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ON SUMMATION PROCESSES OF FOURIER EXPANSIONS IN BANACH SPACES II. SATURATION THEOREMS

Dedicated to Professor G. Sunouchi on the occasion of his 60th birthday on October 29, 1971

P. L. BUTZER, R. J. NESSEL AND W. TREBELS[1)]

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This paper is a sequel to the preceding one with the same title published in this Journal (Note I, Vol. 24, pp. 127). The contents of the first note are assumed to be known. References are in alphabetical order in each paper; they as well as the sections are numbered consecutively throughout this series.

5. The concept of saturation. In this paper the program of embedding results on particular classical orthogonal series in the general framework of orthogonal projections in an arbitrary Banach space is continued by studying saturation problems. As in Note I this is achieved by a discussion of certain multiplier conditions.

The concept of saturation, first introduced by Favard for summation methods of trigonometric series in a lecture in 1947 (cf. [28]), may be formulated as follows (see e.g. [5;, p. 434]):

Let X, $[X]$, and $\{T(\rho)\}$ be defined as in Sec. 1 ($\|\cdot\|$ without any index denoting X-norm). The strong appoximation process $\{T(\rho)\}_{\rho>0}\subset[X]$ is said to possess the saturation property if there exists a positive function $\varphi(\rho)$, $\rho>0$, tending monotonely to zero as $\rho\to\infty$ such that every $f\in X$ for which

$$\|T(\rho)f-f\|=o(\varphi(\rho)) \qquad (\rho\to\infty)$$

is an invariant element of $\{T(\rho)\}$, i.e., $T(\rho)f=f$ for all $\rho>0$, and if the set

$$F[X;\,T(\rho)]=\{f\in X;\ \|T(\rho)f-f\|=O(\varphi(\rho))\ ,\ \rho\to\infty\}$$

contains at least one noninvariant element. In this event, the approximation process $\{T(\rho)\}$ is said to have optimal approximation order $O(\varphi(\rho))$

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or to be saturated in X with order $O(\varphi(\rho))$, and $F[X: T(\rho)]$ is called its Favard or saturation class.

Today there exists a vast literature concerned with saturation for various types of approximation processes. To mention general approaches in regard to solution, there exists an integral transform method in diverse Lebesgue spaces as well as the semi-group method on arbitrary Banach spaces in its extended form (for detailed bibliographical comments one may consult [5], [22], [26]).

In this note we are interested in studying saturation within the frame of Note I. This was again originally envisaged by Favard [29] to whom Theorem 6.1 is essentially due. However, the main results of this paper were mostly inspired by the important work of Sunouchi [20, 31]. In connection with various methods of summation of trigonometric series, it was Sunouchi who stressed the importance of uniform multiplier conditions, in particular using bv_2-spaces. Here we follow up these lines and treat summation processes of Fourier expansions in arbitrary Banach spaces with (C, j)-decompositions, using bv_k-spaces.

For this purpose, the basic conditions (6.1) and (6.4), the standard ones in the concrete case of the trigonometric system, are formulated in Sec. 6: they guarantee the saturation property and allow characterizations of Favard classes in terms of relative completions as well as of conditions which reduce to classical representation theorems in case of the trigonometric system. In Sec. 7 multiplier classes are considered under the assumption that the (C, j) – means (7.1) are uniformly bounded (cf.(7.2)). These results generalize those of Sec. 3 (case $j = 0, 1$). Finally, the general theory presented will allow one to derive in Sec. 8 saturation theorems for summation processes of various kinds of orthogonal expansions such as those into spherical harmonics, ultraspherical polynomials and Hermite functions.

6. A saturation theorem. As in Note I we tacitly suppose that $\{P_k\}_{k\in \boldsymbol{P}} \subset [X]$ is a total sequence of mutually orthogonal (continuous) projections on X. However, in this note we additionally assume that the linear span of $\bigcup_{k=0}^{\infty} P_k(X)$ is dense in X, i.e. $\{P_k\}$ is supposed to be fundamental.

For an approximation process $\{T(\rho)\}$ we set

$$\boldsymbol{T} = \{k \in \boldsymbol{P};\ \tau_k(\rho) = 1 \qquad \text{for all } \rho > 0\}\,,$$

and always assume that $\boldsymbol{T} \neq \boldsymbol{P}$. Then the following condition ensures that $\{T(\rho)\}$ will have the saturation property:

(6.1) Given a (uniformly bounded) strong approximation process $\{T(\rho)\}$ of multiplier operators with associated multiplier sequences $\{\tau(\rho)\}$, let there exist a sequence $\psi \in s$ with $\psi_k \neq 0$ whenever $k \notin \boldsymbol{T}$ and a positive function $\varphi(\rho)$ on $(0, \infty)$ tending monotonely to zero as $\rho \to \infty$ such that

$$\lim_{\rho\to\infty} \varphi^{-1}(\rho)[\tau_k(\rho) - 1] = \psi_k \qquad (k \in \boldsymbol{P}) .$$

Condition (6.1) is a standard one in the study of saturation for summation processes of trigonometric series (cf. [5; p. 435]). In fact, it was already introduced by Favard [29] in connection with fundamental, total biorthogonal systems (cf. Remark in Sec. 2) in arbitrary Banach spaces. As a consequence, the following result is substantially contained in [29].

THEOREM 6.1. *Let $f \in X$ and $\{T(\rho)\}$ satisfy* (6.1).

a) *If there exists $g \in X$ such that*

$$\lim_{\rho\to\infty} \| \varphi^{-1}(\rho)[T(\rho)f - f] - g \| = 0 ,$$

the Fourier expansion of g is given by $g \sim \sum_{k=0}^{\infty} \psi_k P_k f$.

b) $\| T(\rho)f - f \| = o(\varphi(\rho))$ *implies $f \in \bigcup_{m\in \boldsymbol{T}} P_m(X)$ and $T(\rho)f = f$ for all $\rho > 0$, thus f is an invariant element.*

c) *There exists some noninvariant $h \in X$ with $\| T(\rho)h - h\| = O(\varphi(\rho))$.*

PROOF. a) Since $P_k \in [X]$ and

$$P_k(\varphi^{-1}(\rho)[T(\rho)f - f]) = \varphi^{-1}(\rho)[\tau_k(\rho) - 1]P_k f ,$$

one has for each $k \in \boldsymbol{P}$

$$\begin{aligned} \| \psi_k P_k f - P_k g \| &= \lim_{\rho\to\infty} \| \varphi^{-1}(\rho)[\tau_k(\rho) - 1]P_k f - P_k g \| \\ &\leqslant \lim_{\rho\to\infty} \| P_k \|_{[X]} \| \varphi^{-1}(\rho)[T(\rho)f - f] - g \| = 0 , \end{aligned}$$

which proves the assertion.

b) Choosing $g = 0$ in part a) one has $\psi_k P_k f = 0$ for all $k \in \boldsymbol{P}$. In case $k \notin \boldsymbol{T}$ it follows that $P_k f = 0$, whereas for $k \in \boldsymbol{T}$ the normalization $\tau_k(\rho) = 1$ for all $\rho > 0$ gives $P_k T(\rho) f = P_k f$. Thus $P_k T(\rho) f = P_k f$ for all $k \in \boldsymbol{P}$, and since $\{P_k\}$ is total the assertion follows.

c) Since for any $h \in P_k(X)$

$$\| T(\rho)h - h \| = | \tau_k(\rho) - 1 | \, \| h \| ,$$

$h \neq 0$ is noninvariant if $k \notin \boldsymbol{T}$, and the assertion follows by (6.1).

Condition (6.1) and Theorem 6.1 suggest the introduction, for any

$\psi \in s$, of the following subspaces of X:

(6.2) $$X^{\psi} = \{f \in X;\ \text{there exists } f^{\psi} \in X \text{ such that } \psi_k P_k f = P_k f^{\psi} \text{ for all } k \in \boldsymbol{P}\}\,.$$

Obviously, if B^{ψ} is the operator with domain $X^{\psi} \subset X$ and range in X defined by $B^{\psi} f = f^{\psi}$, $f \in X^{\psi}$, then B^{ψ} is a closed linear operator for each $\psi \in s$, $P_k(X) \subset X^{\psi}$ for each $k \in \boldsymbol{P}$ so that B^{ψ} is densely defined, and X^{ψ} is a normalized[1] subspace under $|f|_{\psi} = \|f^{\psi}\|$. Moreover, (6.1) implies the Voronovskaja-type relation

(6.3) $$\operatorname*{s-lim}_{\rho\to\infty} \varphi^{-1}(\rho)\,[T(\rho)f - f] = B^{\psi} f \qquad (f \in P_k(X),\ k \in \boldsymbol{P})$$

on each subset $P_k(X)$.

Having established the saturation property for $\{T(\rho)\}$ in case (6.1) holds, the next problem is to derive equivalent characterizations of the Favard class $F[X; T(\rho)]$. To this end, the following condition is posed; it in fact strengthens (6.1) and admits an extension of (6.3) to the whole space X^{ψ}:

(6.4) Let $\{T(\rho)\}$ satisfy (6.1) with $\psi \in s$, $\varphi(\rho)$, and let there exist a (uniform) multiplier family $\{\eta(\rho)\} \subset M(X; \{P_k\})$ associated with (the strong approximation process) $\{E(\rho)\}$ such that the representation

$$\varphi^{-1}(\rho)\,[\tau_k(\rho) - 1] = \psi_k \eta_k(\rho)$$

holds for every $k \in \boldsymbol{P}$, $\rho > 0$.

Condition (6.4) is again standard, at least in connection with trigonometric series (compare [5; Sec. 12.6] for detailed comments). Obviously by (6.4)

(6.5) $$\varphi^{-1}(\rho)\,[T(\rho)f - f] = B^{\psi} E(\rho) f \qquad (f \in X,\ \rho > 0)$$

for all $f \in X$, implying $E(\rho)(X) \subset X^{\psi}$ for each $\rho > 0$. Since furthermore $T(X^{\psi}) \subset X^{\psi}$ for any $\psi \in s$ and multiplier operator T, in fact, since

$$B^{\psi} T f = T B^{\psi} f \qquad \text{for all } f \in X^{\psi}\,,$$

one may continue formula (6.5) with $B^{\psi} E(\rho) f = E(\rho) B^{\psi} f$ in case $f \in X^{\psi}$, so that (6.3) holds for all $f \in X^{\psi}$. However, Voronovskaja-type relations with a densely defined closed linear operator B are one of the main points

[1] A linear manifold $Y \subset X$ is called a normalized subspace of X if there is a semi-norm $|\cdot|_Y$ on Y such that Y is a Banach space under $\|\cdot\|_Y = \|\cdot\| + |\cdot|_Y$. Two semi-norms $|\cdot|_1$, $|\cdot|_2$ on Y are said to be equivalent: $|\cdot|_1 \sim |\cdot|_2$, if there exist constants $c_1, c_2 > 0$ such that $c_1 |f|_1 \leqslant |f|_2 \leqslant c_2 |f|_1$ for every $f \in Y$.

of the saturation theorem of Berens [22; p. 28] which characterizes Favard classes in terms of relative completions. Thereby the completion of a normalized subspace Y of X relative to X, denoted by $\tilde{Y}^X$, is the set of those elements $f \in X$ for which there is a sequence $\{f_n\} \subset Y$ and a constant $C > 0$ such that $|f_n|_Y \leqslant C$ for all n and $\lim_{n\to\infty} \|f_n - f\| = 0$. With any $f \in \tilde{Y}^X$ one may associate the semi-norm

$$|f|_{Y^{\sim X}} = \inf\{\sup |f_n|_Y;\ \{f_n\} \subset Y,\ \lim_{n\to\infty} \|f_n - f\|\}\,.$$

Recalling that $T(\rho)(X^\psi) \subset X^\psi$ and $E(\rho)(X) \subset X^\psi$, all the assumptions of the above mentioned theorem in the form [22, Bemerkung 3.4] (cf. [5; p. 502]) are satisfied, so that one has

THEOREM 6.2. *If $\{T(\rho)\}$ satisfies (6.4), the following semi-norms are equivalent*[1] *on $(X^\psi)^{\sim X}$:*

$$|f|_{\psi\sim} \sim \sup_{\rho>0} \|\varphi^{-1}(\rho)[T(\rho)f - f]\|\,.$$

If X is reflexive, $|f|_\psi$ is a further equivalent semi-norm.

Thus, $f \in (X^\psi)^{\sim X}$ if and only if $\|T(\rho)f - f\| = O(\varphi(\rho))$, whereas $f \in X^\psi$ if and only if s-$\lim_{\rho\to\infty} \varphi^{-1}(\rho)[T(\rho)f - f]$ *exists.*

Let us conclude with the following additional characterization of the Favard class.

THEOREM 6.3. *Let $\psi \in s$ and $\{G(\rho)\}$ be a (uniformly bounded) strong approximation process of multiplier operators such that $G(\rho)(X) \subset X^\psi$ for each $\rho > 0$. The following semi-norms are equivalent on $(X^\psi)^{\sim X}$:*

$$|f|_{\psi\sim} \sim \sup_{\rho>0} \|B^\psi G(\rho)f\|\,.$$

PROOF. First assume that $|f|_{\psi\sim} < \infty$. Then, by definition, there exists a sequence $\{f_n\} \subset X^\psi$ such that $|f_n|_\psi \leqslant C$ uniformly for all n and $\lim_{n\to\infty} \|f_n - f\| = 0$. Since $B^\psi G(\rho) \in [X]$ by the closed graph theorem, and since B^ψ, $G(\rho)$ commute for each $\rho > 0$, one has

$$\begin{aligned} \|B^\psi G(\rho)f\| &= \lim_{n\to\infty} \|B^\psi G(\rho)f_n\| \\ &= \lim_{n\to\infty} \|G(\rho)B^\psi f_n\| \leqslant \sup_{\rho>0} \|G(\rho)\|_{[X]} \sup_n |f_n|_\psi\,. \end{aligned}$$

However, the left-hand side is independent of the particular choice of the sequence $\{f_n\}$, whereas the right-hand side is independent of ρ. Therefore

$$\sup_{\rho>0} \|B^\psi G(\rho)f\| \leqslant D\,|f|_{\psi\sim}\,,$$

[1] We use the notation $|f|_{(X^\psi)^{\sim X}} = |f|_{\psi\sim}$.

proving one direction of the assertion. The converse one is easily seen by examining the particular sequence $\{G(n)f\} \subset X^{\psi}$.

REMARK. If $G(\rho)$ is not a multiplier operator and B is not generated by some $\psi \in s$ via (6.2), then Theorem 6.3 remains valid provided $G(\rho)$ and B commute.

COROLLARY 6.4. *If* $\{T(\rho)\}$ *satisfies* (6.4), *then the following seminorms are equivalent on* $(X^{\psi})^{\sim X}$:

$$\sup_{\rho>0} \| \varphi^{-1}(\rho) [T(\rho)f - f] \| \sim \sup_{\rho>0} \| B^{\psi} E(\rho) f \| .$$

Let us mention that $\| B^{\psi} G(\rho) f \| = O(1)$ immediately meets standard representation theorems in case of the trigonometric system (cf. [5; p. 233]). For characterizations of the present type in case of semi-groups of operators one may consult [22; p. 43], [26; p. 111] (see also [30a]).

7. **Some multiplier classes.** For a wide range of applications, however, one needs sufficient conditions upon multiplier classes in connection with additional structures of the space X and the system $\{P_k\}$. Here we weaken conditions (3.2) and (3.6), i.e., $\| S_n f \| \leqslant B \| f \|$ and $\| \sigma_n f \| \leqslant C \| f \|$, respectively, to the uniform boundedness of the (C, j)-means.

To this end, let the (C, β)-means be defined for $\beta \geqslant 0$ by

$$(C, \beta)_n f = (A_n^{\beta})^{-1} \sum_{k=0}^{n} A_{n-k}^{\beta} P_k f , \quad A_n^{\beta} = \binom{n+\beta}{n} ; \tag{7.1}$$

thus $(C, \beta)_n$ coincides for $\beta = 0$ with the n-th partial sum operator S_n and for $\beta = 1$ with the n-th Cesàro mean operator σ_n (of order 1). For some fixed $j \in \boldsymbol{P}$ assume that $(C, j)_n$ is uniformly bounded, i.e.,

$$\| (C, j)_n f \| \leqslant C_j \| f \| \qquad (f \in X) , \tag{7.2}$$

the constant $C_j\ (\geqslant 1)$ being independent of $n \in \boldsymbol{P}$ and $f \in X$. Analogous to Sec. 3 the following classes are introduced ($\Delta^{j+1}\alpha_k = \Delta(\Delta^j \alpha_k)$)

$$bv_{j+1} = \left\{ \alpha \in l^{\infty};\ \| \alpha \|_{bv_{j+1}} = \sum_{k=0}^{\infty} \binom{k+j}{j} | \Delta^{j+1} \alpha_k | + \lim_{m\to 0} | \alpha_m | < \infty \right\} . \tag{7.3}$$

Obviously, $bv_1 = bv$ and $bv_2 = bqc$ (cf. (3.3), (3.7)).

REMARK. $\alpha \in l^{\infty}$ and the convergence of the series in (7.3) imply the existence of the limit $\lim_{m\to\infty} \alpha_m = \alpha_{\infty}$. Furthermore, it is known that $bv_{j+1} \subset bv_j$ in the sense of continuous embedding (see e.g. relation (2.2) and Lemma 2, (2) and (3) in [7]).

THEOREM 7.1. *Let* $\{P_k\} \subset [X]$ *be a total sequence of mutually orthogo-*

nal projections satisfying (7.2). *Then every $\alpha \in bv_{j+1}$ is a multiplier and*

(7.4) $$\|\alpha\|_M \leqslant C_j \|\alpha\|_{bv_{j+1}} .$$

PROOF. For each $f \in X$ set

$$f^\alpha = \sum_{k=0}^{\infty} \binom{k+j}{j} \Delta^{j+1}\alpha_k (C, j)_k f + \alpha_\infty f .$$

Then f^α exists in X since by (7.2) and (7.3)

$$\|f^\alpha\| \leqslant C_j \|f\| \sum_{k=0}^{\infty} \binom{k+j}{j} |\Delta^{j+1}\alpha_k| + |\alpha_\infty| \|f\| \leqslant C_j \|\alpha\|_{bv_{j+1}} \|f\| .$$

Thus it remains to show that $f^\alpha \sim \sum \alpha_k P_k f$. Observing that

$$P_n(C, j)_k f = \begin{cases} 0 & , k < n \\ \left(1 - \dfrac{n}{k+1}\right) \cdots \left(1 - \dfrac{n}{k+j}\right) P_n f & , k \geqslant n \end{cases}$$

one obtains

$$\begin{aligned} P_n f^\alpha &= \sum_{k=0}^{\infty} \binom{k+j}{j} \Delta^{j+1}\alpha_k P_n (C, j)_k f + \alpha_\infty P_n f \\ &= \left(\sum_{k=n}^{\infty} \binom{k+j}{j} \left(1 - \frac{n}{k+1}\right) \cdots \left(1 - \frac{n}{k+j}\right) \Delta^{j+1}\alpha_k \right) P_n f + \alpha_\infty P_n f , \end{aligned}$$

and one has to check that the latter sum is equal to $\alpha_n - \alpha_\infty$. However (cf. [7])

$$\alpha_0 - \alpha_\infty = \sum_{m=0}^{\infty} \binom{m+j}{j} \Delta^{j+1}\alpha_m$$

and thus, after shifting,

$$\begin{aligned} \alpha_n - \alpha_\infty &= \sum_{m=0}^{\infty} \binom{m+j}{j} \Delta^{j+1}\alpha_{n+m} \\ &= \sum_{k=n}^{\infty} \binom{k+j}{j} \left(1 - \frac{n}{k+1}\right) \cdots \left(1 - \frac{n}{k+j}\right) \Delta^{j+1}\alpha_k . \end{aligned}$$

REMARK. Let us mention that $bv_{j+1} \subset M$ is characteristic for the fundamental, total sequence $\{P_k\} \subset [X]$ of orthogonal projections to be a (C, j)-decomposition for X; compare the literature cited in Note I in connection with the particular cases $j = 0, 1$.

Let $\{P_k\}$ satisfy (7.2) for some $j \in \boldsymbol{P}$ and consider the sequences $\alpha^n \in s$, $n \in \boldsymbol{P}$, with $\alpha_k^n = \delta_{kn}$ (Kronecker's symbol). Then one has

$$\Delta^{j+1}\alpha^n_{n-\nu} = (-1)^\nu \binom{j+1}{\nu} \quad \text{for } 0 \leqslant \nu \leqslant \min(n, j+1) ,$$

$=0$ otherwise. Hence there are at most $(j+2)$ nonzero terms, bounded by $\binom{j+1}{[(j+1)/2]}$[1] uniformly for n. Therefore by Theorem 7.1

$$\| \alpha^n \|_M \leqslant C_j \binom{j+1}{[(j+1)/2]} \sum_{\max(0;n-j-1)}^{n} \binom{k+j}{j} \leqslant D_j n^j \tag{7.5}$$

with some constant D_j independent of n. However, $\alpha^n \in M$ is the multiplier corresponding to P_n, and since $\| P_n \|_{[X]} = \| \alpha^n \|_M$, one has the following estimate concerning the growth of $\| P_n \|_{[X]}$ for $n \to \infty$:

$$\| P_n \|_{[X]} \leqslant D_j n^j \qquad (n \in \boldsymbol{P}) , \tag{7.6}$$

in case the system $\{P_k\}$ satisfies (7.2). Therefore

COROLLARY 7.2. *Let $\psi \in s$ and $\{G(\rho)\}$ be a (uniformly bounded) strong approximation process of multiplier operators with corresponding multiplier sequences $\{\gamma(\rho)\}$. If $\{P_k\}$ satisfies (7.2) for some $j \in \boldsymbol{P}$ and*

$$\sum_{k=0}^{\infty} k^j \,|\, \psi_k \gamma_k(\rho) \,| < \infty \tag{7.7}$$

for each (fixed) $\rho > 0$, then $G(\rho)(X) \subset X^\psi$ for each $\rho > 0$. The following semi-norms are equivalent on $(X^\psi)^{\sim X}$:

$$| f |_{\psi\sim} \sim \sup_{\rho>0} \left\| \sum_{k=0}^{\infty} \psi_k \gamma_k(\rho) P_k f \right\| . \tag{7.8}$$

Indeed, the convergence of the series (7.7) ensures by (7.6) that for any $f \in X$

$$B^\psi G(\rho) f = \sum_{k=0}^{\infty} \psi_k \gamma_k(\rho) P_k f \qquad (f \in X;\ \rho > 0) ,$$

which completes the proof by Theorem 6.3.

As in Note I it will be very important concerning applications to have convenient sufficient criteria for $\alpha \in s$ to belong to bv_{j+1}, particularly to establish uniform bounds for multiplier families such as those involved in (6.4). For this purpose, we introduce the class $BV_{j+1}[0, \infty)$ consisting of all bounded continuous functions f for which $f, \cdots, f^{(j-1)}$ are locally absolutely continuous on $(0, \infty)$ and $f^{(j)}$ is locally of bounded variation on $(0, \infty)$ such that $\int_0^\infty x^j \,|\, df^{(j)}(x) \,| < \infty$. Then (for the particular cases $j = 0, 1$ compare Lemmas 3.3, 3.5 $(\rho = 1)$).

[1] Here $[\beta]$ denotes the largest integer $\leqslant \beta$.

LEMMA 7.3. *Let $\alpha \in s$ be such that there exists a function $a \in BV_{j+1}$ with $\alpha_k = a(k)$. Then $\alpha \in bv_{j+1}$ and*

$$\sum_{k=0}^{\infty} \binom{k+j}{j} | \Delta^{j+1}\alpha_k | \leqslant D \int_0^\infty x^j \, | da^{(j)}(x) | \, , \tag{7.9}$$

the constant $D > 0$ being uniformly bounded in j (and independent of α).

PROOF. Obviously one has for all $k \in \boldsymbol{P}$

$$\Delta^{j+1}\alpha_k = (-1)^{j+1} \int_k^{k+1} dx_1 \int_{x_1}^{x_1+1} dx_2 \cdots \int_{x_j}^{x_j+1} da^{(j)}(x_{j+1}) \, . \tag{7.10}$$

Since $x_{j+1} \geqslant k$ and $\binom{k+j}{j} \leqslant (3/2)\, x_{j+1}^j$ for $j \in \boldsymbol{P}$, $k = 2, 3, \cdots$, one has

$$\begin{aligned} &\sum_{k=2}^{\infty} \binom{k+j}{j} | \Delta^{j+1}\alpha_k | \\ &\quad \leqslant \sum_{k=2}^{\infty} \binom{k+j}{j} \int_0^1 dx_1 \cdots \int_0^1 dx_j \int_{k+x_1+\cdots+x_j}^{k+1+x_1+\cdots+x_j} | da^{(j)}(x_{j+1}) | \\ &\quad \leqslant \frac{3}{2} \int_2^\infty x_{j+1}^j \, | da^{(j)}(x_{j+1}) | \, . \end{aligned} \tag{7.11}$$

To estimate $| \Delta^{j+1}\alpha_0 |$ and $(j+1) | \Delta^{j+1}\alpha_1 |$, one may show by induction and by interchange of integration as in the proof of Lemma 3.5 that for fixed $u > 0$, $j \in \boldsymbol{P}$

$$\begin{aligned} &\left| \int_u^{u+1} dx_1 \int_{x_1}^{x_1+1} dx_2 \cdots \int_{x_j}^{x_j+1} da^{(j)}(x_{j+1}) \right| \\ &\quad \leqslant \frac{1}{j!} \int_u^{u+1} (x_{j+1} - u)^j \, | da^{(j)}(x_{j+1}) | + \sum_{k=1}^{j} \frac{1}{k\,!} (1+u)^{-j} \int_{u+1}^{\infty} x_{j+1}^j \, | da^{(j)}(x_{j+1}) | \, . \end{aligned}$$

Setting $u = 0+$ and $u = 1$ one obtains

$$\begin{aligned} \sum_{k=0}^{1} \binom{k+j}{j} | \Delta^{j+1}\alpha_k | &\leqslant \frac{1}{j\,!} \int_0^1 x^j \, | da^{(j)}(x) | + \frac{j+1}{j\,!} \int_1^2 x^j \, | da^{(j)}(x) | \\ &\quad + (1 + (j+1)2^{-j}) \sum_{k=1}^{j} \frac{1}{k\,!} \int_1^\infty x^j \, | da^{(j)}(x) | \end{aligned} \tag{7.12}$$

so that the assertion (7.9) follows by (7.11), (7.12). (Probably the exact constant D in (7.9) is $1/j\,!$)

Now let us turn to families $\{\tau(\rho)\} \subset s$. Here we suppose that there is a corresponding family of functions $\{t_\rho\} \subset BV_{j+1}$ with $\tau_k(\rho) = t_\rho(k)$. Then, by Lemma 7.3, for $\{\tau(\rho)\}$ to belong to bv_{j+1} uniformly it is sufficient that the integrals $\int_0^\infty x^j \,|\, dt_\rho^{(j)}(x) \,|$ are uniformly bounded for $\rho > 0$. Let us state this condition in the case that $\{\tau(\rho)\}$ is a family of Fejér's type (cf. Sec. 3).

COROLLARY 7.4. *Let $\{\tau(\rho)\}_{\rho>0} \subset s$ be a family of sequences for which there exists a function $t(x) \in BV_{j+1}\,[0, \infty)$ such that $\tau_k(\rho) = t(k/\rho)$ for all $k \in \boldsymbol{P}$, $\rho > 0$, and let $\{P_k\} \subset [X]$ be a total orthogonal sequence of projections satisfying* (7.2). *Then $\{\tau(\rho)\}$ is a family of uniformly bounded multipliers.*

Let us consider some examples of multiplier families $\{\tau(\rho)\}$ generating (uniformly bounded) strong approximation processes on an arbitrary Banach space X. In any case we assume the system $\{P_k\}$ to satisfy (7.2) for some $j \in \boldsymbol{P}$.

First we pick up the Abel-Cartwright means (4.2). With $w_\kappa(x) = \exp(-x^\kappa)$, $\kappa > 0$, $x \geqslant 0$ one easily checks

$$w_\kappa^{(j+1)}(x) = \sum_{m=1}^{j+1} C_{m,\kappa} x^{m\kappa-j-1} e^{-x^\kappa} ,$$

and therefore $w_\kappa \in BV_{j+1}$ for each $j \in \boldsymbol{P}$. Thus, by Corollary 7.4, $\{w_\kappa(k/(n+1))\}$ is a uniformly bounded multiplier family. Furthermore,

$$\lim_{n\to\infty} w_\kappa(k/(n+1)) = 1 ,$$

and hence one has convergence on the linear span of $\bigcup_{k=0}^\infty P_k(X)$. Since the latter is assumed to be dense in X, the Banach-Steinhaus theorem yields that $\{W_\kappa(n)\}$ is a strong approximation process on X in case $\{P_k\} \subset [X]$ is a fundamental, total, orthogonal system satisfying (7.2).

By straightforward computation one may show that the multiplier family corresponding to the typical means (4.1) do not belong to bv_{j+1}, $j \geqslant 2$, uniformly in n, so that these means cannot be treated in the above way.

However, their generalization, namely the Riesz means

$$R_{\kappa,\beta}(n)f = \sum_{k=0}^{n} r_{\kappa,\beta}(k/(n+1))P_k f , \tag{7.13}$$

$$r_{\kappa,\beta}(x) = \begin{cases} (1 - x^\kappa)^\beta , & 0 \leqslant x \leqslant 1 \\ 0 , & x \geqslant 1 , \end{cases}$$

defines a strong summation method for $\kappa > 0$, $\beta \geqslant j$, the reasoning being the same as for the Abel-Cartwright means.

Next we consider a modification of the Abel-Cartwright means of order $\kappa = 2$ which will reduce (cf. Sec. 8.4) to the classical Gauss-Weierstrass integral in case of an expansion into spherical harmonics, namely

$$W_{2,\zeta}(n)f = \sum_{k=0}^{\infty} e^{-k(k+\zeta)/(n+1)^2} P_k f \qquad (\zeta > 0)\,, \tag{7.14}$$

equality being justified by (7.6). Observe that the convergence factors in (7.14) may be written as $\exp(-k^2/(n+1)^2)\exp(-\zeta k/(n+1)^2)$. Since each of these factors generates a uniformly bounded multiplier family, this holds also for their product, thus $\{W_{2,\zeta}(n)\}$ forms a strong approximation process.

Finally let us examine the (C, β)-means (7.1). By straightforward computation one has $\{A^{\beta}_{n-k}/A^{\beta}_{n}\}_{k=0}^{n} \in bv_{j+1}$ uniformly in n for $\beta \geqslant j$; since $\lim_{n\to\infty} A^{\beta}_{n-k}/A^{\beta}_{n} = 1$, $\{(C, \beta)_n\}$ is a summation method for $\beta \geqq j$.

As a first application of the above results let us give a modest simplification of Theorem 6.2 which, in particular, implies the standard saturation theorems in the trigonometric series case.

THEOREM 7.5. *Let $\{P_k\} \subset [X]$ be a total, fundamental, mutually orthogonal system of projections satisfying* (7.2) *for some $j \in \boldsymbol{P}$.*

a) *If $\{T(\rho)\}$ satisfies* (6.1) *and*

$$\|\varphi^{-1}(\rho)\,[T(\rho)f - f]\,\| = O(1) \qquad (\rho \to \infty)\,,$$

then for some $\beta \geqq j$

$$\left\| \sum_{k=0}^{n} \Big(1 - \frac{k}{n+1}\Big)^{\beta} \psi_k P_k f \right\| = O(1) \qquad (n \to \infty)\,. \tag{7.15}$$

b) *If, furthermore, $\{T(\rho)\}$ satisfies* (6.4), *then the converse is also true.*

PROOF. a) By (6.1) we obtain for $\beta \geqslant j$ (cf. (7.13))

$$\begin{aligned}
\left\| \sum_{k=0}^{n} \Big(1 - \frac{k}{n+1}\Big)^{\beta} \psi_k P_k f \right\| &= \lim_{\rho\to\infty} \left\| \sum_{k=0}^{n} \Big(1 - \frac{k}{n+1}\Big)^{\beta} \frac{\tau_k(\rho) - 1}{\varphi(\rho)} P_k f \right\| \\
&= \lim_{\rho\to\infty} \| R_{1,\beta}(n)\varphi^{-1}(\rho)\,[T(\rho)f - f]\,\| \\
&\leqslant \sup_{n} \| R_{1,\beta}(n) \|_{[X]} \sup_{\rho>0} \| \varphi^{-1}(\rho))\,[T(\rho)f - f]\,\| < \infty\,.
\end{aligned}$$

b) This follows by Theorem 6.2 and (7.8).

8. Applications.

8.1. Abel-Cartwright, Riesz, and (C, β)-means. Let X be a Banach space and $\{P_k\}$ be a sequence of projections as specified in Sec. 2 and 6 satisfying (7.2) for some $j \in \boldsymbol{P}$. First let us determine the saturation behaviour of the Abel-Cartwright means (4.2) and examine condition (6.4). Choosing $\varphi(n) = (n+1)^{-\kappa}$, $\psi_k = -k^{\kappa}$, it is clear that

$$\lim_{n\to\infty} \eta_k(n) = 1,\ \eta_k(n) = e(k/(n+1)),\ e(x) = -x^{-\kappa}[\exp(-x^{\kappa}) - 1]\ .$$

Thus, by the Banach-Steinhaus theorem and Corollary 7.4, it suffices to verify $e(x) \in BV_{j+1}$. By induction one has

$$\begin{aligned} e^{(j+1)}(x) &= D_{\kappa,j+1}[x^{-\kappa-j-1}(e^{-x^{\kappa}} - 1) + x^{-j-1}e^{-x^{\kappa}}] \\ &\quad + \sum_{m=1}^{j} D_{\kappa,m}x^{m\kappa-j-1}e^{-x^{\kappa}}\ , \end{aligned}$$

and thus $e \in BV_{j+1}[0, \infty)$ for any $j \in \boldsymbol{P}$. Now Theorems 6.1–6.3, 7.5 yield

THEOREM 8.1. *Let X be a Banach space, $\{P_k\}_{k=0}^{\infty} \subset [X]$ be a fundamental, total sequence of mutually orthogonal projections satisfying* (7.2) *for some $j \in \boldsymbol{P}$. Then, for each $\kappa > 0$, the Abel-Cartwright means* (4.2) *have the following properties*:

a)
$$\| W_{\kappa}(n)f - f \| = o(n^{-\kappa}) \qquad (n \to \infty)$$

implies $f \in P_0(X)$ and $W_{\kappa}(n)f = f$ for all $n \in \boldsymbol{P}$.

b) *With $\psi = \{-k^{\kappa}\}_{k=0}^{\infty}$ the following semi-norms are equivalent on $(X^{\psi})^{\sim X}$*:

i)
$$\sup_{n\in \boldsymbol{P}} \| n^{\kappa}[W_{\kappa}(n)f - f] \|\ ,$$

ii)
$$| f |_{\psi\sim}\ ,$$

iii)
$$\sup_{n\in \boldsymbol{P}} \left\| \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)^{\beta} k^{\kappa} P_k f \right\| \qquad \textit{for } \beta \geqslant j\ .$$

Hence, in particular, $f \in F[X;\ W_{\kappa}(n)]$ if and only if for $\beta \geqslant j$

$$\left\| \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)^{\beta} k^{\kappa} P_k f \right\| = O(1) \qquad (n \to \infty)\ .$$

Obviously, one may replace the discrete parameter $(n+1)$ by the continuous ρ. Moreover, by setting $t = \rho^{-1/\kappa}$, the resulting family of operators forms a (C_0)-semi-group in t for each fixed $\kappa > 0$, so that Theorem 8.1 may also be derived from a theory concerning general semi-groups for which one may consult [22], [26].

Analogously one may proceed for the Riesz means. The corresponding quotient of (6.4) is again of Fejér's type with ($\psi_k = -\beta k^\kappa, \varphi(n) = (n+1)^{-\kappa}$) the associated function

$$e(x) = \begin{cases} -\beta^{-1}x^{-\kappa}\,[(1-x^\kappa)^\beta - 1]\,, & 0 \leqslant x \leqslant 1 \\ \beta^{-1}x^{-\kappa} \,, & x \geqslant 1\,. \end{cases}$$

By induction one may again deduce $e^{(j+1)}(x) = O(x^{-\kappa-j-1})$ for $x > 1$ and

$$\begin{aligned} e^{(j+1)}(x) = D_{\kappa,\beta,j+1}\{&x^{-\kappa-j-1}\,[(1-x^\kappa)^\beta - 1] + \beta x^{-j-1}(1-x^\kappa)^{\beta-1}\} \\ &+ \sum_{m=1}^{j} D_{\kappa,\beta,m}x^{m\kappa-j-1}(1-x^\kappa)^{\beta-m-1} \end{aligned}$$

for $0 < x < 1$. This formula only holds for $\beta > j$. However, for $\beta = j$ an analogous one is valid. Furthermore, if $j = \beta$ then $e^{(j)}(x)$ has a finite jump at $x = 1$. In any event, $e \in BV_{j+1}\,[0, \infty)$, and $\{\eta(n)\}$ is a uniformly bounded multiplier family by Corollary 7.4.

THEOREM 8.2. *Under the hypotheses of Theorem* 8.1 *the Riesz means* (7.13) *have the following properties for* $\kappa > 0$, $\beta \geqq j$:

a) $$\| R_{\kappa,\beta}(n)f - f \| = o(n^{-\kappa}) \qquad (n \to \infty)$$

implies $f \in P_0(X)$ *and* $R_{\kappa,\beta}(n)f = f$ *for all* $n \in \boldsymbol{P}$.

b) $$\sup_{n \in \boldsymbol{P}} \| n^\kappa\,[R_{\kappa,\beta}(n)f - f]\,\|$$

is a further equivalent semi-norm on $(X^\psi)^{\sim X}$ *with* ψ *as in Theorem* 8.1.

Recalling the results of Sec. 2–4 one may suppose that it would be easier to prove the equivalence of the two processes $\{W_\kappa(n)\}$ and $\{R_{\kappa,\beta}(n)\}$, for in this case one would only have to determine the Favard class of one of these processes. Indeed, this will be possible (and would be more elegant from a theoretical point of view), but though elementary and straightforward, checking condition (2.8) will probably be very tedious for general $j \in \boldsymbol{P}$ since the structure of the multipliers involved in (2.8) is more complicated than that of the multipliers in (6.4). The same remark also holds for the next two examples.

The modified Abel-Cartwright means of order 2 may be treated in the same way as above by verifying (6.4) (using Lemma 7.3 for each n separately since the quotient is not of Fejér's type). Hence

THEOREM 8.3. *Let* $\zeta > 0$. *Under the hypotheses of Theorem* 8.1 *the following assertions are valid*:

a) $$\| W_{2,\zeta}(n)f - f \| = o(n^{-2}) \qquad (n \to \infty)$$

implies $f \in P_0(X)$ *and* $W_{2,\zeta}(n)f = f$ *for all* $n \in \boldsymbol{P}$.

b) $$\sup_{n \in \boldsymbol{P}} \| n^2 [W_{2,\zeta}(n)f - f] \| , \quad \sup_{n \in \boldsymbol{P}} \left\| \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)^{\beta} k(k+\zeta) P_k f \right\| ,$$

$\beta \geqslant j$, *are equivalent semi-norms on* $(X^{\psi})^{\sim X}$, *where* $\psi = \{-k(k+\zeta)\}_{k=0}^{\infty}$.

c) *The Favard classes of* $\{W_{2,\zeta}(n)\}$ *and* $\{W_2(n)\}$ *coincide for each* $\zeta > 0$.

Obviously, in view of the preceding arguments it only remains to prove c). Now, α, $\alpha^{-1} \in s$ defined by

$$\alpha_k = \begin{cases} 0 & , k = 0 \\ k(k+\zeta)/k^2, & k \in \boldsymbol{N} \end{cases} , \qquad \alpha_k^{-1} = \begin{cases} 0 & , k = 0 \\ k^2/k(k+\zeta) & , k \in \boldsymbol{N}, \end{cases}$$

respectively, both belong to bv_{j+1} for each $\zeta > 0$ and $j \in \boldsymbol{P}$, so that c) follows.

Finally let us consider the (C, β)-means, $\beta \geqslant j$.

THEOREM 8.4. *Under the hypotheses of Theorem* 8.1 *one has for* $\beta \geqq j$:

a) $$\| (C, \beta)_n f - f \| = o(n^{-1}) \qquad (n \to \infty)$$

implies $f \in P_0(X)$ *and* $(C, \beta)_n f = f$ *for all* $n \in \boldsymbol{P}$.

b) *The following assertions are equivalent* ($\gamma \geqslant j$):

i) $$\| n [(C, \beta)_n f - f] \| = O(1) \qquad (n \to \infty) ,$$

ii) $$\left\| \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)^{\gamma} \beta k P_k f \right\| = O(1) \qquad (n \to \infty) .$$

Indeed, since (cf. [27; p. 388], [23; p. 248])

$$\lim_{n \to \infty} n [A_{n-k}^{\beta} / A_n^{\beta} - 1] = -\beta k \qquad (k \in \boldsymbol{P}) ,$$

condition (6.1) is satisfied so that a) is given by Theorem 6.1 and b), i) $\Rightarrow$ ii), by Theorem 7.5a). In order to prove b), ii) $\Rightarrow$ i), one may verify the uniform multiplier condition (6.4), but this would be quite hideous. To avoid this, one may proceed via some functional equations just as in [23; p. 248] (up to notation), so that the proof may be omitted.

Let us mention that in the particular instance $\beta = j = \gamma \in \boldsymbol{N}$ Theorem 8.4 coincides with results of Favard [29] (in a Banach space with a biorthogonal system).

8.2 **Trigonometric system.** Let $X_{2\pi} = L_{2\pi}^p$, $1 \leqq p < \infty$, or $C_{2\pi}$ be as in Sec. 4.2 (however, note that $p = \infty$ is excluded) and $\{P_k\}$ be given by

(4.3). Then, by the choice of $X_{2\pi}$, this sequence of projections is also fundamental, and their $(C, 1)_n$-means are uniformly bounded in n by Fejér's theorem. In this particular case, Theorem 7.5 a) is due to Harsiladse [30] and Sunouchi-Watari [32] who then continued with the fact that by standard representation theorems (cf. [5; p. 234]) condition

$$\sup_{n\in P}\left\|\sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)\chi(k)f^\wedge(k)e^{ikx}\right\|_p<\infty \qquad (1\leqslant p\leqslant\infty) \tag{8.1}$$

is equivalent to the existence of a function g such that for

i) $p=1$: g is of bounded variation with $\chi(k)f^\wedge(k)=[dg]^\wedge(k)$, $k\in \boldsymbol{Z}$ ($[dg]^\wedge(k)$ are the Fourier-Stieltjes coefficients of g),

ii) $1<p\leqslant\infty$: g belongs to $L_{2\pi}^p$ with $\chi(k)f^\wedge(k)=g^\wedge(k)$, $k\in\boldsymbol{Z}$.

However, note that (8.1) only corresponds to (7.15) for even χ, i.e.

$$\chi(k)=\chi(-k)\,, \qquad k\in\boldsymbol{Z}\,.$$

Since all hypotheses on $X_{2\pi}$ and $\{P_k\}$ of Sec. 7 are satisfied, one may formulate Theorems 8.1–8.4 for this particular choice of the Banach space X and the system $\{P_k\}$. As a representative example we have

COROLLARY 8.5. *Let $X_{2\pi}$ and $\{P_k\}$ be given as above. Then one has for $\kappa>0$, $\beta\geqslant 1$*

a)
$$\|R_{\kappa,\beta}(n)f-f\|=o(n^{-\kappa}) \qquad (n\to\infty)$$

implies that f is a constant.

b)
$$\|R_{\kappa,\beta}(n)f-f\|=O(n^{-\kappa}) \qquad (n\to\infty)$$

if and only if

$$\left\|\sum_{k=-n}^{n}\left(1-\frac{|k|}{n+1}\right)|k|^\kappa f^\wedge(k)e^{ikx}\right\|=O(1) \qquad (n\to\infty)\,.$$

8.3. Ultraspherical polynomials. Let us recall the definition (4.6) of the ultraspherical polynomials $C_k^\lambda(x)$ of order $0\leqslant\lambda<\infty$ with the corresponding system (4.7) of projections P_k as defined on $X^{\lambda,p}$, $1\leqslant p<\infty$ (cf. (4.8)). Using the result of Askey-Hirschman [2] that $\|(C,\beta)_n f\|_p\leqslant C_\beta\|f\|_p$ for $(2\lambda+1)/(\lambda+1+\beta)<p<(2\lambda+1)/(\lambda-\beta)$, if $0\leqslant\beta\leqslant\lambda$, and for all $1\leqslant p\leqslant\infty$ if $\beta>\lambda$, all hypotheses of Sec. 7 are satisfied. Thus Theorems 8.1–8.4 may be reformulated for this particular case where one has to pay attention to the relation between λ, β, and p.

For $\lambda=1/2$ the ultraspherical polynomials coincide with the Legendre polynomials, thus the corresponding Abel-Cartwright means of order 1 with the solution of Dirichlet's problem for a sphere in the axially symmetrical case. Theorem 8.1 then contains results as given in [24].

8.4. **Surface spherical harmonics.** Let R^N be the N-dimensional Euclidean space with elements $v = (v_1, \cdots, v_N)$, inner product

$$v \cdot v^* = \sum_{k=1}^{N} v_k v_k^* \quad \text{and} \quad |v|^2 = v \cdot v .$$

Let S_N be the surface of the unit sphere in R^N with elements y, z, content $\Omega_N = 2\pi^{N/2}/\Gamma(N/2)$ and surface element ds. Now X denotes one of the spaces $L^p(S_N)$, $1 \leqslant p < \infty$, or $C(S_N)$ with the usual norms

$$\| f \|_p = \left\{ \Omega_N^{-1} \int_{S_N} | f(y) |^p \, ds(y) \right\}^{1/p} (1 \leqslant p < \infty) , \quad \| f \|_C = \max_{y \in S_N} | f(y) | ,$$

respectively. If $Y_k(v)$ is a homogeneous polynomial of degree k in N dimensions which satisfies

$$\Delta Y_k(v) = 0 , \qquad \Delta = \sum_{k=1}^{N} (\partial/\partial v_k)^2 \qquad (v \in R^N) ,$$

then the restriction of Y_k to S_N, denoted by Y_k, too, is called a spherical harmonic of order k. Now it is known that every spherical harmonic of degree k and dimension N satisfies

$$\tilde{\Delta} Y_k(y) = -k(k + N - 2) Y_k(y) , \quad \tilde{\Delta} f(v) = | v |^2 \Delta f(v/| v |);$$

for each k there exist

$$H(k, N) = (2k + N - 2) \frac{(k + N - 3)!}{k!(N - 2)!}$$

linearly independent surface spherical harmonics of degree k; the set

$$\{ Y_k^r(y);\ 1 \leqslant r \leqslant H(k, N),\ k \in \boldsymbol{P} \}$$

is fundamental and may be assumed to be orthonormal in X. Then with each $f \in X$ one may associate its Fourier series expansion (2.1) into spherical harmonics (Laplace series) with the system $\{P_k\}$ of projections defined by (cf. (4.6), $2\lambda = N - 2$).

$$\begin{aligned} P_k f(y) &= \sum_{r=1}^{H(k,N)} \left(\int_{S_N} f(z) Y_k^r(z) ds(z) \right) Y_k^r(y) \\ &= \frac{\Gamma(\lambda)\,(k+\lambda)}{2\pi^{\lambda+1}} \int_{S_N} C_k^\lambda(y \cdot z) f(z) ds(z) = Y_k(f;\, y) . \end{aligned}$$

$\{P_k\}$ satisfies all assumptions required in Sec. 6–7, in particular (7.2) for $j > (N - 2)/2$ (for these properties compare [23] and the literature cited there). Note that $\dim P_k(X) = H(k, N)$, i.e., the dimension of $P_k(X)$

increases with k (the dimension was 1 in all the other examples treated by us apart from the trigonometric system where $\dim P_k(X) = 2$ for $k > 0$; cf. (4.3)).

Now Theorems 8.1–8.4 may be stated. The result on the Abel-Cartwright means then reproduces that of [23; p. 246, p. 253], whereas our result on the Riesz means is not contained in [23]; the result on the (C, β)-means is (naturally) identical with [23; p. 248], whereas that on $\{W_{2,\zeta}(n)\}$ may be found in [23; p. 229]. Observe that not the Abel-Cartwright means of order 2 but their modification

$$W_{2,N-2}(t)f(y) = \Omega_N^{-1} \int_{S_N} w_t(y \cdot z) f(z) ds(z)$$

with

$$w_t(\cos\theta) = \sum_{k=0}^{\infty} \exp(-k(k+N-2)t) \frac{k+N-2}{N-2} C_k^{N-2}(\cos\theta)$$

is called the singular integral of Gauss-Weierstrass on X; it solves the heat equation $(\partial/\partial t) U(y, t) = \tilde{\Delta} U(y, t)$ on the sphere with initial value $\lim_{t\to 0+} U(y, t) = f(y)$ for $t > 0$, $y \in S_N$, and $f \in X$.

Obviously, one can generalize the present means by choosing the coefficients $\{\exp(-[k(k+N-2)]^{\kappa/2} t)\}$, $\kappa > 0$, but the computations necessary for an application of our theory seem to be quite hideous, not to mention the verification of (2.8) by Lemma 7.3 if one wants to compare it with the Abel-Cartwright means.

8.5. **Hermite series.** Choose $X = L^p(-\infty, \infty)$, $1 \leqslant p < \infty$, with

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{1/p}$$

and consider the Hermite polynomials defined by (cf. [8; p. 193])

$$H_k(x) = (-1)^k e^{x^2} (d/dx)^k e^{-x^2} \qquad (k \in \boldsymbol{P}) .$$

Setting

$$\varphi_k(x) = (2^k k! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_k(x) ,$$

$\{\varphi_k\}$ is an orthonormal family of functions on $(-\infty, \infty)$. Thus the projections

$$P_k f(x) = \left[\int_{-\infty}^{\infty} f(u)\varphi_k(u) du\right] \varphi_k(x)$$

are mutually orthogonal. As in the Laguerre series case, the partial sums $(C, 0)_n$ are bounded (cf. [3]) provided $4/3 < p < 4$; moreover, the

$(C, 1)$-means are bounded provided $1 < p < \infty$ (cf. [15a]). Thus, for example,

COROLLARY 8.6. *Let* $f \in L^p(-\infty, \infty)$, $1 < p < \infty$, $\{P_k\}$ *as above, and* $W_\kappa(n)$ *be given by* (4.2).

a) $$\| W_\kappa(n)f - f \| = o(n^{-\kappa}) \qquad (n \to \infty)$$

implies $f(x) = Ae^{-x^2/2}$ *for some constant* A.

b) $$\| W_\kappa(n)f - f \| = O(n^{-k}) \qquad (n \to \infty)$$

if and only if $(\beta = 1)$

$$\left\| \sum_{k=0}^{n} k^\kappa \Big(1 - \frac{k}{n+1}\Big) P_k f \right\| = O(1) \qquad (n \to \infty) .$$

For $\kappa = 1$ this result is contained in [25], where also the extension of a) to $p = 1$ is given and the last condition in b) is characterized for $1 < p < \infty$.

Let us finally emphasize that the essential hypothesis for the application of the general theory of Sec. 6–7 is the boundedness of the (C, β)-means for some $\beta \geqslant 0$. In this case one can treat a number of summation processes in a unified way where other methods may fail. However, for large β the calculations, though elementary and straightforward, may become wearisome.

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LEHRSTUHL A FÜR MATHEMATIK
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