Tôhoku Math. Journ.
25 (1973), 553-556.

A NON-COMPACT RIEMANNIAN MANIFOLD ADMITTING A TRANSITIVE GROUP OF CONFORMORPHISMS

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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(Received March 16, 1973)

1. Introduction. Let (M, g) be a Riemannian n-manifold. Throughout this paper manifolds under consideration are always assumed to be connected and smooth. Furthermore, we assume $\dim M = n > 2$. Let $C(M, g)$ be the group of conformorphisms of (M, g) and $C_0(M, g)$ the connected component of the identity. $I(M, g)$ denotes the group of isometries of (M, g). A subgroup G of $C(M, g)$ is called *essential* if there exists no function ρ such that G is a subgroup of $I(M, g^*)$, $g^* = e^{2\rho}g$.

The following has been known.

THEOREM A [2, 4]. *For a compact (M, g), $C_0(M, g)$ is essential if and only if (M, g) is conformorphic to a Euclidean n-sphere S^n.*

The proof goes as follows. First of all if $C_0(M, g)$ is essential, then there exists a closed essential one-parameter subgroup G of $C(M, g)$. The existence of such G implies conformal flatness of (M, g). Then the following proposition completes the proof.

PROPOSITION B [4]. *Let (M, g) be a conformally flat Riemannian n-manifold. If there is a closed essential one-parameter subgroup of $C(M, g)$, then (M, g) is conformorphic to a Euclidean n-space or a Euclidean n-sphere.*

For a non-compact manifold, the following conjecture is quite probable.

CONJECTURE. *For a non-compact (M, g), $C_0(M, g)$ is essential if and only if (M, g) is conformorphic to a Euclidean n-space.*

In this paper we are going to prove the conjecture affirmatively under the following situations:

(a) $C(M, g)$ is transitive, and

(b) $\dim C(M, g) > \dim M = n$.

In fact, (a) implies conformal flatness of (M, g), and (b) does the existence of a closed essential one-parameter subgroup of $C(M, g)$.

As an application of the main theorem we consider the case where $I(M, g)$ is transitive.

Some results related to ours can be seen in [3].

2. The main theorem. We state the main theorem and give a proof by a series of lemmas.

THEOREM 1. *Let (M, g) be a non-compact connected Riemannian n-manifold, $n > 2$. If the group $C(M, g)$ of conformorphisms is essential and transitive, and if $\dim C(M, g) > n$, then (M, g) is conformorphic to a Euclidean n-space E^n.*

LEMMA 1. *If $C(M, g)$ is essential and transitive, then (M, g) is conformally flat.*

PROOF. For $n > 3$, let W be the Weyl conformal curvature tensor of (M, g), whose vanishing implies conformal flatness. Since W is invariant by any conformorphism of (M, g) and $C(M, g)$ is transitive, W vanishes identically if it does at a point. To prove the lemma, we assume the contrary. Namely, assume that W vanishes nowhere. Then the Riemannian metric $\|W\|g$ is conformal to g and is invariant by $C(M, g)$. Thus $C(M, g)$ is not essential, contrary to the assumption.

For $n = 3$, W is replaced by a tensor field $\tilde{W}$ of type $(0, 3)$ which plays the same role as W for $n > 3$. Then a quite similar argument applies.

LEMMA 2. *Under the assumptions of Theorem 1, the isotropy subgroup of $C(M, g)$ at a point is not compact and contains a closed essential one-parameter subgroup of $C(M, g)$.*

PROOF. Let K be the isotropy subgroup at a point p. If K is compact, since K is a group of conformorphisms, $\bar{g}_p = \int_K k^* g_p dK$, where dK is the Haar measure on K, is invariant by K and is proportional to the original g_p. Therefore, g_p itself is invariant by K. Since $C(M, g)$ is transitive, the K-invariant g_p can be extended to a unique $C(M, g)$-invariant Riemannian metric, which is obviously conformal to the original metric g. Thus $C(M, g)$ is not essential, contrary to the assumption. Therefore, K is not compact.

Since $\dim C(M, g) > \dim M$, we have $\dim K \geqq 1$. Then K contains a closed one-parameter subgroup G which is isomorphic to the group of reals. Since G is closed in K and is not compact, no compact subgroup of K can contain G. Therefore G is essential. In fact, if G is not essential, then G must be contained in the isotropy subgroup $\tilde{K}$ of the group of isometries of $(M, \tilde{g})$ for some $\tilde{g}$ conformal to $\dot{g}$. Obviously $\tilde{K}$ is a compact subgroup of K.

PROOF OF THEOREM 1. From Lemma 1, (M, g) is conformally flat, and from Lemma 2 there is a closed essential one-parameter subgroup of $C(M, g)$. Then Proposition B completes the proof.

REMARK 1. On (E^n, g_0) a conformorphism is a homothetic transformation, where g_0 is a standard Riemannian metric on E^n. Therefore, $C(M, g)$ in the above is isomorphic to the group of homothetic transformations of (E^n, g_0), and $\dim C(M, g) = 1 + n(n+1)/2$.

REMARK 2. Even if $C(M, g)$ is not transitive, any compact subgroup of the isotropy subgroup K at a point p leaves invariant the original metric g_p at p. This can be seen in the above argument.

3. A homogeneous Riemannian manifold. As a special case of Theorem 1, we consider the case where $I(M, g)$ is transitive. In this case the essentiality of $C(M, g)$ becomes simpler, and "conformorphic" is replaced by "isometric" in Theorem 1.

LEMMA 3. *Suppose that the group $I(M, g)$ of isometries is transitive. Then $C_0(M, g)$ is essential if and only if $C_0(M, g) \neq I_0(M, g)$, where G_0 denotes the connected component of the identity of G.*

PROOF. If $C_0(M, g)$ is essential, then obviously $C_0(M, g) \neq I_0(M, g)$. Conversely assume this. Since $I(M, g)$ is transitive, so are $I_0(M, g)$ and $C_0(M, g)$ as well. Let K and H be the isotropy subgroups of $C_0(M, g)$ and $I_0(M, g)$ respectively at a point. Then H is compact and is a subgroup of K. Since if $C_0(M, g)$ is not essential, K must be compact, we have only to show that K is not compact. Assume the contrary, namely assume that K is compact. Then by Remark 2 the metric g_p at p is K-invariant. Since $I_0(M, g)$ as well as $C_0(M, g)$ are transitive, g_p can be extended uniquely to a $C_0(M, g)$-invariant Riemannian metric $\tilde{g}$ on M. Since $I_0(M, g)$ is a subgroup of $C_0(M, g)$, $\tilde{g}$ is $I_0(M, g)$-invariant as well. Therefore, $\tilde{g} = g$, which implies $C_0(M, g) = I_0(M, g)$, contradicting the assumption.

THEOREM 2. *Let (M, g) be a non-compact connected Riemannian n-manifold, $n > 2$. If $I(M, g)$ is transitive, and $C_0(M, g) \neq I_0(M, g)$, then (M, g) is isometric to a Euclidean n-space (E^n, g_0) with standard metric g_0.*

PROOF. From Lemma 3 and our assumptions it follows that $C_0(M, g)$ is essential and $\dim C_0(M, g) > \dim I_0(M, g) \geqq n$. Therefore by Theorem 1 there is a conformorphism f of (M, g) onto a Euclidean n-space (E^n, g_0). The pull-back f^*g_0 is denoted by g^*. Then g^* is conformal to g. At a point p the isotropy subgroup H is a compact subgroup of $C(M, g) = C(M, g^*)$. Then by Remark 2 we may assume that g_p^* is H-invariant and $g_p^* = g_p$. It follows then that $I_0(M, g)$ leaves g^* invariant and $g^* = g$. Thus f is an isometry of (M, g) onto (E^n, g_0).

Theorem 2 is the non-compact version of a theorem of Goldberg and Kobayashi [1].

REFERENCES

[1] S. I. GOLDBERG and S. KOBAYASHI, The conformal transformation group of a compact Riemannian manifold, Amer. J. Math., 84 (1962), 170-174.

[2] J. LELONG-FERRAND, Transformations conformes et quasiconformes des variétés riemanniennes compactes, Mémoires de Acad. Royale Belg., 39 (1971), n° 5.

[3] T. NAGANO, On conformal transformations of Riemannian spaces, J. Math. Soc. Japan, 10 (1958), 79-93.

[4] M. OBATA, The conjectures on conformal transformations of Riemannian manifolds, J. Differential Geometry, 6 (1971), 247-258.

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Added in proof· The conjecture in this paper has been proved affirmatively by D. J. Alekseevikii, *Groups of conformal transformations of Riemannian spaces*, Mat. Sbornik, 89 (131) (1972), 280-296; Math. USSR Sbornik, 18 (1972), 285-301.