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EXT-PRODUCTS AND EDGE-MORPHISMS

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In a previous paper, we constructed products for certain Ext-functors, in particular for $\mathrm{Ext}^*(A_X, -) = H^*(X, -)$ where A_X denotes the structure sheaf of an object X in a ringed site. If U is a covering of X, it must be asked whether our pairings have anything to do with the usual cup-product in the ČECH cohomology $H^*(U, -)$. The answer is that the two are related by the edge-morphism $H^*(U, -) \to H^*(X, -)$ of the CARTAN-LERAY spectral sequence.

To see this, we proceed in two stages. In section 2, we show that our products (whose construction is sketched at the beginning of that section) are compatible with edge-morphisms arising from an adjoint situation. The main work is in section 1, which gives a very explicit description of such edge-morphisms.

In section 3, we look at $H^q(U, -)$ as an Ext-functor on the category of presheaves and show that the pairing defined by our constructions coincides with the simplicial cup-product.

We shall adhere to the notational conventions of [2]. For the sake of completeness, let us list the most important ones again. Consider an abelian category $\mathcal{A}$. By $K(\mathcal{A})$ we denote the category of (co-chain) complexes in $\mathcal{A}$ and homotopy classes of maps, specifying the full subcategory of complexes bounded below (resp. above) by a superscript $+$ (resp. $-$). $D(\mathcal{A})$ stands for the *derived category* of A: it is obtained from $K(\mathcal{A})$ by formally inverting those $K(\mathcal{A})$-arrows which induce isomorphisms in the cohomology. As in [2], the latter will be called *quisos* and represented by double arrows. Thus, morphisms in $D(\mathcal{A})$ are given as left or right fractions with quisos as "denominators." If u: $\mathcal{A} \to \mathcal{B}$ is an additive functor to another abelian category, a *left u-acyclic class* $\mathcal{P} \subset \mathcal{A}$ is a full subcategory such that u respects quisos between objects of $K^-(\mathcal{P})$ and such that every object M^* of $K^-(\mathcal{A})$ admits a quiso $P^* \Longrightarrow M^*$ with P^* in $K^-(\mathcal{P})$. In the usual way, $\mathcal{P}$ permits the construction of a *left derived functor* $\mathcal{L}u: D^-(\mathcal{A}) \to D(\mathcal{B})$. The obvious dual notion of *right u-acyclic* class is analogously related to the existence of a *right derived functor* $\mathcal{R}u: D^+(\mathcal{A}) \to D(\mathcal{B})$.

1. Some edge morphisms. Throughout this section, we fix our attention on a pair u, v of additive functors between abelian categories

$$\mathcal{A} \underset{u}{\overset{v}{\rightleftarrows}} \mathcal{B}$$

supposing u to be left adjoint to v and assuming the existence of a left u-acyclic class $\mathcal{P} \subset \mathcal{A}$ and of a right v-acyclic class $\mathcal{J} \subset \mathcal{B}$.

We begin by extending the given adjointness

$$\mathrm{Hom}_{\mathcal{A}}(X, vY) \simeq \mathrm{Hom}_{\mathcal{B}}(uX, Y)$$

to the derived categories. Given a morphism

$$(1) \qquad P^* \Longleftarrow Q^* \to vN^*$$

in $D(\mathcal{A})$ with P^*, $Q^* \in D^-(\mathcal{P})$ and $N^* \in D^+(\mathcal{B})$, we apply u and the canonical adjunction map $uv \to Id_{\mathcal{B}}$ to obtain the morphism

$$(2) \qquad uP^* \Longleftarrow uQ^* \to uvN^* \to N^* .$$

This procedure gives us a natural map

$$\alpha_{P^*, N^*} : \mathrm{Hom}_{D(\mathcal{A})}(P^*, vN^*) \to \mathrm{Hom}_{D(\mathcal{B})}(uP^*, N^*)$$

for left u-acyclic P^* and arbitrary N^*(bounded to the right and to the left respectively). Dually, for $M^* \in D^-(\mathcal{A})$ and $J^* \in D^+(\mathcal{J})$ we have

$$\beta_{M^*, J^*} : \mathrm{Hom}_{D(\mathcal{B})}(uM^*, J^*) \to \mathrm{Hom}_{D(\mathcal{A})}(\mathrm{M}^*, vJ^*) .$$

PROPOSITION 1.1. *If $P^* \in D^-(\mathcal{P})$ and $J^* \in D^+(\mathcal{J})$, the respective maps α and β are inverses of each other. Hence α induces natural isomorphisms*

$$\gamma_{M^*, N^*} : \mathrm{Hom}_{D(\mathcal{B})}(\mathcal{L}u\mathrm{M}^*, N^*) \simeq \mathrm{Hom}_{D(\mathcal{A})}(M^*, \mathfrak{R}\, vN^*) .$$

PROOF. Consider the morphism (1) with $N^* = J^*$, apply α to obtain (2), and represent the lattter as a left fraction $uP^* \to K^* \Longleftarrow J^*$ with $K^* \in D^+(\mathcal{B})$. This yields the commutative square

$$\begin{array}{ccc} uP^* & \longrightarrow & K^* \\ \Uparrow & & \Uparrow \\ uQ^* \to uvJ^* & \to & J^* \end{array}$$

Applying v and the adjunction map $Id_{\mathcal{A}} \to vu$, we get

$$
(3) \qquad
\begin{tikzcd}
P^* \arrow[r] & vuP^* \arrow[rr] & & vK^* \\
Q^* \arrow[u, Rightarrow] \arrow[r] & vuQ^* \arrow[u] \arrow[r] & vuvJ^* \arrow[r] & vJ^* \arrow[u, Rightarrow]
\end{tikzcd}
$$

a morphism in $D(\mathcal{A})$ from P^* to vJ^* which represents the application of $\beta \cdot \alpha$ to (1). Since the bottom line in (3) is the same as the original arrow $Q^* \to vJ^*$, we indeed have $\beta \cdot \alpha =$ identity. The other composite $\alpha \cdot \beta$ is treated analogously. The second half of the proposition follows immediately from the definition of $\mathcal{R}v$ and $\mathcal{L}u$ in terms of $\mathcal{J}$ and $\mathcal{P}$(cf. [3] or [2]).

The aim of this section is to identify α with a certain edge-morphism. For this purpose we recall (cf. [3]) that

$$\mathrm{Ext}^p_{\mathcal{A}}(M^*, L^*) \cong \mathrm{Hom}_{D\mathcal{A}}(M^*, L^*[p])$$

where $M^* \in D^-(\mathcal{A})$, $L^* \in D^+(\mathcal{A})$ and $[p]$ denotes a p-fold shift: $(L^*[p])^n = L^{n+p}$. We now restrict ourselves to objects P of $\mathcal{P}$ and N of $\mathcal{B}$ and consider the commutative diagram

$$
(4) \qquad
\begin{tikzcd}
\mathrm{Ext}^p_{\mathcal{B}}(uP, N) \arrow[r, "\gamma^p", "\sim"'] & \mathrm{Ext}^p_{\mathcal{A}}(P, RvN) \\
& \mathrm{Ext}^p_{\mathcal{A}}(P, vN) \arrow[ul, "\alpha^p"] \arrow[u]
\end{tikzcd}
$$

in which the vertical arrow comes from the canonical morphism $v \to \mathcal{R}v$, and α^p, γ^p correspond to the previously constructed α, γ in an obvious way ($\mathcal{L}uP \to uP$ being a quiso).

PROPOSITION 1.2. *If $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, there are natural convergent spectral sequences*

$$(5) \qquad \mathrm{Ext}^p_{\mathcal{A}}(P, R^q vN) \Longrightarrow \mathrm{Ext}^{q+p}_{\mathcal{B}}(uP, N)$$

provided that P is left u-acyclic. The morphisms α^p of (4) coincide with the corresponding edge-morphisms of (5).

PROOF. By Proposition 1.1, we have for injectives I of $\mathcal{B}$:

$$\begin{aligned} 0 = \operatorname{Ext}^p_{\mathcal{B}}(uP, I) &= \operatorname{Hom}_{D\mathcal{B}}(uP, I[p]) \\ &\cong \operatorname{Hom}_{D\mathcal{A}}(P, vI[p]) = \operatorname{Ext}^p_{\mathcal{A}}(P, vI)\,. \end{aligned}$$

Hence vI is acyclic for $\operatorname{Hom}_{\mathcal{A}}(P, \text{-})$. Considering $\operatorname{Hom}_{\mathcal{B}}(uP, \text{-})$ to be the composite $\operatorname{Hom}_{\mathcal{A}}(P, \text{-}) \circ v$, we obtain (5) as usual via a CARTAN-EILENBERG resolution.

In terms of derived functors, we have

$$(6)\qquad \begin{tikzcd} \mathfrak{R}\operatorname{Hom}_{\mathcal{B}}(uP, N) \arrow[r, "\delta", "\sim"'] & \mathfrak{R}\operatorname{Hom}_{(\mathcal{A})}(P, \mathfrak{R}vN) \\ & \mathfrak{R}\operatorname{Hom}_{(\mathcal{A})}(P, vN)\,, \arrow[u] \arrow[ul, "\varepsilon"] \end{tikzcd}$$

where the horizontal arrow δ represents the canonical morphism

$$\mathfrak{R}(\operatorname{Hom}_{\mathcal{A}}(P, \text{-}) \cdot v) \to \mathfrak{R}\operatorname{Hom}_{\mathcal{A}}(P, \text{-}) \cdot \mathfrak{R}v\,.$$

In this situation it is well-known that the maps induced in cohomology by the morphism ε of (6) are edge-morphisms of (5). To see that these coincide with the α^p of (4), it is necessary to identify (4) as the diagram induced by (6) in cohomology; i.e. to show that δ induces the γ^p. This is easily checked by viewing δ as

$$\delta : \operatorname{Hom}^*_{\mathcal{B}}(uP, I^*) \xrightarrow{\sim} \operatorname{Hom}^*_{\mathcal{B}}(P, vI^*),$$

where $N \Longrightarrow I^*$ is an injective resolution (remember that vI^* is acyclic for $\operatorname{Hom}_{\mathcal{A}}(P, \text{-})$).

2. Compatibility with Ext-products. We keep the basic situation

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift right, "u"'] & \mathcal{B} \arrow[l, shift right, "v"'] \end{tikzcd}$$

of section 1 and add the following hypotheses:

P1: Both $\mathcal{A}$ and $\mathcal{B}$ have tensor-products in the sense of [2] (right-exact, bi-additive bifunctors with the usual symmetries).

P 2: Both $\mathcal{A}$ and $\mathcal{B}$ have enough flats as well as enough injectives.

Under these conditions, we can tensor morphisms in $D(\mathcal{A})$ and $D(\mathcal{B})$. Given for instance

$$P_1^* \Longleftarrow Q_1^* \rightarrow L_1^* \text{ and } P_2^* \Longleftarrow Q_2^* \rightarrow L_2^*$$

with P_i^*, Q_i^* flat, we can make

$$P_1^* \otimes P_2^* \Longleftarrow Q_1^* \otimes Q_2^* \rightarrow L_1^* \otimes L_2^* .$$

In particular, we get products for $\mathrm{Ext}^*_{(\mathcal{A})}$ by its interpretation as morphisms in $D(\mathcal{A})$.

To apply the results of section 1, we make the additional assumptions:

C 1: There are natural isomorphisms

$$u(M_1 \otimes M_2) \simeq uM_1 \otimes uM_2$$

C 2: The flats of $\mathcal{A}$ are left u-acyclic.

C 3: u takes flats of $\mathcal{A}$ into flats of $\mathcal{B}$.

By C1 and adjointness, one obtains natural pairings

$$vN_1 \otimes vN_2 \rightarrow v(N_1 \otimes N_2)$$

for arbitrary N_1, N_2 of $\mathcal{B}$. Hence for flat objects P_1, P_2 of $\mathcal{A}$ one has products

$$\mathrm{Ext}^p_{\mathcal{A}}(P_1, vN_1) \times \mathrm{Ext}^q_{\mathcal{A}}(P_2, vN_2) \rightarrow \mathrm{Ext}^{p+q}_{\mathcal{A}}(P_1 \otimes P_2, v(N_1 \otimes N_2)) \tag{7}$$

and also(on account of C1 and C 3)

$$\mathrm{Ext}^p_{\mathcal{B}}(uP_1, N_1) \times \mathrm{Ext}^q_{\mathcal{B}}(uP_2, N_2) \rightarrow \mathrm{Ext}^{p+q}_{\mathcal{B}}(u(P_1 \otimes P_2), N_1 \otimes N_2) . \tag{8}$$

PROPOSITION 2.1. *The edge-morphisms $\alpha^p, \alpha^q, \alpha^{p+q}$ of Proposition* 1.2 *map the product* (7) *onto the product* (8).

The proof is a routine verification using the product-construction given at the beginning of this section and the explicit from of α given prior to Proposition 1.1.

3. Application to sheaves. The situation of sections 1 and 2 arises in a natural way from a morphism

$$(u, \varphi): (\mathcal{X}, A) \rightarrow (, \mathcal{Y} B)$$

of ringed sites by way of the associated inverse and direct image functors

$$u^*: \mathcal{A} \to \mathcal{B},\ u_*: \mathcal{B} \to \mathcal{A}$$

between the categories $\mathcal{A}$ and $\mathcal{B}$ of sheaves of modules. It is well-known [5], that u^* is left adjoint to u_* and that conditions P 1, P 2, C 1 of section 2 are satisfied. Elsewhere [1] it will be shown that C 2 and C 3 also hold.

Applying our results to this set-up, we obtain natural convergent spectral sequences

$$\mathrm{Ext}^p_{\mathcal{A}}(P, R^q u_* F) \Longrightarrow \mathrm{Ext}^{p+q}_{\mathcal{B}}(u^* P, F) \tag{9}$$

for flat A-modules P and arbitrary B-modules F. This generalizes proposition 5.5 of exposé $\overline{\mathrm{V}}$ in [5]. Moreover, the edge-morphisms

$$\mathrm{Ext}^p_{\mathcal{A}}(P, u_* F) \to \mathrm{Ext}^p_{\mathcal{B}}(u^* P, F)$$

respect the Ext-products defined in [2]. In particular, since $u^* B = A$, the products induced on the sheaf-cohomology are compatible via the maps

$$H^p(X, u_* F) \to H^p(Y, F).$$

For the remainder of this section, we shall be concerned with the transition from presheaves to sheaves on a fixed ringed site $(\mathcal{X}, A)$, where $\mathcal{X}$ has pull-backs. Denoting presheaves by $\mathcal{A}$ and sheaves by $\mathcal{B}$, the canonical pair

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift right, "a"'] & \mathcal{B} \arrow[l, shift right, "i"'] \end{tikzcd}$$

"inclusion" and "association" can be viewed as direct image and inverse image (respectively) of the morphism of sites

$$(1_{\mathcal{X}}, \Phi): (\mathcal{X}_C, iA) \to (\mathcal{X}, A),$$

where $\mathcal{X}_C$ denotes the category $\mathcal{X}$ equipped with the "chaotic" topology and $\Phi: A \to aiA$ an adjunction morphism.

The forgetful functor from $\mathcal{A}$ to presheaves of sets has a left adjoint $f: H \to A_H$ whose values are flat iA-modules. We are interested in $\mathrm{Ext}^p_{\mathcal{A}}(A_R, F)$ where R denotes a covering crible for an object $X \in \mathcal{X}$. Remembering that, for representable H, $f(H) = A_H$ is a *projective* iA-module, we can compute this Ext-group via the projective resolution $R_*(U) \Longrightarrow A_R$ obtained from a generating family $U = \{U_\iota \to X\}$ for R by applying f to the complex

$$\ldots \coprod U_i \underset{X}{\times} U_j \underset{X}{\times} U_k \rightrightarrows \coprod U_i \underset{X}{\times} U_j \rightrightarrows \coprod U_i \to R\,.$$

Thus, $\mathrm{Ext}^p_{\mathcal{A}}(A_R, F) = H^p(\mathrm{Hom}^*_{\mathcal{A}}(R_*(U), F)$ is the "Cech-cohomology" of the covering U which as usual is denoted by $H^p(U, F)$. Therefore (9) turns into the spectral sequence

$$H^p(U, \mathcal{H}^q(F)) \Longrightarrow H^{p+q}(X, F) \tag{10}$$

of CARTAN-LERAY.

To interpret our results concerning products, we use the canonical isomorphisms

$$A_H \otimes A_K \xrightarrow{\sim} A_{H\times K}$$

(cf. 2.13 in exposé IV of [5]) and the diagonal

$$A_H \to A_{H\times H}$$

to obtain pairings

$$H^p(U, iF) \otimes H^q(U, iG) \to H^{p+q}(U, i(F\otimes G)) \tag{11}$$

$$H^p(X, F) \otimes H^q(X, G) \to H^{p+q}(X, F\otimes G) \tag{12}$$

from the Ext-products of [2]. In these terms we have:

PROPOSITION 3.1.

a) *The edge-morphism* $H^p(U, iF) \to H^p(X, F)$ *of* (10) *maps the product* (11) *into the product* (12).

b) *The product* (11) *coincides with the usual cup-product.*

PROOF. Since only (b) remains to be proved, we shall stay in $\mathcal{A}$ from now on. Whenever we can work with projective resolutions, our Ext-product clearly coincides with the "external product" of [4], Chp. VIII, § 4. To see this, one has only to recall the connection between Ext and morphisms in the derived category as exhibited by means of a projective resolution. In the present case, the external product turns pairs of cocycles of $\mathrm{Hom}^*(R_*(U), -)$ into cocycles of $\mathrm{Hom}^*(R_*(U) \otimes R_*(U), -)$. From this same process, the usual cup-product is obtained by composition with the morphism

$$R_*(U) \xrightarrow{\Delta} R_*(U\times U) \xrightarrow{\omega} R_*(U) \otimes R_*(U),$$

where δ is the diagonal and ω the map produced by the ALEXANDER-WHITNEY formula (cf. [4], Chp. VIII, formula (8. 7)). The commutativity of

$$\begin{array}{ccc} R_*(U)\otimes R_*(U) & \longrightarrow & A_{R\times R} \\ {\scriptstyle \omega\cdot\delta}\uparrow & & \uparrow{\scriptstyle \delta} \\ R_*(U) & \longrightarrow & A_R \end{array}$$

now completes the proof.

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