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ON POSITIVELY CURVED RIEMANNIAN MANIFOLDS WITH BOUNDED VOLUME

Dedicated to Professor Shigeo Sasaki on his 60th birthday

TAKATOSHI NAGAYOSHI AND YÔTARÔ TSUKAMOTO*

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It is very interesting and important to investigate the relations among curvatures, volumes and topological structures on Riemannian manifolds of positive curvature. The following theorems are well known.

THEOREM A. (Bishop-Crittenden [5]) *Let M be an n-dimensional complete Riemannian manifold with sectional curvature $K \geqq 1$. Then we have*

$$\operatorname{vol} M \leqq \operatorname{vol} S^n ,$$

and equality holds only if M is isometric to a sphere S^n with constant curvature 1, *where we denote the volume of M by* vol M.

THEOREM B. (Heim [7]) *Let M be an n-dimensional complete Riemannian manifold with sectional curvature $K \geqq 1$ and* $\operatorname{vol} M > (1/2) \operatorname{vol} S^n$. *Then M is a homotopical sphere.*

In this paper we give a simple proof of Theorem B and prove the following theorem.

THEOREM C. *Let M be an n-dimensional complete Riemannian manifold with sectional curvature $K \geqq 1$ and* $\operatorname{vol} M \geqq (1/2) \operatorname{vol} S^n$. *Then M is a homotopical sphere or isometric to the real projective space with constant curvature* 1.

1. Preliminaries.

(a) **Volumes** (cf. Berger-Gauduchon-Mazet [4]). Let M be a compact Riemannian manifold, g be its Riemannian metric and v_g be the canonical measure on M. For a point $m \in M$ let v_m be the volume element of the tangent space M_m to M at m, $\exp_m$ be the exponential mapping of M_m onto M and let U_m be the maximal open neighborhood around the origin of M_m which $\exp_m$ maps diffeomorphically onto its image. We call

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U_m the *injective neighborhood* of m. Let $\theta(x)$ be the Jacobian determinant of $\exp_x$ for any point $x \in U_m$. Then we have

$$\text{vol}\, M = \int_{U_m} \theta \cdot v_m \,. \tag{1}$$

We use Jacobi fields to calculate volume. Let $U_\varepsilon(m)$ be a normal neighborhood around m with radius $\varepsilon > 0$. For $0 < r < \varepsilon$ and a unit vector u on M we give the expression of $\theta(ru)$. Let c_u be the geodesic starting from m, with the initial direction u. Let $\{y_2, \cdots, y_n\}$ be an orthonormal basis of orthogonal complement $u^\perp$ of u. Let $Y_i(s)$ be the Jacobi field along c_u such that $Y_i(0) = 0$, $Y_i'(0) = y_i/r$. Then we have

$$\theta(ru) = \det(\langle Y_i(r),\ Y_j(r)\rangle)^{1/2},\ i, j = 2, \cdots, n \tag{2}$$

where $\langle\ ,\ \rangle$ is the inner product on M.

(b) **Rauch's comparison theorem** (cf. [6]). For a point $m \in M$ let G_m be the set of all 2-dimensional linear subspaces of M_m and we put $G_M = \bigcup_{m \in M} G_m$. If c: $[0, l] \to M$ is a differentiable curve, we denote by G_c the set of all 2-dimensional linear subspaces of $M_{c(t)}$ each of which contains a tangent vector to c. For any $\sigma \in G_M$ let (u, v) be a basis of σ. Then we denote the sectional curvature by $K_\sigma = K(u, v)$.

THEOREM. (Rauch) *Let S^n be an n-dimensional sphere with constant curvature* 1 *and M be an n-dimensional Riemannian manifold* $(n \geqq 2)$. *Let c: $[0, l] \to M$ and $\tilde{c}$; $[0, l] \to S^n$ be geodesics. Let Y (resp. $\tilde{Y}$) be the Jacobi field along c (resp. $\tilde{c}$) such that*

$$Y(0) = \tilde{Y}(0) = 0\,,$$
$$\langle Y', \dot{c}\rangle(0) = \langle Y', \dot{\tilde{c}}\rangle(0) = 0\,,$$
$$\|Y'(0)\| = \|\tilde{Y}'(0)\|\,,$$

where Y' is the covariant derivative with respect to the direction $\dot{c}$ and $\|\ \|$ is the norm. Furthermore we assume $K_\sigma \geqq 1$ for all $t \in [0, l]$ and $\sigma \in G_{c(t)}$, and c has no conjugate points on the interval $(0, l)$. *Under these conditions we have the inequality*

$$\|Y(t)\| \leqq \|\tilde{Y}(t)\|$$

for all $t \in [0, l]$.

If we have $\|Y(t_0)\| = \|\tilde{Y}(t_0)\| \neq 0$ *for some* $t_0 \in (0, l]$, *the equality* $K(Y(t), \dot{c}(t)) = 1$ *holds good for all* $t \in [0, t_0]$.

(c) **Diameter.** Let $d(M)$ be the diameter of M. Then the following two theorems are well known.

THEOREM. (Myers) *Let M be a complete Riemannian manifold with sectional curvature $K \geqq k > 0$, $k =$ constant. Then M is compact and $d(M) \leqq \pi/\sqrt{k}$ holds good.*

THEOREM. (Berger [3]) *Let M be a complete Riemannian manifold with sectional curvature $K \geqq k > 0$, $k =$ constant. If $d(M)$ is larger than $\pi/2\sqrt{k}$, then M is a homotopical sphere.*

We give a new proof of Berger's theorem in the appendix.

2. **Proof of Theorem B.** We assume that M is not a homotopical sphere. By Berger's theorem we have $d(M) \leqq \pi/2$. Let m be an arbitrary point of M and U_m be the injective neighborhood of m. Then U_m is contained in the open ball with center 0 in M_m and radius $\pi/2$. By using (1), (2), Rauch's comparison theorem and a property of Gramian determinant we have

$$\operatorname{vol} M = \int_{Um} \theta \cdot v_m \leqq \frac{1}{2} \operatorname{vol} S^n .$$

This contradicts our assumption: $\operatorname{vol} M > (1/2) \operatorname{vol} S^n$. Hence M is a homotopical sphere.

3. **Proof of Theorem C.** We assume that M is not a homotopical sphere. By Theorem B and Berger's theorem we have $\operatorname{vol} M = (1/2) \operatorname{vol} S^n$ and $d(M) \leqq \pi/2$. Let m be an arbitrary point of M and U_m be the injective neighborhood of m. Then U_m is contained in the open ball with center 0 in M_m and radius $\pi/2$. By using (1), (2), Rauch's comparison theorem and a property of Gramian determinant we have

$$\operatorname{vol} M \leqq \frac{1}{2} \operatorname{vol} S^n .$$

On the other hand we have $\operatorname{vol} M = (1/2) \operatorname{vol} S^n$. Hence it follows from (1), (2) and Rauch's comparison theorem that U_m coincides with the open ball with center 0 and radius $\pi/2$, and for any geodesic arc $c: [0, \pi/2] \rightarrow M$ starting from m we have $K_\sigma = 1$ for all $\sigma \in G_c$. In particular we have $K_\sigma = 1$ for all $\sigma \in G_m$. So M is a space of constant curvature 1. Since $\operatorname{vol} M$ is equal to $(1/2) \operatorname{vol} S^n$, M is isometric to a real projective space with constant curvature 1.

4. **Appendix: Proof of Berger's theorem.** The second author gave previously a new proof of Berger's theorem in Japanese [9], which we reproduce here. We divide the proof into several steps.

(i) *M is simply connected.*

PROOF OF (i). We assume that M is not simply connected. Let $\widetilde{M} \xrightarrow{\pi} M$ be the universal covering manifold and $\widetilde{M}$ have the natural Riemannian metric induced by the Riemannian metric on M. Then we have the inequalities $K \geqq k > 0$ for all $\sigma \in G_{\widetilde{M}}$. Let p and q be two points on M such that $d(p, q) = d(M)$. Since M is not simply connected, the set $\pi^{-1}(p)$ contains at least two different points $\widetilde{p}_1$ and $\widetilde{p}_2$. Let $\widetilde{G}$ be a shortest geodesic arc joining $\widetilde{p}_1$ to $\widetilde{p}_2$. By Myer's theorem we have the inequality $L(\widetilde{G}) \leqq \pi/\sqrt{k}$, where $L(\widetilde{G})$ is the length of $\widetilde{G}$. Then the curve $G = \pi(\widetilde{G})$ is a geodesic loop starting at p and satisfies the inequalities $L(G) = L(\widetilde{G}) \leqq \pi/\sqrt{k}$. We denote G by $c\colon [0, l] \to M$, $l = L(G)$, $c(0) = c(l) = p$. From now on we mean the parameter of geodesic by the arc length measured from its initial point. Because of $d(p, q) = d(M)$ there exists a shortest geodesic $a\colon [0, m] \to M$ which joins p to q and satisfies $a(0) = p$, $a(m) = q$ and $\langle \dot{c}(0), \dot{a}(0) \rangle \geqq 0$ (c.f [2], [8]), where $\dot{a}(0)$ is the unit tangent vector to the curve a at $a(0)$. Let r_0 be a nearest point on G to q, i.e, $d(q, r_0) = d(q, G)$. Then we may assume $r_0 \neq p$, because of $d(p, q) = d(M)$. We denote a shortest geodesic between r_0 and q by $b\colon [0, u] \to M$, $b(0) = c(s_0) = r_0$, $b(u) = q$. Then we have $\langle \dot{b}(0), \dot{c}(s_0) \rangle = 0$ and $u \leqq \pi/2\sqrt{k}$, (cf. [1]). Two points p and r_0 divide G into two subarcs. We denote the shorter one by G_1. For instance we assume that G_1 is $c\colon [0, s_0] \to M$. Let $0 = s_l < s_{l-1} < \cdots < s_1 < s_0$ be a subdivision such that each subarc $c|[s_i, s_{i-1}]$ is a shortest geodesic. We put $c(s_i) = r_i$ $(i = 0, 1, \cdots, l)$. In particular we have $r_l = p$. Now we construct geodesic triangles $\varDelta\hat{q}\hat{r}_0\hat{r}_1, \varDelta\hat{q}\hat{r}_1\hat{r}_2, \cdots, \varDelta\hat{q}\hat{r}_{l-1}\hat{r}_l$ on a 2-dimensional sphere $S^2(k)$ with constant curvature k, each of which is isometric to geodesic triangles $\varDelta qr_0r_1, \varDelta qr_1r_2, \cdots, \varDelta qr_{l-1}r_l$ on M: the corresponding sides of the corresponding triangles have same length respectively. And we attach geodesic triangles $\varDelta\hat{q}\hat{r}_0\hat{r}_1, \varDelta\hat{q}\hat{r}_1\hat{r}_2, \cdots, \varDelta\hat{q}\hat{r}_{l-1}\hat{r}_l$ on $S^2(k)$ and obtain a geodesic polygon $\hat{q}\hat{r}_0\hat{r}_1 \cdots \hat{r}_{l-1}\hat{r}_l$ on $S^2(k)$. We may consider the point $\hat{q}$ as the north pole of $S^2(k)$. By using Toponogov's comparison theorem (c.f [6]) we have the following relations of the angles:

(a) $\measuredangle \hat{q}\hat{r}_0\hat{r}_1 \leqq \measuredangle qr_0r_1 = \pi/2$,

(b) $\measuredangle \hat{r}_0\hat{r}_1\hat{r}_2 = \measuredangle \hat{r}_0\hat{r}_1\hat{q} + \measuredangle \hat{q}\hat{r}_1\hat{r}_2 \leqq \measuredangle r_0r_1q + \measuredangle qr_1r_2 = \pi$, $\measuredangle \hat{r}_1\hat{r}_2\hat{r}_3 \leqq \pi, \cdots, \measuredangle \hat{r}_{l-2}\hat{r}_{l-1}\hat{r}_l \leqq \pi$,

(c) the length of the geodesic polygon $\hat{r}_0\hat{r}_1 \cdots \hat{r}_l = L(G_1) \leqq \pi/2\sqrt{k}$.

By the relations (a), (b) and (c) we can see that the point $\hat{r}_l$ is contained in the northern hemisphere of $S^2(k)$. Hence we have $d(p, q) = d(\hat{q}, \hat{r}_l) \leqq \pi/2\sqrt{k}$. This contradicts our assumption $d(p, q) = d(M) > \pi/2\sqrt{k}$. M is simply connected.

(ii) *For two points r and s on M let $\Omega_{r,s}$ be the set of all piecewise* *differentiable curves joining r to s. Let r and s be two points on M such that* $\Omega_{r,s}$ *be non-degenerate. If M is not a homotopical sphere,* $\Omega_{r,s}$ *contains a geodesic with length* $l, \pi/\sqrt{\alpha} \leqq l \leqq \pi/\sqrt{k}$, *where* $\alpha = \mathrm{Max}_{\sigma \in G_M} K_\sigma$.

PROOF OF (ii). If $\Omega_{r,s}$ contains only geodesics with length l, $l > \pi/\sqrt{k}$ or $l < \pi/\sqrt{\alpha}$, then their indices are not less than $n-1$ or equal to 0. By the fundamental theorem of Morse theory we have $\pi_i(\Omega_{r,s}) = 0, 1 \leqq i \leqq n-2$. On the other hand, by homotopy exact sequence of path space we have $\pi_i(M) = \pi_{i-1}(\Omega_{r,s})$. Hence we have $\pi_i(M) = 0, 2 \leqq i \leqq n-1$. Since M is simply connected, we have $\pi_i(M) = 0, 1 \leqq i \leqq n-1$. Hence M is a homotopical sphere. This contradicts our assumption. Hence $\Omega_{r,s}$ contains a geodesic with length $l, \pi/\sqrt{\alpha} \leqq l \leqq \pi/\sqrt{k}$.

(iii) *If M is not a homotopical sphere, we have a geodesic loop at p with length* $l \leqq \pi/\sqrt{k}$ *for all points* $p \in M$.

PROOF OF (iii). The set $\{q \in M \mid \Omega_{p,q}$ is non-degenerate$\}$ is dense in M. Let $\{q_i\}_{i=1,2,\dots}$ be a sequence of points on M such that each Ω_{p,q_i} is non-degenerate and the sequence $(q_i)_{i=1,2,\dots}$ converges to the point p. By (ii) we have a geodesic c_i of Ω_{p,q_i}, $i = 1, 2, \cdots$ whose length l satisfies $\pi/\sqrt{\alpha} \leqq l \leqq \pi/\sqrt{k}$. We can choose a converging subsequence of $\{c_i\}$. The limit geodesic c is a geodesic loop at p with length $l, \pi/\sqrt{\alpha} \leqq l \leqq \pi/\sqrt{k}$.

(iv) *Proof of Berger's theorem.*

We assume that M is not a homotopical sphere. Let p and q be two points on M such that $d(p, q) = d(M)$. By (iii) we have a geodesic loop at p with length $l, \pi/\sqrt{\alpha} \leqq l \leqq \pi/\sqrt{k}$. By the same argument as (i), we have $d(M) \leqq \pi/2\sqrt{k}$. This contradicts our assumption. Hence M is a homotopical sphere.

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DEPARTMENT OF SCIENCE
KYUSHU UNIVERSITY
AND
DEPARTMENT OF TECHNOLOGY
KYUSHU UNIVERSITY
FUKUOKA, JAPAN