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ON ALMOST COTANGENT STRUCTURES

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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In two previous papers [7], [8], the present authors studied a manifold which admits a structure called homogeneous contact structure and at the same time an almost product structure. See also [1], [2], [3], [4], [5], and [6].

On the other hand, Patterson and one of the present authors studied in [9], [10] the cotangent bundles, especially, vertical, complete and horizontal lifts from a manifold to its cotangent bundle.

The main purpose of the present paper is to define the almost cotangent structure and to study its properties in the light of the papers quoted above.

1. Cotangent bundle and cotangent structure. Let M be an n-dimensional differentiable manifold, $T_P(M)$ the tangent space at $P \in M$, and $T_P^*(M)$ the dual space of $T_P(M)$. Then the fibre bundle ${}^cT(M)$ with the base space M and the fibre $T_P^*(M)$ on P is called the cotangent bundle of M. The cotangent bundle ${}^cT(M)$ is a $2n$-dimensional differentiable manifold with a special structure.

We cover the M by a system of coordinate neighborhoods $\{U; x^h\}(h, i, j, \cdots = 1, 2, \cdots, n)$. Then an element of $T_P^*(M)$ at P being a covariant vector at P, it is represented by its components p_i with respect to the natural frame defined by the local coordinate system. Thus a point of the cotangent bundle ${}^cT(M)$ is represented by (x^h, p_i) in terms of the local coordinate system thus introduced. We call this local coordinate system that naturally induced from the local coordinate system in M. When we fix the values of x^h and give arbitrary values to p_i, we get a fibre of ${}^cT(M)$ at $P(x^h)$.

We put $p_i = x^{i^*}$ and represent a point of ${}^cT(M)$ by $(x^1, x^2, \cdots, x^n, x^{1^*}, x^{2^*}, \cdots, x^{n^*})$, or (x^A), where, here and in the sequel, the indices h^*, $i^*, j^*, \cdots$ run over the range $\{1^*, 2^*, \cdots, n^*\}$ and the indices $A, B, C, \cdots$ the range $\{1, 2, \cdots, n, n+1, \cdots, 2n\}$, i^* being equal to $n+i$. We use, in addition to the ordinary summation convention, also the summation convention such as $a^{i^*}b^i = \sum a^{i^*}b^i = a^{n+1}b^1 + a^{n+2}b^2 + \cdots + a^{2n}b^n$.

Now, in ${}^cT(M)$ regarded as a $2n$-dimensional differentiable manifold, its fibre can be interpreted as the integral manifold of the n-dimensional distribution given by

$$dx^1 = 0, dx^2 = 0, \cdots, dx^n = 0 . \tag{1.1}$$

To this distribution, corresponds a simple pseudo-n-form

$$\tilde{\omega} = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n , \tag{1.2}$$

where a pseudo-n-form means an n-form which is defined only up to a non-vanishing scalar multiple.

In general, when a simple pseudo-n-form $\tilde{\omega}$ is written locally in the form $\tilde{\omega} = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n$ of a product of n 1-forms, then the form $\tilde{\omega}$ defines a distribution given by $\omega^1 = 0, \omega^2 = 0, \cdots, \omega^n = 0$, and conversely, when a global distribution is defined in each neighborhood by n linearly independent equations $\omega^1 = 0, \omega^2 = 0, \cdots, \omega^n = 0$, then a simple pseudo-$n$-form $\tilde{\omega}$ is defined.

Let π be the projection ${}^cT(M) \to M$ of the fibre bundle ${}^cT(M)$. To each point $b = (x^h, x^{h^*})$ of ${}^cT(M)$ corresponds a covariant vector p_i ($p_i = x^{i^*}$) at a point πb whose coordinates are (x^h). Thus there exists in ${}^cT(M)$ a 1-form

$$\omega = x^{i^*} dx^i . \tag{1.3}$$

We call this 1-form *the natural 1-form* of ${}^cT(M)$. In the sequel, we exclude from our considerations the points of ${}^cT(M)$ such that $p_1 = p_2 = \cdots = p_n = 0$.

Thus, there exist in ${}^cT(M)$ the natural 1-form ω and a simple pseudo-n-form $\tilde{\omega}$. The natural 1-form ω is completely determined but $\tilde{\omega}$ is determined only up to a non-vanishing scalar multiple. As is easily seen, they satisfy

$$\omega \neq 0, (d\omega)^n \neq 0, \tilde{\omega} \neq 0 , \tag{1.4}$$

$$\omega \wedge \tilde{\omega} = 0 . \tag{1.5}$$

When we regard the cotangent bundle ${}^cT(M)$ as a $2n$-dimensional differentiable manifold with a special structure, we call this structure the *cotangent structure*.

The cotangent structure contains the natural 1-form ω and a simple pseudo-n-form $\tilde{\omega}$ satisfying (1.4) and (1.5), but it satisfies also other conditions.

2. Almost cotangent structure and canonical coordinates. Since the cotangent structure is a structure which a cotangent bundle possesses, if we use the local coordinate system (x^h, p_i) in the cotangent bundle, the properties of ω and $\tilde{\omega}$ are completely determined by (1.2) and (1.3). The properties (1.4) and (1.5) are obtained in this way. Though we have from (1.2) $d\tilde{\omega} = 0$ locally, since we can multiply $\tilde{\omega}$ by an arbitrary non-vanishing scalar α, we have only $d\tilde{\omega} = \alpha \wedge \tilde{\omega}$ globally.

As a structure more general than the cotangent structure, we consider a structure which satisfies only (1.4) and (1.5) and will call it an almost cotangent structure. In the sequel we shall study properties of an almost cotangent structure and try to characterize the cotangent structure as almost cotangent structure satisfying certain conditions. We state

DEFINITION. Suppose that, in a $2n$-dimensional differentiable manifold, there are given globally a 1-form ω and a simple pseudo-n-form $\tilde{\omega}$ which satisfy

$$\omega \doteqdot 0,\ \tilde{\omega} \doteqdot 0,\ (d\omega)^n \neq 0 \tag{2.1}$$

and

$$\omega \wedge \tilde{\omega} = 0 . \tag{2.2}$$

Then the structure defined by $(\omega, \tilde{\omega})$ is called an *almost cotangent structure.*

This means that a $2n$-dimensional differentiable manifold M with an almost cotangent structure is a manifold with a global 1-form ω satisfying $\omega \doteqdot 0$, $(d\omega)^n \doteqdot 0$ everywhere and such that M is covered by a system of open neighborhoods U_λ, $\lambda \in \{\lambda\}$, in each of which there exists a simple n-form $\tilde{\omega}_\lambda$ satisfying $\tilde{\omega}_\lambda \doteqdot 0$, $\omega \wedge \tilde{\omega}_\lambda = 0$ in U_λ and satisfying $\tilde{\omega}_\lambda = f_{\lambda\kappa}\tilde{\omega}_\kappa$ (κ: not summed) if $U_\lambda \cap U_\kappa \doteqdot \varnothing$, $f_{\lambda\kappa}$ being a differentiable function such that $f_{\lambda\kappa}f_{\kappa\lambda} = 1$ in $U_\lambda \cap U_\kappa$. Besides, we are concerned only with properties which are preserved by any reversible change of $\tilde{\omega}_\lambda$, $\tilde{\omega}_\lambda \to f_\lambda\tilde{\omega}_\lambda$.

If $(\omega, \tilde{\omega})$ is an almost cotangent structure, then ω and $\tilde{\omega}$ satisfy $\omega \doteqdot 0$, $(d\omega)^n \doteqdot 0$, and consequently, following E. Cartan, we can choose a local coordinate system (x^A) in which ω can be written as

$$\omega = x^{i^*}dx^i . \tag{2.3}$$

In this local coordinate system, ω^i in $\tilde{\omega} = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n$ can be expressed in the form

$$\omega^i = X_j^i dx^j + X_{j^*}^i dx^{j^*} . \tag{2.4}$$

In this case, if $\det (X_j^i) \neq 0$, then we can choose $\omega^1, \omega^2, \cdots, \omega^n$ in such a way that ω^i is of the form

$$\omega^i = dx^i + A^{ih}dx^{h^*} , \tag{2.5}$$

because $\tilde{\omega}$ is determined up to a non-vanishing scalar factor.

If the rank m of the matrix (X_j^i) is less than n, we can not put ω^i in the form (2.5) only choosing ω^i suitably. But we can prove the following theorem.

THEOREM 2.1. *In a $2n$-dimensional differentiable manifold M^{2n} with an almost cotangent structure $(\omega, \tilde{\omega})$, there exists a system of coordinate neighborhoods U_λ, $\lambda \in \{\lambda\}$, in each of which we have*

$$
\begin{aligned}
\omega &= x_\lambda^{i^*} dx_\lambda^i , \\
\omega_\lambda^i &= dx_\lambda^i + \Lambda_\lambda^{ih} dx_\lambda^{h^*} , \qquad (2.6)\\
\tilde{\omega}_\lambda &= \omega_\lambda^1 \wedge \omega_\lambda^2 \wedge \cdots \wedge \omega_\lambda^n ,
\end{aligned}
$$

where $\tilde{\omega}_\lambda$ is a simple n-form representing the structure $\tilde{\omega}$ in U_λ.

We prove Theorem 2.1 in §3. To state the following Theorem 2.2, we give here the definition of canonical coordinates and that of almost cotangent manifold.

DEFINITION. A local coordinate system in which equations (2.6) of Theorem 2.1 hold is called a canonical coordinate system of the almost cotangent structure.

DEFINITION. A manifold in which an almost cotangent structure is defined is called an almost cotangent manifold.

We now state

THEOREM 2.2. *If we take a canonical coordinate system in an almost cotangent manifold, then $\omega = x^{i^*}\omega^i$ and Λ^{ih} appearing in* (2.6) *satisfy*

$$x^{i^*}\Lambda^{ih} = 0 . \qquad (2.7)$$

PROOF. Since $\omega \wedge \tilde{\omega} = 0$, ω can be written as

$$\omega = X_i \omega^i .$$

Substituting

$$\omega^i = dx^i + \Lambda^{ih} dx^{h^*}$$

into this, we find

$$x^{i^*} dx^i = X_i dx^i + X_i \Lambda^{ih} dx^{h^*} .$$

Comparing first the coefficients of dx^i, we find $x^{i^*} = X_i$. Comparing next the coefficients of dx^{h^*}, we find

$$X_i \Lambda^{ih} = 0 .$$

Thus (2.7) is proved.

DEFINITION. Λ^{ih} appearing in (2.6) are called coefficients of the almost cotangent struture.

3. Proof of Theorem 2.1.

1^0 **Preliminaries.** Suppose that the rank of the matrix $X = (X^i_j)$ be $m < n$. Then we can represent X as

$$X^i_j = P^i_r Q^r_j \qquad (r = 1, 2, \cdots, m)$$

where $P = (P^i_j)$ and $Q = (Q^i_j)$ are regular matrices. In this section, we assume that

$$p, q, r = 1, 2, \cdots, m\,,$$
$$t, u, v = m+1, m+2, \cdots, n$$

and use the summation convention also for these indices. Let S be the inverse matrix of P and change the choice of ω^i so that $S^i_k \omega^k$ may now be written as ω^i, then, in stead of (2.4), we have

$$\begin{aligned} \omega^p &= Q^p_j dx^j + X^p_{j^*} dx^{j^*}\,, \\ \omega^t &= \phantom{Q^p_j dx^j +{}} X^t_{j^*} dx^{j^*}\,. \end{aligned}$$

Also, since, changing suitably the order of the coordinates $x^1, x^2, \cdots, x^n$, we can assume that $\det(Q^p_q) \neq 0$, we can use the inverse matrix of (Q^p_q) and can put the above equations in the form

$$\begin{aligned} \omega^p &= dx^p + X^p_u dx^u + X^p_{j^*} dx^{j^*}\,, \\ \omega^t &= \phantom{dx^p + X^p_u dx^u +{}} X^t_{j^*} dx^{j^*}\,. \end{aligned} \tag{3.1}$$

Thus, since $\tilde{\omega} \neq 0$, $\omega \wedge \tilde{\omega} = 0$, we see that the rank of the matrix

$$\begin{pmatrix} x^{1^*} \cdots x^{m^*} & x^{(m+1)^*} \cdots x^{n^*} & 0 \cdots\cdots\cdots 0 \\ 1 \cdots\cdots 0 & X^1_{m+1} \cdots X^1_n & X^1_{1^*} \cdots\cdots X^1_{n^*} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0 \cdots\cdots 1 & X^m_{m+1} \cdots X^m_n & X^m_{1^*} \cdots\cdots X^m_{n^*} \\ 0 \cdots\cdots & \cdots\cdots 0 & X^{m+1}_{1^*} \cdots X^{m+1}_{n^*} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0 \cdots\cdots & \cdots\cdots 0 & X^n_{1^*} \cdots\cdots X^n_{n^*} \end{pmatrix} \tag{3.2}$$

is n. Also, multiplying the second row of the matrix (3.2) by $-x^{1^*}$, the third row by $-x^{2^*}, \cdots$, the $(m+1)$st row by $-x^{m^*}$, and adding these rows thus obtained to the first row, we see that

$$x^{t^*} - x^{p^*} X^p_t = 0\,, \tag{3.3}$$

because the matrix thus obtained is still of rank n and the rank of the matrix $(X^t_{j^*})$ is $n - m$. Thus, since the rank of the matrix

$$\begin{bmatrix} 0\cdots\cdots 0 & 0\cdots\cdots\cdots 0 & -x^{p^*}X^p_{1^*}\cdots\cdots-x^{p^*}X^p_{n^*} \\ 1\cdots\cdots 0 & X^1_{m+1}\cdots X^1_n & X^1_{1^*}\cdots\cdots\cdots\cdots\cdots X^1_{n^*} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0\cdots\cdots 1 & X^m_{m+1}\cdots X^m_n & X^m_{1^*}\cdots\cdots\cdots\cdots\cdots X^m_{n^*} \\ 0\cdots\cdots 0 & 0\cdots\cdots\cdots 0 & X^{m+1}_{1^*}\cdots\cdots\cdots\cdots X^{m+1}_{n^*} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0\cdots\cdots 0 & 0\cdots\cdots\cdots 0 & X^n_{1^*}\cdots\cdots\cdots\cdots\cdots X^n_{n^*} \end{bmatrix}$$

is also n, we see that there exist $X_{m+1}, \cdots, X_n$ satisfying

$$x^{p^*}X^p_{j^*} = X_t X^t_{j^*} . \tag{3.4}$$

The condition that (2.3) keeps the same form, that is,

$$x^{i^*}dx^i = x^{i^{*\prime}}dx^{i'} \tag{3.5}$$

under the transformation of local coordinates

$$(x^i, x^{i^*}) \longrightarrow (x^{i'}, x^{i^{*\prime}})$$

is given by

$$x^{i^{*\prime}}\frac{\partial x^{i'}}{\partial x^{j^*}} = 0, \quad x^{i^{*\prime}}\frac{\partial x^{i'}}{\partial x^j} = x^{j^*} . \tag{3.6}$$

The transformation of variables $(x^i, x^{i^*}) \longrightarrow (x^{i'}, x^{i^{*\prime}})$ satisfying these conditions is called a homogeneous contact transformation.

It is well known [2] that for a homogeneous contact transformation we have

$$\frac{\partial x^{i'}}{\partial x^j} = \frac{\partial x^{j^*}}{\partial x^{i^{*\prime}}}, \quad \frac{\partial x^{i'}}{\partial x^{j^*}} = -\frac{\partial x^j}{\partial x^{i^{*\prime}}}, \quad \frac{\partial x^{i^{*\prime}}}{\partial x^j} = -\frac{\partial x^{j^*}}{\partial x^{i'}} . \tag{3.7}$$

If ω^i do not change under this transformation, from

$$X^i_j dx^j + X^i_{j^*}dx^{j^*} = X^i_{j'}dx^{j'} + X^i_{j^{*\prime}}dx^{j^{*\prime}} ,$$

we obtain

$$X^i_{j'} = X^i_k\frac{\partial x^k}{\partial x^{j'}} + X^i_{k^*}\frac{\partial x^{k^*}}{\partial x^{j'}} . \tag{3.8}$$

Thus the rank m of the matrix (X^i_j) changes depending on the choice of local coordinate system. To express this fact clearly we denote by $m(x^A)$ the rank of (X^i_j) when we use the local coordinate system (x^A).[1)]

2^0 **A lemma.** We prove the following

[1)] Any local coordinate system we consider in the sequel is assumed to keep the form (2.3).

LEMMA 3.1. *In a neighborhood of a point P of an almost cotangent manifold, let* (x^A) *be a local coordinate system in which* (3.1) *holds for a certain m such that* $m < n$. *If* $x^{m^*} \neq 0$ *and* $X^{m+1}_{(m+1)^*} \neq 0$, *then there exists a local coordinate system* $(x^{A'})$ *in a neighborhood of P such that the rank* $m(x^{A'})$ *of the matrix* $(X^i_{j'})$ *is greater than m.*

PROOF. We here use indices having the following ranges:

$$\kappa, \lambda, \mu = 1, 2, \cdots, m-1; \quad \xi, \eta, \zeta = m+2, \cdots, n .$$

We assume that for $2n-4$ variables x^κ, x^{κ^*}, x^ξ, and x^{ξ^*}, we have

$$x^{\kappa'} = x^\kappa, \quad x^{\kappa^{*\prime}} = x^{\kappa^*}, \quad x^{\xi'} = x^\xi, \quad x^{\xi^{*\prime}} = x^{\xi^*}$$

and for remaining 4 variables, $x^{m'}$, $x^{(m+1)'}$, $x^{m^{*\prime}}$ and $x^{(m+1)^{*\prime}}$ are functions of x^m, x^{m+1}, x^{m^*} and $x^{(m+1)^*}$ only, but the functions $x^{A'}$ satisfy (3.5). Then, from (3.1) and (3.8), we have the following equations:

$$X^\lambda_{\mu'} = \delta^\lambda_\mu ,$$

$$X^\lambda_{m'} = X^\lambda_{m+1}\frac{\partial x^{m+1}}{\partial x^{m'}} + X^\lambda_{m^*}\frac{\partial x^{m^*}}{\partial x^{m'}} + X^\lambda_{(m+1)^*}\frac{\partial x^{(m+1)^*}}{\partial x^{m'}} ,$$

$$X^\lambda_{(m+1)'} = X^\lambda_{m+1}\frac{\partial x^{m+1}}{\partial x^{(m+1)'}} + X^\lambda_{m^*}\frac{\partial x^{m^*}}{\partial x^{(m+1)'}} + X^\lambda_{(m+1)^*}\frac{\partial x^{(m+1)^*}}{\partial x^{(m+1)'}} ,$$

$$X^\lambda_{\xi'} = X^\lambda_\xi ,$$

$$X^m_{\mu'} = 0 ,$$

$$X^m_{m'} = \frac{\partial x^m}{\partial x^{m'}} + X^m_{m+1}\frac{\partial x^{m+1}}{\partial x^{m'}} + X^m_{m^*}\frac{\partial x^{m^*}}{\partial x^{m'}} + X^m_{(m+1)^*}\frac{\partial x^{(m+1)^*}}{\partial x^{m'}} ,$$

$$X^m_{(m+1)'} = \frac{\partial x^m}{\partial x^{(m+1)'}} + X^m_{m+1}\frac{\partial x^{m+1}}{\partial x^{(m+1)'}} + X^m_{m^*}\frac{\partial x^{m^*}}{\partial x^{(m+1)'}} + X^m_{(m+1)^*}\frac{\partial x^{(m+1)^*}}{\partial x^{(m+1)'}} ,$$

$$X^m_{\xi'} = X^m_\xi ,$$

$$X^{m+1}_{\mu'} = 0 ,$$

$$X^{m+1}_{m'} = X^{m+1}_{m^*}\frac{\partial x^{m^*}}{\partial x^{m'}} + X^{m+1}_{(m+1)^*}\frac{\partial x^{(m+1)^*}}{\partial x^{m'}} ,$$

$$X^{m+1}_{(m+1)'} = X^{m+1}_{m^*}\frac{\partial x^{m^*}}{\partial x^{(m+1)'}} + X^{m+1}_{(m+1)^*}\frac{\partial x^{(m+1)^*}}{\partial x^{(m+1)'}} ,$$

$$X^{m+1}_{\xi'} = 0\,,$$
$$X^{\eta}_{\mu'} = 0\,,$$
$$X^{\eta}_{m'} = X^{\eta}_{m^*}\frac{\partial x^{m^*}}{\partial x^{m'}} + X^{\eta}_{(m+1)^*}\frac{\partial x^{(m+1)^*}}{\partial x^{m'}}\,,$$
$$X^{\eta}_{(m+1)'} = X^{\eta}_{m^*}\frac{\partial x^{m^*}}{\partial x^{(m+1)'}} + X^{\eta}_{(m+1)^*}\frac{\partial x^{(m+1)^*}}{\partial x^{(m+1)'}}\,,$$
$$X^{\eta}_{\xi'} = 0\,.$$

If $m < n-1$, then the matrix $(X^i_{j'})$ can be written as

$$\begin{bmatrix} \delta^\lambda_\mu & X^\lambda_{m'} & X^\lambda_{(m+1)'} & X^\lambda_\xi \\ 0 & X^m_{m'} & X^m_{(m+1)'} & X^m_\xi \\ 0 & X^{m+1}_{m'} & X^{m+1}_{(m+1)'} & 0 \\ 0 & X^\eta_{m'} & X^\eta_{(m+1)'} & 0 \end{bmatrix},$$

and if $m = n-1$, it can be written as

$$\begin{bmatrix} \delta^\lambda_\mu & X^\lambda_{m'} & X^\lambda_{n'} \\ 0 & X^m_{m'} & X^m_{n'} \\ 0 & X^n_{m'} & X^n_{n'} \end{bmatrix}.$$

Thus in each of cases, if

$$\begin{vmatrix} X^m_{m'} & X^m_{(m+1)'} \\ X^{m+1}_{m'} & X^{m+1}_{(m+1)'} \end{vmatrix} \neq 0\,, \tag{3.9}$$

then the rank of $(X^i_{j'})$ is greater than m. Computing the first member of (3.9), we obtain

$$\begin{aligned} &X^{m+1}_{m^*}\frac{\partial(x^m, x^{m^*})}{\partial(x^{m'}, x^{(m+1)'})} \\ &+ X^{m+1}_{(m+1)^*}\frac{\partial(x^m, x^{(m+1)^*})}{\partial(x^{m'}, x^{(m+1)'})} \\ &+ X^m_{m+1}X^{m+1}_{m^*}\frac{\partial(x^{m+1}, x^{m^*})}{\partial(x^{m'}, x^{(m+1)'})} \\ &+ X^m_{m+1}X^{m+1}_{(m+1)^*}\frac{\partial(x^{m+1}, x^{(m+1)^*})}{\partial(x^{m'}, x^{(m+1)'})} \\ &+ (X^m_{m^*}X^{m+1}_{(m+1)^*} - X^m_{(m+1)^*}X^{m+1}_{m^*})\frac{\partial(x^{m^*}, x^{(m+1)^*})}{\partial(x^{m'}, x^{(m+1)'})}\,. \end{aligned} \tag{3.10}$$

We now assume that a local coordinate system in a neighborhood of a point P satisfies in addition to (3.1), $x^{m^*} \neq 0$, $X^{m+1}_{(m+1)^*} \neq 0$ at the point P. We then consider a coordinate transformation under which x^κ, x^{κ^*}, x^ξ, x^{ξ^*}

are not changed and x^m, x^{m+1}, x^{m*}, $x^{(m+1)*}$ are changed by

$$
\begin{aligned}
x^m &= x^{m'} + \frac{x^{(m+1)*'}}{x^{m*'}} x^{(m+1)'} , \\
x^{m+1} &= \frac{x^{(m+1)*'}}{x^{m*'}} , \\
x^{m*} &= x^{m*'} , \\
x^{(m+1)*} &= -x^{(m+1)'} x^{m*'} .
\end{aligned} \tag{3.11}
$$

It will be easily seen that this coordinate transformation satisfies (3.5). As among the functional determinants appearing in (3.10), the only non-zero one is

$$\frac{\partial(x^m, x^{(m+1)*})}{\partial(x^{m'}, x^{(m+1)'})} = -x^{m*'} = -x^{m*} ,$$

(3.10) becomes $-x^{m*}X^{m+1}_{(m+1)*}$ and is consequently not zero. Thus by the transformation (3.11), the rank of the matrix (X^i_j) increases at least one. Thus the Lemma 3.1 is proved.

3^0 **Proof of Theorem 2.1.** We prove another lemma:

LEMMA 3.2. *When we introduce, in a neighborhood of a point P of an almost cotangent manifold M^{2n}, a local coordinate system (x^A) which satisfies* (3.1) *for an integer m, there exists another local coordinate system $(x^{A'})$ which satisfies equations having the same form as* (3.1) *and $x^{m*'} \neq 0$, $X^{m+1}_{(m+1)*'} \neq 0$ at P.*

PROOF. Since $\tilde{\omega} \neq 0$, there exists a non-zero one among $X^{m+1}_{1*}, \cdots, X^{m+1}_{n*}$. If $X^{m+1}_{(m+1)*} \neq 0$, then we do nothing. But if $X^{m+1}_{(m+1)*} = 0$, we change the coordinate system by the following equations:

$$
\begin{aligned}
x^i &= x^{i'}, \quad (i \neq m+1) , \\
x^{m+1} &= x^{(m+1)'} - {\sum}' a_i x^{i'} , \\
x^{i*} &= x^{i*'} + a_i x^{(m+1)*'} , \qquad (i \neq m+1) , \\
x^{(m+1)*} &= x^{(m+1)*'} ,
\end{aligned}
$$

where a_i are constant and $\sum'$ denotes the summation excluding the term corresponding to $i = m+1$. It is easily seen that this coordinate transformation satisfies (3.5). Also, since we have, in this case,

$$
\begin{aligned}
\omega^p &= dx^p + X^p_u dx^u + X^p_{i*} dx^{i*} \\
&= dx^{p'} + X^p_{m+1}(dx^{(m+1)'} - {\sum}' a_i dx^{i'}) \\
&\quad + X^p_\xi dx^{\xi'} + X^p_{i*} dx^{i*} ,
\end{aligned}
$$

ω^p can be written as

$$\omega^p = (\delta^p_q - X^p_{m+1}a_q)dx^{q'} + X^p_{u'}dx^{u'} + X^p_{i^{*\prime}}dx^{i^{*\prime}}$$

and ω^t as

$$\omega^t = X^t_{i^*}dx^{i^*} = X^t_{i^{*\prime}}dx^{i^{*\prime}} .$$

Since $X^{m+1}_{(m+1)^*} = 0$, especially for ω^{m+1}, from

$$\omega^{m+1} = \sum{}' X^{m+1}_{i^*}dx^{i^*} ,$$

we have

$$\omega^{m+1} = \sum{}' X^{m+1}_{i^*}dx^{i^{*\prime}} + \sum{}' X^{m+1}_{i^*}a_i dx^{(m+1)^{*\prime}} ,$$

and consequently, in $\omega^{m+1} = X^{m+1}_{i^{*\prime}}dx^{i^{*\prime}}$, we have

$$X^{m+1}_{(m+1)^{*\prime}} = \sum{}' a_i X^{m+1}_{i^*} .$$

Thus if we take a_i $(i \neq m+1)$ suitably, we get $X^{m+1}_{(m+1)^{*\prime}} \neq 0$. Also, if we take $|a_i|$ sufficiently small, then $\det(\delta^p_q - X^p_{m+1}a_q)$ is sufficiently close to 1, and consequently, taking suitably a linear combination $'\omega^p = p^p_q\omega^q$ instead of ω^p, we obtain

$$'\omega^p = dx^{p'} + X^{p'}_{u'}dx^{u'} + X^{p'}_{i^{*\prime}}dx^{i^{*\prime}} ,$$
$$\omega^t = X^t_{i^{*\prime}}dx^{i^{*\prime}}, \qquad X^{m+1}_{(m+1)^{*\prime}} \neq 0 .$$

We rewrite this result as

$$\omega^p = dx^p + X^p_u dx^u + X^p_{i^*}dx^{i^*} ,$$
$$\omega^t = X^t_{i^*}dx^{i^*}, \qquad X^{m+1}_{(m+1)^*} \neq 0 .$$

Since, in such a coordinate system, we can change the order of $\omega^1, \omega^2, \cdots, \omega^m; x^1, x^2, \cdots, x^m; x^{1^*}, x^{2^*}, \cdots, x^{m^*}$ arbitrarily but in the same way, we can assume that $x^{m^*} \neq 0$ unless $x^{1^*} = x^{2^*} = \cdots = x^{m^*} = 0$. But the case in which $x^{1^*} = x^{2^*} = \cdots = x^{m^*} = 0$ cannot happen, because if this happens, then we get, from (3.3), $x^{1^*} = x^{2^*} = \cdots = x^{n^*} = 0$ and consequently $\omega = 0$, which contradicts the assumption $\omega \neq 0$. Thus the Lemma 3.2 is proved.

Theorem 2.1 is a consequence of Lemmas 3.1 and 3.2.

4. Examples of almost cotangent structures.

1° Almost cotangent structure in the cotangent bundle of an n-dimensional differentiable manifold M^n. As we may easily see ${}^cT(M)$ admits an almost cotangent structure such that, with respect to a suitable local coordinate system, we have $\Lambda^{ih} = 0$. That is, the ${}^cT(M)$ is covered by a system of canonical coordinates with respect to which $\Lambda^{ih} = 0$.

Conversely is such an almost cotangent structure a cotangent structure? The answer is no in general. Suppose that a $2n$-dimensional differentiable manifold is covered by a canonical coordinate system with respect to which we have $\Lambda^{ih} = 0$. The distribution defined by $dx^1 = dx^2 = \cdots = dx^n = 0$ is locally integrable, but, as the following example shows, does not give globally a closed integral manifold.

Let X be a 2-dimensional topological space such that a point P of X is represented as $P(a, b)$ by an ordered pair (a, b) of two real numbers a and b where $0 \leqq a \leqq 1; 0 \leqq b \leqq 1$. We assume that $P(a, b) = P(a', b')$ if and only if $a = a'$ and $b = b'$ for $0 < a, a' < 1, 0 < b, b' < 1$ and $P(0, b) = P(1, b)$, $P(a, 1) = P(a + \alpha, 0)$, or $P(a + \alpha - 1, 0)$ where α is an irrational number such that $0 < \alpha < 1$. We fix suitably the ranges of two variables x^1 and x^{1*} and choose also suitably two integers l^1 and l^{1*}, then $(x^1 + l^1 - \alpha l^{1*}, x^{1*} + l^{1*})$ represents a point of X and consequently (x^1, x^{1*}) is a local coordinate system of X. We take a similar 2-dimensional topological space Y and let (x^2, x^{2*}) be its local coordinate system in the above sense. In the product space $X \times Y$, we can use $(x^1, x^2, x^{1*}, x^{2*})$ as a local coordinate system. The transformation between two different local coordinate systems is of the form

$$x^1 = x^{1'} + l^1 - \alpha l^{1*}, x^2 = x^{2'} + l^2 - \alpha l^{2*} ,$$
$$x^{1*} = x^{1*'} + l^{1*} , \qquad x^{2*} = x^{2*'} + l^{2*} ,$$

where l^1, l^2, l^{1*}, l^{2*} are $0, +1,$ or -1. Since α is an irrational number, the distribution $dx^1 = dx^2 = 0$ can not have a closed submanifold in $X \times Y$ as its integral submanifold. Consequently, $\omega = x^{1*}dx^1 + x^{2*}dx^2$, $\tilde{\omega} = dx^1 \wedge dx^2$ does not give the almost cotangent structure of a cotangent bundle.

2^0 **Almost cotangent structure of an $(\omega, \tilde{\omega}, \tilde{\omega}^*)$-structure.** In a previous paper [7, 8], we have studied $2n$-dimensional differentiable manifold M having the following properties.

In M, there exist globally a 1-form ω and two simple n-forms $\tilde{\omega}, \tilde{\omega}^*$, $\tilde{\omega}$ and $\tilde{\omega}^*$ being determined only up to non-vanishing scalar multiples, such that

$$\omega \neq 0, (d\omega)^n \neq 0, \tilde{\omega} \wedge \tilde{\omega}^* \neq 0, \omega \wedge \tilde{\omega}^* \neq 0 \tag{4.1}$$

everywhere and there exist locally decompositions of $\tilde{\omega}$ and $\tilde{\omega}^*$ satisfying

$$\begin{aligned} \sigma\tilde{\omega} &= \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n \qquad (\sigma \neq 0) , \\ \rho\tilde{\omega}^* &= \omega^{1*} \wedge \omega^{2*} \wedge \cdots \wedge \omega^{n*} \qquad (\rho \neq 0) , \\ d\omega &= \omega^{i*} \wedge \omega^i . \end{aligned} \tag{4.2}$$

In fact, (4.2) means that there exist a covering of M by a system of open neighborhoods U_λ, $\lambda \in \{\lambda\}$, and local n-forms $\tilde{\omega}_\lambda$ and $\tilde{\omega}^*_\lambda$ representing $\tilde{\omega}$ and $\tilde{\omega}^*$ respectively such that

$$\tilde{\omega}_\lambda = \omega_\lambda^1 \wedge \omega_\lambda^2 \wedge \cdots \wedge \omega_\lambda^n ,$$
$$\tilde{\omega}_\lambda^* = \omega_\lambda^{1^*} \wedge \omega_\lambda^{2^*} \wedge \cdots \wedge \omega_\lambda^{n^*} ,$$
$$d\omega = \omega_\lambda^{i^*} \wedge \omega_\lambda^i$$

in U_λ.

This manifold is called an even-dimensional contact manifold with a contact almost product structure. Choosing a suitable local coordinate system, we have

$$\begin{aligned} \omega &= \xi^{i^*} d\xi^i , \\ \omega^i &= d\xi^i + \Pi^{ih}\omega^{h^*} , \\ \omega^{i^*} &= d\xi^{i^*} - \Gamma_{ih} d\xi^h , \end{aligned} \tag{4.3}$$

where

$$\Pi^{ih} = \Pi^{hi},\ \Gamma_{ih} = \Gamma_{hi} . \tag{4.4}$$

Moreover we have put the condition

$$\xi^{i^*}\Pi^{ih} = 0 \tag{4.5}$$

(see, for example, [8] p. 28). In this case, it will be easily seen that ω and $\tilde{\omega}$ satisfy $\omega = \xi^{i^*}\omega^i$ and consequently

$$\omega \wedge \tilde{\omega} = 0 . \tag{4.6}$$

Conversely, if (4.6) holds, then ω can be written as

$$\omega = X_i \omega^i$$

and consequently substituting this into (4.3), we obtain

$$\xi^{i^*} d\xi^i = X_i(d\xi^i + \Pi^{ih}\omega^{h^*}) ,$$

hence

$$\begin{aligned} (\xi^{i^*} - X_i) d\xi^i &= X_i \Pi^{ih}\omega^{h^*} \\ &= X_i \Pi^{ih}(d\xi^{h^*} - \Gamma_{hk} d\xi^k) . \end{aligned}$$

Then comparing the coefficients of $d\xi^{h^*}$, we obtain

$$X_i \Pi^{ih} = 0 , \tag{4.7}$$

and consequently,

$$X_i = \xi^{i^*} .$$

Substituting this into (4.7), we obtain (4.5).

Thus the even-dimensional contact manifold with a contact almost product structure we considered satisfies (4.6) in addition to (4.1) and

(4.2). We call such a structure $(\omega, \tilde{\omega}, \tilde{\omega}^*)$-structure.

Evidently an $(\omega, \tilde{\omega}, \tilde{\omega}^*)$-structure contains an almost cotangent structure $(\omega, \tilde{\omega})$.

5. Integrable almost cotangent structure. We now consider the case in which an almost cotangent structure $(\omega, \tilde{\omega})$ satisfies

$$d\tilde{\omega} = \alpha \wedge \tilde{\omega} \tag{5.1}$$

for a 1-form α. Here and often in the sequel $\tilde{\omega}$ means a local n-form representing the pseudo-n-form. This property (5.1) is preserved by a change of $\tilde{\omega}$, $\tilde{\omega} \to f\tilde{\omega}$, if α is simultaneously changed by $\alpha \to d\log f + \alpha$.

In the canonical coordinate system, from

$$\omega^i = dx^i + \Lambda^{ih}dx^{h^*} ,$$

we have

$$\begin{aligned} d\omega^i &= d\Lambda^{ih} \wedge dx^{h^*} \\ &= \partial_j \Lambda^{ih} dx^j \wedge dx^{h^*} + \partial_{j^*}\Lambda^{ih}dx^{j^*} \wedge dx^{h^*} \\ &= \partial_j \Lambda^{ih}\omega^j \wedge dx^{h^*} + (\partial_{j^*}\Lambda^{ih} - \Lambda^{kj}\partial_k \Lambda^{ih})dx^{j^*} \wedge dx^{h^*} \end{aligned}$$

and consequently

$$\begin{aligned} & d(\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n) \\ = & -\partial_i \Lambda^{ih} dx^{h^*} \wedge \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n \\ & + \Omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n - \omega^1 \wedge \Omega^2 \wedge \omega^3 \wedge \cdots \wedge \omega^n \\ & + \cdots + (-1)^{n-1}\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^{n-1} \wedge \Omega^n , \end{aligned}$$

where

$$\Omega^k = (\partial_{j^*}\Lambda^{ki} - \Lambda^{lj}\partial_l \Lambda^{ki})dx^{j^*} \wedge dx^{i^*} .$$

Thus, a necessary and sufficient condition in order that (5.1) holds is, as we can see it comparing the coefficients of $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{n-1} \wedge dx^{j^*} \wedge dx^{i^*}$ and so on, that $\Omega^k = 0$, that is,

$$\partial_{j^*}\Lambda^{ki} - \partial_{i^*}\Lambda^{kj} + \Lambda^{li}\partial_l \Lambda^{kj} - \Lambda^{lj}\partial_l \Lambda^{ki} = 0 . \tag{5.2}$$

Thus we have proved the following

LEMMA 5.1. *A necessary and sufficient condition that an almost cotangent structure $(\omega, \tilde{\omega})$ locally satisfies*

$$d\tilde{\omega} = \alpha \wedge \tilde{\omega}$$

for a certain 1-form α is given by (5.2).

We next consider the case in which, for an almost cotangent structure $(\omega, \tilde{\omega})$, there exist local scalar fields $\beta, f^1, f^2, \cdots, f^n$ such that

(5.3) $$\tilde{\omega} = \beta df^1 \wedge df^2 \wedge \cdots \wedge df^n .$$

Then there exists a regular matrix A_i^h such that

(5.4) $$df^h = A_i^h \omega^i .$$

In canonical coordinate system, this can be written as

$$\partial_k f^h dx^k + \partial_{k^*} f^h dx^{k^*} = A_i^h (dx^i + \Lambda^{ij} dx^{j^*}) ,$$

from which we get

$$\partial_i f^h = A_i^h ,$$
$$\partial_{i^*} f^h = A_k^h \Lambda^{ki} .$$

Thus we have the following

LEMMA 5.2. *In an almost cotangent structure* $(\omega, \tilde{\omega})$, *local scalar fields* $\sigma, f^1, f^2, \cdots, f^n$ *satisfying*

(5.5) $$\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n = \sigma df^1 \wedge df^2 \wedge \cdots \wedge df^n ,$$

where

$$\omega^i = dx^i + \Lambda^{ih} dx^{h^*} ,$$

satisfy

(5.6) $$\sigma \det (\partial_i f^h) = 1 ,$$

(5.7) $$\partial_{i^*} f^h - \Lambda^{ki} \partial_k f^h = 0 .$$

Conversely, if there exist local scalar fields $f^1, f^2, \cdots, f^n$ *satisfying* (5.7) *and such that* $\det (\partial_i f^h) \neq 0$, *then we can determine* σ *by* (5.6) *and get* (5.5).

If there are n local scalar fields f^h which satisfy (5.7) and $\det (\partial_i f^h) \neq 0$, then the system of partial differential equations

(5.8) $$\partial_{i^*} f - \Lambda^{ki} \partial_k f = 0$$

is completely integrable. If $f^1, f^2, \cdots, f^n$ satisfying (5.7) satisfy $\det (\partial_i f^h) = 0$, then there exist $\varphi_h (\neq 0)$ such that $\varphi_h \partial_i f^h = 0$, but, following (5.7), φ_h also satisfy $\varphi_h \partial_{i^*} f^h = 0$ and consequently $f^1, f^2, \cdots, f^n$ are not independent functions. Thus the existence of f^h satisfying (5.7) and $\det (\partial_i f^h) \neq 0$ is equivalent to the complete integrability of (5.8). The complete integrability of (5.8) is found to be

(5.9) $$\partial_{j^*} \Lambda^{hi} - \partial_{i^*} \Lambda^{hj} + \Lambda^{li} \partial_l \Lambda^{hj} - \Lambda^{lj} \partial_l \Lambda^{hi} = 0$$

and is equivalent to (5.2). Thus we have

THEOREM 5.3. *In an almost cotangent structure* $(\omega, \tilde{\omega})$, *the existence of local scalar fields* $\sigma, f^1, f^2, \cdots, f^n$ *such that*

$$\tilde{\omega} = \sigma df^1 \wedge df^2 \wedge \cdots \wedge df^n$$

and the existence of a local 1-*form* α *such that*

$$d\tilde{\omega} = \alpha \wedge \tilde{\omega}$$

are equivalent.

If the condition of Theorem 5.3 is satisfied, then from $\omega \wedge \tilde{\omega} = 0$, we see that there exist local scalar fields φ_i such that $\omega = \varphi_i df^i$. Since $(d\omega)^n \neq 0$, $2n$ functions $f^1, \cdots, f^n, \varphi_1, \cdots, \varphi_n$ are independent and consequently we can determine a local coordinate system putting $x^i = f^i$, $x^{i^*} = \varphi_i$. This is a canonical coordinate system satisfying

$$\omega = x^{i^*} dx^i \,, \qquad \tilde{\omega} = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \,.$$

Thus we have the following

COROLLARY. *If the condition in Theorem* 5.3 *is satisfied, then there exists a canonical coordinate system such that*

$$\tilde{\omega} = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \,.$$

A differentiable manifold satisfying this condition is locally a cotangent bundle, but globally not in general. We say that this manifold has an integrable almost cotangent structure since the local distribution $\omega^1 = \cdots = \omega^n = 0$ determined by $\tilde{\omega}$ is completely integrable.

6. Symmetric almost cotangent structure. We now consider the case in which an almost cotangent structure $(\omega, \tilde{\omega})$ satisfies

$$d\omega \wedge \tilde{\omega} = 0 \,. \tag{6.1}$$

In this case, substituting $dx^k = \omega^k - \Lambda^{kh} dx^{h^*}$ into

$$dx^{k^*} \wedge dx^k \wedge \omega^1 \wedge \cdots \wedge \omega^n = 0 \,,$$

we find

$$\Lambda^{ji} = \Lambda^{ij} \,. \tag{6.2}$$

DEFINITION. An almost cotangent structure in which (6.1), that is, (6.2) holds, is called a symmetric almost cotangent structure.

Integrable almost cotangent structure is a special symmetric almost cotangent structure and the almost cotangent structure in an $(\omega, \tilde{\omega}, \tilde{\omega}^*)$ structure is also a symmetric almost cotangent structure.

In an almost cotangent manifold M^{2n}, we can derive, from ω, the 2-form

$$d\omega = dx^{i^*} \wedge dx^i$$

and consequently a tensor field

$$(6.3) \qquad (\varepsilon_{BA}) = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$$

of type (0, 2) where E_n is the $n \times n$ unit matrix. Using this tensor we can make correspond a covector (W_i, W_{i^*}) to a vector (X^i, X^{i^*}) of M^{2n} by

$$W_i = -X^{i^*}, \qquad W_{i^*} = X^i .$$

These $X = (X^i, X^{i^*})$ and $W = (W_i, W_{i^*})$ are incident, that is, they satisfy

$$X^i W_i + X^{i^*} W_{i^*} = 0 .$$

DEFINITION. Two vectors X and Y in M^{2n} are said to be ε-orthogonal when they satisfy

$$\varepsilon_{BA} X^B Y^A = 0 ,$$

that is,

$$X^{i^*} Y^i - X^i Y^{i^*} = 0 .$$

DEFINITION. Suppose that there is given a vector $X = (X^A)$ in M^{2n}. The covector $W = (W_A)$ given by $W_B = \varepsilon_{BA} X^A$ is said to be associated with the vector X.

The fact that X and Y are ε-orthogonal is equivalent to the fact that the covector W associated with X and the vector Y are incident.

DEFINITION. A vector X in M^{2n} is said to be incident with $\tilde{\omega}$ if it satisfies

$$X^i + \Lambda^{ih} X^{h^*} = 0 .$$

We also say that, in this case, X is a tangent vector of the distribution $\tilde{\omega}$.

A vector Y ε-orthogonal to a vector X which is incident with $\tilde{\omega}$ satisfies

$$X^{i^*} Y^i + \Lambda^{ih} X^{h^*} Y^{i^*} = 0 ,$$

that is,

$$(Y^i + \Lambda^{hi} Y^{h^*}) X^{i^*} = 0 ,$$

and consequently, a necessary and sufficient condition that a vector Y is ε-orthogonal to all the vector X incident with $\tilde{\omega}$ is given by

$$Y^i + \Lambda^{hi} Y^{h^*} = 0 .$$

We also have

THEOREM 6.1. *In an almost cotangent structure* $(\omega, \tilde{\omega})$, *a necessary* *and sufficient condition that all the vectors incident with $\tilde{\omega}$ are mutually ε-orthogonal is that the almost cotangent structure is symmetric.*

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