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ON THE HOLONOMY GROUPS OF KÄHLERIAN MANIFOLDS WITH VANISHING BOCHNER CURVATURE TENSOR

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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1. Introduction. Let (M, J, g) be a Kählerian manifold of complex dimension n with the almost complex structure J and the Kählerian metric g.

S. Bochner [1] introduced so called Bochner curvature tensor B on M as follows;

$$\begin{aligned} B(X, Y) = R(X, Y) &- \frac{1}{2n+4}[R^1X \wedge Y + X \wedge R^1Y + R^1JX \wedge JY \\ &+ JX \wedge R^1JY - 2g(JX, R^1Y)J - 2g(JX, Y)R^1 \circ J] \\ &+ \frac{\operatorname{trace} R^1}{(2n+4)(2n+2)}[X \wedge Y + JX \wedge JY - 2g(JX, Y)J] \end{aligned}$$

for any tangent vectors X and Y, where R and R^1 are the Riemannian curvature tensor of M and a field of symmetric endomorphism which corresponds to the Ricci tensor R_1 of M, that is, $g(R^1X, Y) = R_1(X, Y)$, respectively. $X \wedge Y$ denotes the endomorphism which maps Z upon $g(Y, Z)X - g(X, Z)Y$.

But we do not know what kind of transformations in M leave B invariant [10].

The purpose of the present paper is to classify the restricted homogeneous holonomy group of M with vanishing B.

THEOREM. *Let (M, J, g) be a connected Kählerian manifold of complex dimension n $(n \geqq 2)$ with vanishing Bochner curvature tensor. Then its restricted homogeneous holonomy group H_{x_0} at some point $x_0 \in M$ is in general the unitary group $U(n)$* [10]. *If H_{x_0} is not $U(n)$, then we can classify into the following two cases:*
(I) *H_{x_0} is identity and M is locally flat.*
(II) *H_{x_0} is $U(k) \times U(n-k)$ and M is a locally product manifold of an k-dimensional space of constant holomorphic sectional curvature K and an $(n-k)$-dimensional space of constant holomorphic sectional curvature $-K$ $(K \neq 0)$.*

The above theorem seems to be a Kählerian analogue of Kurita's theorem for the holonomy groups of conformally flat Riemannian manifolds [6].

2. Preliminaries. Let (M, J, g) be a Kählerian manifold with vanishing B. Then its curvature tensor R is written as follows;

$$(2.1)\quad \begin{aligned} R(X, Y) = \frac{1}{2n+4}[&R^1X \wedge Y + X \wedge R^1Y + R^1JX \wedge JY \\ &+ JX \wedge R^1JY - 2g(JX, R^1Y)J - 2g(JX, Y)R^1 \circ J] \\ &- \frac{\operatorname{trace} R^1}{(2n+4)(2n+2)}[X \wedge Y + JX \wedge JY - 2g(JX, Y)J] . \end{aligned}$$

There are following relations among g, J and R^1:

$$J^2 = -I ,$$
$$g(JX, Y) + g(X, JY) = 0 ,$$
$$R^1 \circ J = J \circ R^1 ,$$
$$g(R^1X, Y) = g(X, R^1Y) .$$

Then, at a point $x \in M$, we can take an orthonormal basis $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ of tangent space $T_x(M)$ such that J and R^1 are represented by the following $2n \times 2n$ matrices with respect to the basis;

$$(2.2)\qquad J = \begin{pmatrix} & & & -1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ & & & & & & -1 \\ 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \end{pmatrix} , \quad R^1 = \begin{pmatrix} \lambda_1 & & & & & & \\ & \lambda_2 & & & & & \\ & & \ddots & & & & \\ & & & \lambda_n & & & \\ & & & & \lambda_1 & & \\ & & & & & \lambda_2 & \\ & & & & & & \ddots \\ & & & & & & & \lambda_n \end{pmatrix} .$$

And we have

$$(2.3)\quad \begin{cases} R(e_i, Je_i) = \sigma_i e_i \wedge Je_i + \tau_i J - \dfrac{1}{n+2} R^1 \circ J & (i = 1, \cdots, n) , \\ R(e_i, e_j) = \sigma_{ij}(e_i \wedge e_j + Je_i \wedge Je_j) , & \\ R(e_i, Je_j) = \sigma_{ij}(e_i \wedge Je_j - Je_i \wedge e_j) & (i, j = 1, \cdots, n, i \neq j) , \end{cases}$$

where we have put

$$
(2.4)\qquad
\begin{cases}
\sigma_{ij} = \dfrac{1}{2(n+1)(n+2)}[(n+1)(\lambda_i+\lambda_j) - \Lambda]\ , \\
\sigma_i = \dfrac{1}{(n+1)(n+2)}[2(n+1)\lambda_i - \Lambda]\ , \\
\tau_i = \dfrac{1}{(n+1)(n+2)}[\Lambda - (n+1)\lambda_i]\ , \\
\Lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_n\ .
\end{cases}
$$

Considering $R(X, Y)$ for $X, Y \in T_x(M)$ as a linear endomorphism of $T_x(M)$, $R(e_i, e_j)$, $R(e_i, Je_j)$ and $R(e_i, Je_i)$ are represented by the following $2n \times 2n$ matrices with respect to the above basis:

$$
(2.5)\qquad R(e_i, e_j) = \sigma_{ij} M_{ij}^{(1)}\ ,
$$

where

$$
M_{ij}^{(1)} =
\begin{array}{c}
\begin{array}{cccc} i & j & n+i & n+j \end{array} \\
\begin{pmatrix}
0 & 1 & & \\
-1 & 0 & & \\
 & & 0 & 1 \\
 & & -1 & 0
\end{pmatrix}
\end{array}
\begin{array}{l} i \\ j \\ n+i \\ n+j \end{array}
$$

$$
(2.6)\qquad R(e_i, Je_j) = \sigma_{ij} M_{ij}^{(2)}\ ,
$$

where

$$
M_{ij}^{(2)} =
\begin{array}{c}
\begin{array}{cccc} i & j & n+i & n+j \end{array} \\
\begin{pmatrix}
 & & 0 & 1 \\
 & & 1 & 0 \\
0 & -1 & & \\
-1 & 0 & &
\end{pmatrix}
\end{array}
\begin{array}{l} i \\ j \\ n+i \\ n+j \end{array}
$$

$$
(2.7)\qquad R(e_i, Je_i) = 2
\begin{array}{c}
\begin{array}{cc} i & n+i \end{array} \\
\begin{pmatrix}
 & \begin{matrix} \sigma_{i1} & & & & & & \\ & \ddots & & & & & \\ & & \sigma_{ii-1} & & & & \\ & & & \sigma_i & & & \\ & & & & \sigma_{ii+1} & & \\ & & & & & \ddots & \\ & & & & & & \sigma_{in} \end{matrix} \\
\begin{matrix} -\sigma_{i1} & & & & & & \\ & \ddots & & & & & \\ & & -\sigma_{ii-1} & & & & \\ & & & -\sigma_i & & & \\ & & & & -\sigma_{ii+1} & & \\ & & & & & \ddots & \\ & & & & & & -\sigma_{in} \end{matrix} &
\end{pmatrix}
\end{array}
\begin{array}{l} i \\ n+i \end{array}\ ,
$$

where $2(\sigma_{i1} + \cdots + \sigma_{ii-1} + \sigma_i + \sigma_{ii+1} + \cdots + \sigma_{in}) = \lambda_i$. Taking the bracket

$$[R(e_i, e_j), R(e_i, Je_j)] = R(e_i, e_j) \circ R(e_i, Je_j) - R(e_i, Je_j) \circ R(e_i, e_j) ,$$

we get

$$[R(e_i, e_j), R(e_i, Je_j)] = 2\sigma_{ij}^2 M_{ij}^{(3)} , \tag{2.8}$$

where

$$M_{ij}^{(3)} = \begin{array}{c} \begin{array}{cccc} i & j & n+i & n+j \end{array} \\ \begin{pmatrix} & & 1 & 0 \\ & & 0 & -1 \\ -1 & 0 & & \\ 0 & 1 & & \end{pmatrix} \end{array} \begin{array}{l} \\ i \\ j \\ n+i \\ n+j \end{array} .$$

The real representation of the Lie algebra $u(k)$ of a unitary group $U(k)$ consists of real $2k \times 2k$ matrices in the form

$$\begin{pmatrix} P & Q \\ -Q & P \end{pmatrix}$$

where P and Q are $k \times k$ matrices satisfying ${}^tP = -P$ and ${}^tQ = Q$. The element

$$\begin{pmatrix} P & Q \\ -Q & P \end{pmatrix}$$

of $u(k)$ is an element of the Lie algebra $su(k)$ of a special unitary group $SU(k)$ if and only if $\operatorname{trace} Q = 0$.

We denote by h_x the Lie algebra of the restricted homogeneous holonomy group H_x at $x \in M$. h_x and H_x are a Lie algebra of linear endomorphisms and a group of linear transformations of $T_x(M)$, respectively. When the elements of h_x and H_x are represented by $2n \times 2n$ matrices with respect to the basis $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$, they are considered as a Lie subalgebra of $u(n)$ and a closed connected Lie subgroup of $U(n)$, respectively [2].

We denote by $U[i_1, \cdots, i_k]$ and $SU[i_1, \cdots, i_k]$ subgroups of $U(n)$ which are represented by $2n \times 2n$ matrices

$$\begin{pmatrix} U(k) & \\ & I_{n-k} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} SU(k) & \\ & I_{n-k} \end{pmatrix}$$

with respect to the basis

$$\{e_{i_1}, \cdots, e_{i_k}, Je_{i_1}, \cdots, Je_{i_k}, e_{i_{k+1}}, \cdots, e_{i_n}, Je_{i_{k+1}}, \cdots, Je_{i_n}\} ,$$

and, by $u[i_1, \cdots, i_k]$ and $su[i_1, \cdots, i_k]$, we denote the Lie algebras of $U[i_1, \cdots, i_k]$ and $SU[i_1, \cdots, i_k]$, respectively.

3. Proof of theorem. In this section, the complex dimension n of M is assumed to be greater than 2. The case $n = 2$ will be treated in the next section.

LEMMA 3.1. *At a point $x \in M$, we take a basis $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ of $T_x(M)$ satisfying* (2.2). *If σ_{ij} defined in* (2.4) *is equal to zero for any i, j $(i \neq j)$, then $R = 0$ at x.*

PROOF. The assumption of the lemma is equivalent to

$$\Lambda - (n+1)(\lambda_i + \lambda_j) = 0 \quad \text{for any} \quad i, j\ (i \neq j)\,.$$

This implies $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$ as $n \geqq 3$, that is, $R^1 = 0$. Then $R = 0$ by (2.1). q.e.d.

To prove the theorem, we first assume that M is not locally flat. By lemma 3.1, there exists at least one point $x_0 \in M$ where σ_{ij} does not vanish for some i, j $(i \neq j)$. Then, H_{x_0} contains $SU[i, j]$ by (2.5), (2.6), and (2.8). Hence, there are following two cases:

(1) H_{x_0} contains $SU(n)$.

(2) H_{x_0} does not contain $SU(n)$.

Case (1): In this case, H_{x_0} must be equal to $U(n)$ or $SU(n)$ itself, because $SU(n)$ is the only closed connected subgroup of dimension $n^2 - 1$ in $U(n)$; in fact, let us assume that $U(n)$ contains a closed connected subgroup G of dimension $n^2 - 1$ which does not coincide with $SU(n)$. Then, the dimension of $su(n) \cap \mathfrak{g}$ is $n^2 - 2$ where $\mathfrak{g}$ is the Lie algebra of G. As $SU(n)$ is compact and simple, the Killing form φ of $su(n)$ is negative definite. Thus, we can take an orthonormal (with respect to $-\varphi$) basis $\{f_1, \cdots, f_{m-1}, f_m\}$ of $su(n)$ such that $\{f_1, \cdots, f_{m-1}\}$ is a basis of $su(n) \cap \mathfrak{g}$ where $m = n^2 - 1$. Then we have

$$\varphi([f_a, f_m], f_m) = \varphi(f_a, [f_m, f_m]) = 0 \quad (1 \leqq a \leqq m-1)$$

which implies that $[f_a, f_m] \in su(n) \cap \mathfrak{g}$ as φ is definite. Of course, $[f_a, f_b] \in su(n) \cap \mathfrak{g}$ $(1 \leqq a, b \leqq m-1)$. This means that $su(n) \cap \mathfrak{g}$ is an ideal of $su(n)$ which contradicts the fact that $su(n)$ is simple[1)].

On the other hand, $H_{x_0} = SU(n)$ occurs if and only if the Ricci tensor R_1 vanishes identically by the following lemma.

LEMMA 3.2. [4] *For a Kählerian manifold M of dimension n, the restricted homogeneous holonomy group is contained in $SU(n)$ if and only if the Ricci tensor vanishes identically. But, by* (2.1), *this contradicts the assumption that M is not locally flat. Therefore, the case* (1) *occurs when and only when $H_{x_0} = U(n)$.*

1) This proof is due to T. Sakai. The authors wish to express their hearty thanks to him.

Case (2): In this case, there exist k $(2 \leqq k \leqq n-1)$ and $i_1, \cdots, i_k$ such that H_{x_0} contains $SU[i_1, \cdots, i_k]$ but does not contain $SU[i_1, \cdots, i_k, j]$ for any j. We change the indices suitably and assume that H_{x_0} contains $SU[1, \cdots, k]$ but does not contain $SU[1, \cdots, k, j]$ for any $j, j > k$.

LEMMA 3.3. *If h_{x_0} contains $su[1, \cdots, k]$ and $su[i, j]$ for some i, j satisfyiug $1 \leqq i \leqq k$ and $k+1 \leqq j \leqq n$, then h_{x_0} contains $su[1, \cdots, k, j]$.*

PROOF. We can take as bases of $su[1, \cdots, k]$ and $su[i, j]$ the sets of matrices

$$\{M_{ab}^{(1)}, M_{ab}^{(2)}, M_{12}^{(3)}, \cdots, M_{1k}^{(3)} ; \quad 1 \leqq a < b \leqq k\}$$

and

$$\{M_{ij}^{(1)}, M_{ij}^{(2)}, M_{ij}^{(3)}\} ,$$

respectively. On the other hand, we have the following equalities:

$$\begin{aligned} &[M_{pq}^{(1)}, M_{qr}^{(2)}] = -M_{pr}^{(2)} && (1 \leqq p < q < r \leqq n) , \\ &M_{1p}^{(3)} + M_{pq}^{(3)} = M_{1q}^{(3)} && (1 < p < q \leqq n) , \\ &[M_{pr}^{(2)}, M_{pr}^{(3)}] = 2M_{pr}^{(1)} && (1 \leqq p < r \leqq n) . \end{aligned}$$

This means that if h_{x_0} contains $su[1, \cdots, k]$ and $su[i, j]$, then it contains

$$\{M_{ab}^{(1)}, M_{cj}^{(1)}, M_{ab}^{(2)}, M_{cj}^{(2)}, M_{12}^{(3)}, \cdots, M_{1k}^{(3)}, M_{1j}^{(3)} ; \quad 1 \leqq a < b \leqq k, 1 \leqq c \leqq k\}$$

which is a basis of $su[1, \cdots, k, j]$. q.e.d.

By Lemma 3.3, H_{x_0} can not contain $SU[a, u]$ $(a=1, \cdots, k, u = k+1, \cdots, n)$ and we get

$$\sigma_{au} = 0 \qquad (a = 1, \cdots, k, u = k+1, \cdots, n) . \tag{3.1}$$

Then, by (2.4), we have

$$\lambda_1 = \cdots = \lambda_k (= \lambda), \lambda_{k+1} = \cdots = \lambda_n (= \mu) \tag{3.2}$$

and

$$(n+1-k)\lambda + (k+1)\mu = 0 , \tag{3.3}$$

from which we have $\lambda \neq \mu$. Hence, we have

$$\sigma_{ab} = \frac{1}{2(n+1)(n+2)}[(2n+2-k)\lambda - (n-k)\mu] \qquad (1 \leqq a < b \leqq k) ,$$

$$\sigma_{uv} = \frac{1}{2(n+1)(n+2)}[-k\lambda + (n+2+k)\mu] \qquad (k+1 \leqq u < v \leqq n) ,$$

which cannot vanish by (3.3). Hence, H_{x_0} contains $U[1, \cdots, k] \times U[k+1, \cdots, n]$ by (2.5), (2.6), (2.7) and (2.8).

Next, we take a point x in the neighborhood of x_0 and choose a basis $\{e_1, \cdots, e_n, Je_1, \cdots Je_n\}$ of $T_x(M)$ satisfying (2.2) and hence (2.3). By the continuity of characteristic roots of R^1, when x is sufficiently near x_0, we may conclude that

$$\begin{cases} \sigma_{ab} \neq 0 & (1 \leqq a < b \leqq k) , \\ \sigma_{uv} \neq 0 & (k+1 \leqq u < v \leqq n) , \end{cases} \tag{3.4}$$

as they are so at x_0. Hence, H_x contains $SU[1, \cdots, k]$ and $SU[k+1, \cdots, n]$ by (2.5), (2.6) and (2.8). If H_x contains none of $SU[1, \cdots, k, j]$, (3.1) holds good. In the case $k = n-1$, H_x contains $SU[1, \cdots, n-1]$ but does not contain $SU[1, \cdots, n]$ as H_x is isomorphic to H_{x_0} by the connectivity of M. Therefore, we consider the case $k < n-1$.

We change the indices of $e_{k+1}, \cdots, e_n, Je_{k+1}, \cdots, Je_n$, in such a way that H_x contains $SU[1, \cdots, k, k+1, \cdots, k+r]$ $(k + r \leqq n-1)$ and non of $SU[1, \cdots, k, k+1, \cdots, k+r, k+r+s]$, because H_x is isomorphic to H_{x_0}. Then we get by the repetition of the above process

$$\sigma_{uv} = 0 \quad (u = k+1, \cdots, k+r \colon v = k+r+1, \cdots, n) .$$

This contradicts (3.4). Thus we can take bases at each point of a neighborhood V of x_0 in such a way that (3.1) and (3.4) hold good with same k.

Let W be the set of the point $x \in M$ such that for a suitable basis of $T_x(M)$ satisfying (2.2), σ_{ij} does not vanish for some i, j $(i \neq j)$, which is an open set. Let W_0 be the connected component of x_0 in W. Then it follows that k (in the above argument) is constant on W_0 and that $\lambda(x)$ and $\mu(x)$ are differentiable functions on W_0 by (3.3) and the fact that $k\lambda + (n-k)\mu = (1/2)\,\mathrm{trace}\, R^1$ or $\mathrm{trace}\,(R^1 \circ R^1)$ is a differentiable function on W_0. It should be remarked that $\lambda(x) \neq \mu(x)$ at each point $x \in W_0$. We define two distributions on W_0 as follows:

$$T_1(x) = \{X \in T_x(M) \colon R^1 X = \lambda(x) X\} ,$$
$$T_2(x) = \{X \in T_x(M) \colon R^1 X = \mu(x) X\} ,$$

which are mutually orthogonal and J-invariant.

Let $X, Y \in T_1$ and $X', Y' \in T_2$. Then we have

$$\begin{cases} R(X, Y) = K[X \wedge Y + JX \wedge JY - 2g(JX, Y)J_1] , \\ R(X', Y') = -K[X' \wedge Y' + JX' \wedge JY' - 2g(JX', Y')J_2] , \\ R(X, Y') = 0 , \end{cases} \tag{3.5}$$

by (2.1), (2.3), (3.1), (3.2) and (3.3), where we have put

$$K = \frac{1}{2(n+1)(n+2)}[(2n+2-k)\lambda - (n-k)\mu]$$

which does not vanish by (3.3). J_1 and J_2 are defined by $J_1X = JX$, $J_1X' = 0$ and $J_2X = 0$, $J_2X' = JX'$, respectively.

LEMMA 3.4. *T_1 and T_2 are parallel and K is constant.*

PROOF. For any $x \in W_0$, we may choose a differentiable field of orthonormal basis $\{X_1, \cdots, X_n, JX_1, \cdots, JX_n\}$ near x in W_0 in such a way that $\{X_1, \cdots, X_k, JX_1, \cdots, JX_k\}$ and $\{X_{k+1}, \cdots, X_n, JX_{k+1}, \cdots, JX_n\}$ are bases near x in W_0 for T_1 and T_2, respectively. This choice is possible by virtue of the property $J \circ R^1 = R^1 \circ J$.

Now, in general, for a differentiable field of orthonormal basis $\{Y_1, \cdots, Y_n\}$ in a Riemannian manifold (M, g), we may put

$$\nabla_i Y_j = \nabla_{Y_i} Y_j = \sum_{k=i}^{n} A_{ijk} Y_k \,, \tag{3.6}$$

where $\nabla_i = \nabla_{Y_i}$ denotes the covariant differentiation for the Riemannian connection, and $A_{ijk} = -A_{ikj}$.

Hereafter, the indices run as follows:

$$a, b, c, \cdots = 1, \cdots, k, \quad u, v, w, \cdots = k+1, \cdots, n \,.$$

Put $X_{i^*} = JX_i$ for any i, then $A_{ijk} = A_{ij^*k^*}$, $A_{ijk^*} = -A_{ij^*k}$ and etc. by the property $\nabla J = 0$ for the Kählerian manifold M. First, we shall prove the case $2 \leqq k \leqq n-2$. Taking account of (3.5), (3.6), we have (3.7):

$$\begin{aligned}
\frac{1}{K}(\nabla_a R)(X_b, X_u) &= 2A_{abu^*}J \\
&\quad + \sum_{v=k+1}^{n} [A_{abv}(X_v \wedge X_u + X_{v^*} \wedge X_{u^*}) + A_{abv^*}(X_{v^*} \wedge X_u - X_v \wedge X_{u^*})] \\
&\quad - \sum_{c=1}^{k} [A_{auc}(X_b \wedge X_c + X_{b^*} \wedge X_{c^*}) + A_{auc^*}(X_b \wedge X_{c^*} - X_{b^*} \wedge X_c)] \,, \\
\frac{1}{K}(\nabla_b R)(X_u, X_a) &= -2A_{bau^*}J \\
&\quad + \sum_{v=k+1}^{n} [A_{bav}(X_u \wedge X_v + X_{u^*} \wedge X_{v^*}) + A_{bav^*}(X_u \wedge X_{v^*} - X_{u^*} \wedge X_v)] \\
&\quad - \sum_{c=1}^{k} [A_{buc}(X_c \wedge X_a + X_{c^*} \wedge X_{a^*}) + A_{buc^*}(X_{c^*} \wedge X_a - X_c \wedge X_{a^*})] \,, \\
\frac{1}{K}(\nabla_u R)(X_a, X_b) &= \frac{1}{K}(X_u K)(X_a \wedge X_b + X_{a^*} \wedge X_{b^*}) \\
&\quad + \sum_{v=k+1}^{n} [A_{uav}(X_v \wedge X_b + X_{v^*} \wedge X_{b^*}) + A_{uav^*}(X_{v^*} \wedge X_b - X_v \wedge X_{b^*}) \\
&\quad + A_{ubv}(X_a \wedge X_v + X_{a^*} \wedge X_{v^*}) + A_{ubv^*}(X_a \wedge X_{v^*} - X_{a^*} \wedge X_v)] \,,
\end{aligned}$$

where

$$J = -\sum_{c=1}^{k} X_c \wedge X_{c^*} - \sum_{v=k+1}^{n} X_v \wedge X_{v^*} .$$

By the second Bianchi identity, we have

$$A_{uav} = A_{uav^*} = 0 ,$$

and hence

$$A_{uva} = A_{uva^*} = A_{uv^*a} = A_{uv^*a^*} = 0 .$$

If we replace u by u^* in (3.7), we have

$$A_{u^*va} = A_{u^*va^*} = A_{u^*v^*a} = A_{u^*v^*a^*} = 0 .$$

If we replace (u, a, b) by (a, u, v) or (a^*, u, v) in (3.7), we have

$$A_{abu} = A_{abu^*} = A_{ab^*u} = A_{ab^*u^*} = 0$$

and

$$A_{a^*bu} = A_{a^*bu^*} = A_{a^*b^*u} = A_{a^*b^*u^*} = 0 .$$

Then we have $X_uK = 0$ by (3.7). Similary $X_aK = 0$. These facts show that the lemma is valid for $2 \leqq k \leqq n - 2$.

Next, we prove the case $2 \leqq k = n - 1$. The proof is accomplished, applying the second Bianchi identity to the following equalities:

$$\begin{aligned} \frac{1}{K}(\nabla_a R)(X_b, X_n) &= 2A_{abn^*}X_{n^*} \wedge X_n + 2A_{abn^*}J \\ &\quad -\sum_{c=1}^{n-1}[A_{anc}(X_b \wedge X_c + X_{b^*} \wedge X_{c^*}) + A_{anc^*}(X_b \wedge X_{c^*} - X_{b^*} \wedge X_c)] , \\ \frac{1}{K}(\nabla_b R)(X_n, X_a) &= 2A_{ban^*}X_n \wedge X_{n^*} - 2A_{ban^*}J \\ &\quad - \sum_{c=1}^{n-1} [A_{bnc}(X_c \wedge X_a + X_{c^*} \wedge X_{a^*}) \\ &\quad + A_{bnc^*}(X_{c^*} \wedge X_a - X_c \wedge X_{a^*})] , \\ \frac{1}{K}(\nabla_n R)(X_a, X_b) &= \frac{1}{K}(X_nK)(X_a \wedge X_b + X_{a^*} \wedge X_{b^*}) \\ &\quad + [A_{nan}(X_n \wedge X_b + X_{n^*} \wedge X_{b^*}) + A_{nan^*}(X_{n^*} \wedge X_b - X_n \wedge X_{b^*}) \\ &\quad + A_{nbn}(X_a \wedge X_n + X_{a^*} \wedge X_{n^*}) + A_{nbn^*}(X_a \wedge X_{n^*} - X_{a^*} \wedge X_n)] , \\ \frac{1}{K}(\nabla_a R)(X_n, X_{n^*}) &= -\frac{4}{K}(X_aK)X_n \wedge X_{n^*} \\ &\quad - 4\sum_{c=1}^{n-1} [A_{anc}X_c \wedge X_{n^*} + A_{anc^*}X_{c^*} \wedge X_{n^*} \\ &\quad + A_{an^*c}X_n \wedge X_c + A_{anc}X_n \wedge X_{c^*}] , \end{aligned}$$

$$\frac{1}{K}(\nabla_n R)(X_{n^*}, X_a) = 2A_{nan}(X_{n^*} \wedge X_n + J)$$
$$-\sum_{c=1}^{n-1}[A_{nn^*c}(X_c \wedge X_a + X_{c^*} \wedge X_{a^*}) + A_{nn^*c^*}(X_{c^*} \wedge X_a - X_c \wedge X_{a^*})]$$
$$\frac{1}{K}(\nabla_{n^*} R)(X_a, X_n) = 2A_{n^*an^*}(X_{n^*} \wedge X_n + J)$$
$$-\sum_{c=1}^{n-1}[A_{n^*nc}(X_a \wedge X_c + X_{a^*} \wedge X_{c^*})$$
$$+ A_{n^*nc^*}(X_a \wedge X_{c^*} - X_{a^*} \wedge X_c)]\,,$$

where

$$J = -X_n \wedge X_{n^*} - \sum_{c=1}^{n-1} X_c \wedge X_{c^*}\,. \qquad \text{q.e.d.}$$

Thus, W_0 is a locally product manifold of a k-dimensional space of constant holomorphic sectional curvature $4K$ and an $(n-k)$-dimensional space of constant holomorphic sectional curvature $-4K$ [3]. Therefore, by the connectivity of M and the continuity argument for the characteristic roots of R^1, it follows that $W_0 = M$. In particular, M is locally symmetric. On the other hand, it is easily seen that the restricted homogeneous holonomy group of an m-dimensional space of non-zero constant holomorphic sectional curvature is $U(m)$. Then, $H_{x_0} = U(k) \times U(n-k)$ [7], [5; vol. 1, p. 263].

4. **Case $n = 2$.** To prove the theorem for $n = 2$, we assume that M is not locally flat and that H_x at $x \in M$ does not coincide with $U(2)$. Then, H_x can not contain $SU(2)$ by the same argument as in the last section. Then, we have $\sigma_{12} = (1/12)(\lambda_1 + \lambda_2) = 0$ at any point of M. And there exists at least one point x_0 such that $\lambda_1\lambda_2 < 0$. Let W_0 be the connected component containing x_0 of $W = \{x \in M;\ \lambda_1\lambda_2 < 0 \text{ at } x\}$. $\lambda_1\ (= -\lambda_2 \neq 0)$ is a differentiable function on W_0. We have following two distributions on W_0:

$$T_1(x) = \{X \in T_x(M)\ ;\quad R^1X = \lambda_1 X\}$$
$$T_2(x) = \{X' \in T_x(M)\ ;\quad R^1X' = \lambda_2 X'\}$$

which are J-invariant. Let $X, Y \in T_1$ and $X', Y' \in T_2$. Then we have

$$\begin{cases} R(X, Y) = 4\lambda_1 X \wedge Y\,, \\ R(X', Y') = -4\lambda_1 X' \wedge Y'\,, \\ R(X, X') = 0\,. \end{cases}$$

From the last equations, we can easily see that T_1 and T_2 are parallel and λ_1 is constant. Hence, $W_0 = M$ and $H_{x_0} = U(1) \times U(1)$.

References

[1] S. Bochner, Curvature and Betti numbers II. Ann. of Math., 50 (1949), 77-93.

[2] A. Borel and A. Lichnerowicz, Groupes d'holonomie des variétés Riemanniennes, C. R. Acad. Sci. Paris, 234 (1952), 1835-1837.

[3] J-I. Hano and Y. Matsushima, Some studies of Kählerian homogeneous spaces, Nagoya Math. J., 11 (1957), 77-92.

[4] H. Iwamoto, On the structure of Riemannion spaces whose holonomy groups fix a null system, Tôhoku Math. J., 1 (1950), 109-135.

[5] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vols. I, II, Intersci. Publ., 1963, 1969.

[6] M. Kurita, On the holonomy group of conformally flat Riemannian manifold, Nagoya Math. J., 9 (1955), 161-171.

[7] A. Nijenhuis, On the holonomy group of linear connections IA, IB, Indag. Math., 15 (1953), 233-249; II, 16 (1954), 17-25.

[8] K. Sekigawa and H. Takagi, On conformally flat spaces satisfying a certain condition on the Ricci tensor, Tôhoku Math. J., 23 (1971), 1-11.

[9] S. Tachibana, On the Bochner curvature tensor, Nat. Sci. Rep. of the Ochanomizu Univ., 18 (1967), 15-19.

[10] S. Tachibana and R. C. Liu, Notes on Kählerian metric with vanishing Bochner curvature tensor, Kōdai Math. Sem. Rep., 22 (1970), 313-321.

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