M)'$. Let $\mathfrak{M}_3 = \mathfrak{M}_1 \cap \mathfrak{M}_1^*$ and $\mathfrak{M}_4 = \mathfrak{M}_2 \supset \mathfrak{M}_2^*$, where $\mathfrak{A}^* = \{x^*, x \in \mathfrak{A}\}$ for any subset $\mathfrak{A}$ of $\mathcal{B}(\mathfrak{M})$, then $\mathfrak{M}_3'' \subset \pi_2(M)'$ and $\mathfrak{M}_4'' \subset \pi_1(M)'$. Next we shall show that $\mathfrak{M}_3'' = \pi_2(M)'$. In fact, for any $T \in \pi_2(M)'$ and $T_1 \in \mathfrak{M}_3'$, $T_1\pi_1(b)T\pi_1(a) = \pi_1(b)T \cdot \pi_1(a)T_1$ for any a and b in $M \cap \mathfrak{M}$, so that from the above argument, we have $T_1T = TT_1$, that is, $\pi_2(M)' = \mathfrak{M}_3''$. By the same way, $\pi_1(M)' = \mathfrak{M}''^4$. To prove Lemma 5.2. it suffices to show $\mathfrak{M}_3 \subset \mathfrak{M}_4'$. In fact, let $B_1(a) \in \mathfrak{M}_3$ and $B_2(b) \in \mathfrak{M}_4$, then $B_1(a)^* = B_1(c)$(resp. $B_2(b)^* = B_2(d)$) for some left (resp. right) bounded element c (resp. d). Therefore, by a standard calculation shows that for any x and y in $M \cap \mathfrak{M}$, $(a, xy)_\Phi = (c^*, xy)_\Phi$. By lemma 5.2, it follows that $a = c^*$. By the same way $b = d^*$. Again by Lemma 5.2, there exist sequences $\{x_n\}$ and $\{y_n\}$ in $M \cap \mathfrak{M}$ such that $|x_n - a|_\Phi = |x_n^* - c|_\Phi \to 0(0)$, $|y_n - b|_\Phi = |y_n^* - d|_\Phi \to 0(0)$, $|||x_n|||_2 \leqq |||a|||_2$ and $|||y_n|||_2 \leqq |||b|||_2$ for each n. Therefore, by Lemma 1.1.1, from the similar arguments ([2, p. 68, Lemma 3]) it follows that $(B_1(a)B_2(b)x, y)_\Phi = (B_2(b)B_1(a)x, y)_\Phi$ for any x and y in $M \cap \mathfrak{M}$. From

Lemma 5.2, we have $B_1(a)B_2(b)=B_2(b)B_1(a)$, which Implies $\mathfrak{M}_3''\subset\mathfrak{M}_4'$. This completes the proof.

For any $a\in\mathfrak{M}$, let $\vee\{\pi_1(M)'a\}$ be the AW^*-submodule generated by $\{\pi_1(M)'a\}$ and E_a be the projection on $\vee\{\pi_1(M)'a\}$ ([9, Theorem 3]), then $E_a\in\pi_1(M)''$. In fact, for any $A\in\pi_1(M)'$, $A\{\pi_1(M)'a\}\subset\vee\{\pi_1(M)'a\}$. Let $\{e_\alpha\}$ be an orthogonal family of projections in Z with $\Sigma_\alpha e_\alpha=1$ and let $\{y_\alpha\}$ be a uniformly bounded subset of $\{\pi_1(M)'a\}$, then [9, p. 842, Definition], $A(\Sigma_\alpha e_\alpha y_\alpha)=\Sigma_\alpha e_\alpha Ay_\alpha$ in $\mathfrak{M}$, so that $A(\Sigma_\alpha e_\alpha y_\alpha)\in\vee\{\pi_1(M)'a\}$. The continuity of A implies $A(\vee\{\pi_1(M)'a\})\subset\vee\{\pi_1(M)'a\}$, that is, $AE_a=E_aAE_a$ for all $A\in\pi_1(M)'$, so that $E_a\in\pi_1(M)''$. E_a is called a cyclic projection relative to a.

Now we are in the position to state

THEOREM 5.1. *$\pi_1(M)''=\pi_1(M)$, that is, M can be imbedded as a double commutator in a type* 1 *AW^*-algebra $\mathcal{B}(\mathfrak{M})$ with the center which is $*$-isomorphic with Z.*

PROOF. By the spectral theorem, it suffices to show that $\pi_1(M)_p''=\pi_1(M_p)$. For any $P\in\pi_1(M)_p''$, let $\{E_x\}$ be a maximal family of orthogonal cyclic projections in $\pi_1(M)''$ majorized by P. By the definition of E_x, the standard argument shows that $P=\Sigma_x E_x$ in $\mathcal{B}(\mathfrak{M})$. Since $\pi_1(M)$ is an AW^*-subalgebra of $\mathcal{B}(\mathfrak{M})$, by [14, Lemma 4.5], in order to prove $P\in\pi_1(M)_p$, we have only to show that $E_x\in\pi_1(M)$ for all $x\in\mathfrak{M}$.

Let $x=u|x|$ be the polar decomposition of x in C, then $E_x=\pi_1(u)E_{|x|}\pi_1(u)^*$. In fact, observe that $x=\pi_1(u)|x|$ and $|x|=\pi_1(u)^*x$, $Ax=\pi_1(u)A|x|$ and $\pi_1(u)^*Ax=A|x|$ for any $A\in\pi_1(M)'$, so that $\vee\{\pi_1(M)'x\}\supset\pi_1(u)(\vee\{\pi_1(M)'|x|\})$. For any $y\in\vee\{\pi_1(M)'x\}$ and for any positive real number ε, we can choose an orthogonal set $\{e_\alpha\}$ of projections in Z and a family $\{B_\alpha\}$ in $\pi_1(M)'$ such that $\Sigma_\alpha e_\alpha=1$, $\operatorname{Sup}_\alpha |||B_\alpha x|||_2<\infty$ and $|||y-\Sigma_\alpha e_\alpha B_\alpha x|||_2<\varepsilon$. Since $e_\alpha\pi_1(u)\pi_1(u)^*B_\alpha x=e_\alpha B_\alpha x$ for each α, we have $|||y-\pi_1(u)\pi_1(u)^*y|||_2<2\varepsilon$, that is, $y=\pi_1(u)\pi_1(u)^*y$. On the other hand, $\pi_1(u)^*B_\alpha x=B_\alpha|x|$ and $|||B_\alpha|x|\,|||_2\leqq|||B_\alpha x|||_2$ for each α implies that $|||\pi_1(u)^*y-\Sigma_\alpha e_\alpha B_\alpha|x|\,|||_2<\varepsilon$ and $\pi_1(u)^*y\in\vee\{\pi_1(M)'|x|\}$. Therefore combining the above results, $y\in\pi_1(u)(\vee\{\pi_1(M)'|x|\})$, that is, $\vee\{\pi_1(M)'x\}=\pi_1(u)(\vee\{\pi_1(M)'|x|\})$. By the same way, it follows that $\pi_1(Rp(x))(\vee\{\pi_1(M)'|x|)=\vee\{\pi_1(M)'|x|\}$. From these facts, we get that $E_x=\pi_1(u)E_{|x|}\pi_1(u)^*$. Hence to prove that $E_x\in\pi_1(M)$, we may assume $x\geqq0$ without loss of generality.

Let $x\in\mathfrak{M}$ with $x\geqq0$, then there exist a projection e_n and f_n in $\{x\}''$ satisfying the properties described in the proof of Lemma 5.2. Let $a_n=xe_nf_n(\in\mathcal{F})$, then $a_n\uparrow$, $a_n\leqq x$ and $|a_n-x|_\Phi\to0(0)$. Since $a_n=\pi_1(e_nf_n)x=\pi_2(e_nf_n)x$, $E_{a_n}\leqq E_x$ and $E_{a_n}\uparrow$. Moreover $|a_n-x|_\Phi\to0(0)$ implies $E_{a_n}\uparrow E_x$ in $\mathcal{B}(\mathfrak{M})$. Thus by [14, Lemma 4.5], to prove $E_x\in\pi_1(M)$, we have only to show that $E_{a_n}\in\pi_1(M)$ for each n.

Now we shall prove that $E_a \in \pi_1(M)$ for all $a \in \mathcal{F}$. Since $\pi_1(M)$ is an AW^*-subalgebra of $\mathcal{B}(\mathfrak{M})$, it is sufficient to show that $E_a = LP(\mathcal{B}(\mathfrak{M}))\pi_1(a))$ ([8, Lemma 2]). Observe that for any $b \in M \cap \mathfrak{M}$, $\pi_2(b)a = ab = \pi_1(a)b \in \vee\{\pi_1(a)\mathfrak{M}\}$, let E be the projection in $\mathcal{B}(\mathfrak{M})$ corresponding to $\vee\{\pi_1(a)\mathfrak{M}\}$, then $E\pi_2(b)a = \pi_2(b)a$ for all $b \in M \cap \mathfrak{M}$. The semi-finiteness of Φ implies that for any $A \in \pi_2(M)$, there is a net $\{a_\alpha\}$ in $M \cap \mathfrak{M}$ such that $\|\pi_2(a_\alpha)\| \leqq \|A\|$ for each α and $\pi_2(a_\alpha) \to A$ strongly in $\mathcal{B}(\mathfrak{M})$. Therefore $E\pi_2(b)a = \pi_2(b)a$ for all $b \in M$. For any $A \in \pi_2(M)''$ $(=\pi_1(M)')$, since $\pi_2(M)$ is an AW^*-subalgebra of $\mathcal{B}(\mathfrak{M})$, by [14, Lemma 4.2], there is a bounded net $\{A_\lambda\} \subset \pi_2(M)$ such that $A_\lambda \to A$ strongly in $\mathcal{B}(\mathfrak{M})$, thus $EAa = Aa$, which implies $\vee\{\pi_1(M)'a\} \subset \vee\{\pi_1(a)\mathfrak{M}\}$, that is $E_a \leqq E$. For any $x \in \mathfrak{M}$, by Lemma 5.2, there is a sequence $\{b_n\}$ in $M \cap \mathfrak{M}$ such that $|x - b_n|_\Phi \to 0(0)$ and $|||b_n|||_2 \leqq |||x|||_2$ for each n, so that $E_a\pi_1(a)b_n = \pi_1(a)b_n$ implies $E_a\pi_1(a)x = \pi_1(a)x$, that is, $E = E_a$. An easy calculation shows that $E = LP(\mathcal{B}(\mathfrak{M}))(\pi_1(a))$ and the proof is now completed.

COROLLARY. *Let $\mathcal{B}$ be an AW^*-algebra of type* 1 *with center $\mathcal{Z}$ and let $\mathcal{A}$ be a semi-finite AW^*-subalgebra of $\mathcal{B}$ which contains $\mathcal{Z}$, then $\mathcal{A} = \mathcal{A}''$ in $\mathcal{B}$.*

By Theorem 5.1, the proof proceeds in entire analogy to that of [14, Theorem 4.4], so we omit the details.

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