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SOME CRITICAL MAPPINGS

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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Let (N, h) and (M, g) be connected compact orientable Riemannian manifolds of dimension n and m respectively, where $n \geqq m$. Let μ be a differentiable mapping of rank m everywhere. There exists on M an m-form φ naturally induced from the metric g. The pull back ψ of φ is a closed m-form on N and the integral

$$J[\mu] = \int_N (\psi, \psi) dV_N$$

is a functional of the mapping μ. A critical point $\bar{\mu}$ of $J[\mu]$ is called in the present paper a critical mapping. The purpose of the present paper is to study some properties of such critical mappings.

Let (M_1, f), (M_2, g), and (M_3, h) be connected compact orientable Riemannian manifolds where $\dim M_1 = \dim M_2 \leqq \dim M_3$. If $\mu_{12}: M_2 \to M_1$ and $\mu_{23}: M_3 \to M_2$ are critical mappings, then $\mu_{13} = \mu_{12}\mu_{23}$ is a critical mapping of M_3 onto M_1. If a critical mapping μ is homeomorphic, its inverse μ^{-1} is also a critical mapping. When a set of Riemannian manifolds $\{(M_\lambda, g_\lambda), \lambda \in \Lambda\}$ of the same dimension is given where each manifold is connected, compact and orientable, the set of homeomorphic critical mappings forms a groupoid. Some examples of critical mappings are also given.

1. The functional $J[\mu]$ and its critical point. Let us consider connected compact orientable Riemannian manifolds (M, g) and (N, h) of dimension m and n respectively, such that $n \geqq m$ and admitting differentiable mappings $\mu: N \to M$ of rank m everywhere. Local coordinates in M are denoted by x^h and those in N by y^κ. We use indices $h, i, j, \cdots = 1, \cdots, m$ for M and indices $\kappa, \lambda, \mu, \cdots = 1, \cdots, n$ for N. The metric tensors of M and N are denoted by g_{ji} and $h_{\mu\lambda}$ respectively. When a mapping μ is expressed locally by $x^h = x^h(y^1, \cdots, y^n)$, we get connecting tensors

$$B^h_\kappa = \frac{\partial x^h}{\partial y^\kappa}, \qquad B^{h_1 \cdots h_p}_{\kappa_1 \cdots \kappa_p} = B^{h_1}_{\kappa_1} \cdots B^{h_p}_{\kappa_p}$$

of the mapping.

The m-form

$$\sqrt{\det(g_{ji})}dx^1 \cdots dx^m$$

is a closed form on M and

$$B_{\lambda_1\cdots\lambda_m}^{[1\cdots m]}\sqrt{\det(g_{ji})}dy^{\lambda_1} \cdots dy^{\lambda_m}$$

is its pull back with respect to the mapping μ. From this m-form let us define a functional $J[\mu]$ by

$$J[\mu] = \int_N B_{\mu_1\cdots\mu_m}^{[1\cdots m]}B_{\lambda_1\cdots\lambda_m}^{[1\cdots m]}h^{\mu_1\lambda_1} \cdots h^{\mu_m\lambda_m} \det(g_{ji})dV_N , \tag{1.1}$$
$$dV_N = \sqrt{\det(h_{\mu\lambda})}dy^1 \cdots dy^n .$$

Let us consider the space $\mathfrak{M} = \mathfrak{M}(N, M)$ of all such mappings μ. Let μ_0 be a point of $\mathfrak{M}$ and let $\mathscr{C}$: $[0, 1] \to \mathfrak{M}$ be a curve $\{\mu(t), 0 \leqq t \leqq 1\}$ in $\mathfrak{M}$ where $\mu(0) = \mu_0$ and $\mu(t)$ is expressed locally by differentiable functions $x^h(y^1, \cdots, y^n; t)$. For this curve $\mathscr{C}$ we define $\eta^h = Dx^h$ by

$$Dx^h = \left(\frac{\partial x^h}{\partial t}\right)_{t=0}$$

and $DJ[\mu]$ by

$$DJ[\mu] = \left(\frac{dJ[\mu(t)]}{dt}\right)_{t=0} .$$

If μ_0 is such that for all such curves $\mathscr{C}$ $J[\mu]$ vanishes, then μ_0 is a critical point $\bar{\mu}$ of the functional $J[\mu]$ defined by (1.1).

A critical point of $J[\mu]$ is called a critical mapping with respect to the integral $J[\mu]$. Let us define a necessary and sufficient condition of a critical mapping μ in tensor form.

For this purpose we use the connecting tensor $H_{\mu\lambda}{}^h$ defined as the van der Waerden-Bortolotti derivative of B_λ^h, namely

$$H_{\mu\lambda}{}^h = \partial_\mu B_\lambda^h + \begin{Bmatrix} h \\ ji \end{Bmatrix} B_{\mu\lambda}^{ji} - \begin{Bmatrix} \kappa \\ \mu\lambda \end{Bmatrix} B_\kappa^h \tag{1.2}$$

where $\begin{Bmatrix} h \\ ji \end{Bmatrix}$ and $\begin{Bmatrix} \kappa \\ \mu\lambda \end{Bmatrix}$ are the Christoffels of g_{ji} and $h_{\mu\lambda}$ respectively.

As we have

$$DJ[\mu] = \int_N D[B_{\mu_1\cdots\mu_m}^{[1\cdots m]}B_{\lambda_1\cdots\lambda_m}^{[1\cdots m]}h^{\mu_1\lambda_1}\cdots h^{\mu_m\lambda_m} \det(g_{ji})]dV_N$$

and

$$
\begin{aligned}
DB^h_\kappa &= \partial_\kappa \eta^h , \\
DB^{[1\cdots m]}_{\mu_1\cdots\mu_m} &= m(DB^{[1}_{[\mu_1})B^{2\cdots m]}_{\mu_2\cdots\mu_m]} \\
&= m(\partial_{[\mu_1}\eta^{[1})B^{2\cdots m]}_{\mu_2\cdots\mu_m]} , \\
Dg &= gg^{ji}\eta^k\partial_k g_{ji} ,
\end{aligned}
$$

where $g = \det(g_{ji})$, we get

$$
\begin{aligned}
DJ[\mu] = \int_N [&2m(\partial_{[\mu_1}\eta^{[1})B^{2\cdots m]}_{\mu_2\cdots\mu_m]}B^{[1\cdots m]}_{\lambda_1\cdots\lambda_m} h^{\mu_1\lambda_1}\cdots h^{\mu_m\lambda_m}g \\
&+ B^{[1\cdots m]}_{\mu_1\cdots\mu_m} B^{[1\cdots m]}_{\lambda_1\cdots\lambda_m} h^{\mu_1\lambda_1}\cdots h^{\mu_m\lambda_m}gg^{ji}\eta^k\partial_k g_{ji}]dV_N .
\end{aligned}
$$

Since $\partial_{[\mu_1}B^{2\cdots m}_{\mu_2\cdots\mu_m]}$ always vanish, we get from the above expression

$$
DJ[\mu] = \int_N F dV_N
$$

where

$$
\begin{aligned}
F = &-2m\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}(\partial_{\mu_1]}B^{[1\cdots m]}_{\lambda_1\cdots\lambda_m})h^{\mu_1\lambda_1}\cdots h^{\mu_m\lambda_m}g \\
&- 2m\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}(\partial_{\mu_1]}(h^{\mu_1\lambda_1}\cdots h^{\mu_m\lambda_m}))B^{[1\cdots m]}_{[\lambda_1\cdots\lambda_m]}g \\
&- 2m\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}B^k_{\mu_1]}g^{ji}(\partial_k g_{ji})B^{[1\cdots m]}_{\lambda_1\cdots\lambda_m}h^{\mu_1\lambda_1}\cdots h^{\mu_m\lambda_m}g \\
&- m\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}(\partial_{\mu_1]}h_{\omega\nu})h^{\omega\nu}B^{[1\cdots m]}_{\lambda_1\cdots\lambda_m}h^{\mu_1\lambda_1}\cdots h^{\mu_m\lambda_m}g \\
&+ B^{[1\cdots m]}_{\mu_1\cdots\mu_m}B^{[1\cdots m]}_{\lambda_1\cdots\lambda_m}h^{\mu_1\lambda_1}\cdots h^{\mu_m\lambda_m}gg^{ji}\eta^k\partial_k g_{ji} .
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\partial_\mu B^{[1\cdots m]}_{\lambda_1\cdots\lambda_m} &= \partial_\mu B^{[1\cdots m]}_{[\lambda_1\cdots\lambda_m]} \\
&= m(\partial_\mu B^{[1}_{[\lambda_1})B^{2\cdots m]}_{\lambda_2\cdots\lambda_m]} \\
&= mH_{\mu[\lambda_1}{}^{[1}B^{2\cdots m]}_{\lambda_2\cdots\lambda_m]} + m\begin{Bmatrix}\kappa \\ \mu[\lambda_1\end{Bmatrix}B^{[1\,2\cdots m]}_{|\kappa|\lambda_2\cdots\lambda_m]} \\
&\quad - B^k_\mu\begin{Bmatrix} j \\ kj\end{Bmatrix}B^{[1\cdots m]}_{\lambda_1\cdots\lambda_m}
\end{aligned}
$$

where we have used (1.2) and the identity

$$
m\begin{Bmatrix}[1 \\ kj\end{Bmatrix}B^{|j|2\cdots m]}_{[\lambda_1\cdots\lambda_m]} = \begin{Bmatrix} j \\ kj\end{Bmatrix}B^{[1\cdots m]}_{[\lambda_1\cdots\lambda_m]} .
$$

We also have

$$
\begin{aligned}
&m\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}B^k_{\mu_1]} \\
&= \eta^1 B^{2\cdots m}_{[\mu_2\cdots\mu_m}B^k_{\mu_1]} - \sum_{t=2}^{m}\eta^t B^{[2\cdots 1\cdots m]}_{[\mu_2\cdots\mu_m}B^k_{\mu_1]} \\
&= \eta^k B^{[1\cdots m]}_{\mu_1\cdots\mu_m} .
\end{aligned}
$$

Substituting these identities into the expression of F we get

$$
\begin{aligned}
F = & -2m^2\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}H_{\mu_1][\lambda_1}{}^{[1}B^{2\cdots m]}_{\lambda_2\cdots\lambda_m]}h^{\mu_1\lambda_1}\cdots h^{\mu_m\lambda_m}g \\
& - 2m^2\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}\left\{ {\kappa \atop \mu_1][\lambda_1} \right\}B^{[12\cdots m]}_{|\kappa|\lambda_2\cdots\lambda_m]}h^{\mu_1\lambda_1}\cdots h^{\mu_m\lambda_m}g \\
& + 2\eta^k B^{[1\cdots m]}_{\mu_1\cdots\mu_m}\left\{ {j \atop kj} \right\}B^{[1\cdots m]}_{\lambda_1\cdots\lambda_m}h^{\mu_1\lambda_1}\cdots h^{\mu_m\lambda_m}g \\
& - 2m^2\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}(\partial_{\mu_1]}h^{\mu_1\lambda_1})h^{\mu_2\lambda_2}\cdots h^{\mu_m\lambda_m}B^{[1\cdots m]}_{[\lambda_1\cdots\lambda_m]}g \\
& - B^{[1\cdots m]}_{[\mu_1\cdots\mu_m]}B^{[1\cdots m]}_{\lambda_1\cdots\lambda_m}h^{\mu_1\lambda_1}\cdots h^{\mu_m\lambda_m}g^{ji}\eta^k(\partial_k g_{ji})g \\
& - 2m\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}\left\{ {\omega \atop \mu_1]\omega} \right\}B^{[1\cdots m]}_{\lambda_1\cdots\lambda_m}h^{\mu_1\lambda_1}\cdots h^{\mu_m\lambda_m}g\ .
\end{aligned} \tag{1.3}
$$

Since we have

$$
\begin{aligned}
& \left\{ {\kappa \atop \mu[\lambda_1} \right\}B^{[1\cdots m]}_{|\kappa|\lambda_2\cdots\lambda_m]}h^{[\mu_1|\lambda_1|}\cdots h^{\mu_m]\lambda_m} \\
& \quad = \left\{ {\lambda_1 \atop \mu\alpha} \right\}B^{[1\cdots m]}_{\lambda_1\cdots\lambda_m}h^{[\mu_1|\alpha|}h^{\mu_2|\lambda_2|}\cdots h^{\mu_m]\lambda_m} \\
& \quad = \left\{ {\lambda_1 \atop \mu\alpha} \right\}B^{[1\cdots m]}_{[\lambda_1\cdots\lambda_m]}h^{[\mu_1|\alpha|}h^{\mu_2|\lambda_2|}\cdots h^{\mu_m]\lambda_m}
\end{aligned}
$$

and $\partial_\nu h^{\mu\lambda} = -\left\{ {\mu \atop \nu\alpha} \right\}h^{\alpha\lambda} - \left\{ {\lambda \atop \nu\alpha} \right\}h^{\mu\alpha}$, we get

$$
\begin{aligned}
& -2m^2\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}\left\{ {\kappa \atop \mu_1][\lambda_1} \right\}B^{[12\cdots m]}_{|\kappa|\lambda_2\cdots\lambda_m]}h^{\mu_1\lambda_1}\cdots h^{\mu_m\lambda_m} \\
& -2m^2\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}(\partial_{\mu_1]}h^{\mu_1[\lambda_1})h^{|\mu_2|\lambda_2}\cdots h^{|\mu_m|\lambda_m]}B^{[1\cdots m]}_{[\lambda_1\cdots\lambda_m]} \\
& = -2m^2\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}\left\{ {\lambda_1 \atop \mu_1]\alpha} \right\}B^{[1\cdots m]}_{[\lambda_1\cdots\lambda_m]}h^{\mu_1\alpha}h^{\mu_2\lambda_2}\cdots h^{\mu_m\lambda_m} \\
& \quad + 2m^2\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}\left\{ {\mu_1 \atop \mu_1]\alpha} \right\}h^{\alpha\lambda_1}h^{\mu_2\lambda_2}\cdots h^{\mu_m\lambda_m}B^{[1\cdots m]}_{[\lambda_1\cdots\lambda_m]} \\
& \quad + 2m^2\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}\left\{ {\lambda_1 \atop \mu_1]\alpha} \right\}h^{\mu_1\alpha}h^{\mu_2\lambda_2}\cdots h^{\mu_m\lambda_m}B^{[1\cdots m]}_{[\lambda_1\cdots\lambda_m]} \\
& = 2m^2\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}\left\{ {\mu_1 \atop \mu_1]\alpha} \right\}h^{\alpha\lambda_1}h^{\mu_2\lambda_2}\cdots h^{\mu_m\lambda_m}B^{[1\cdots m]}_{[\lambda_1\cdots\lambda_m]} \\
& = 2m\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m]}\left\{ {\omega \atop \omega\alpha} \right\}h^{\alpha\lambda_1}h^{\mu_2\lambda_2}\cdots h^{\mu_m\lambda_m}B^{[1\cdots m]}_{[\lambda_1\cdots\lambda_m]} \\
& \quad - 2m\sum_{t=2}^{m}\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_1\cdots\mu_m]}\left\{ {\mu_1 \atop \mu_t\alpha} \right\}h^{\alpha\lambda_1}h^{\mu_2\lambda_2}\cdots h^{\mu_m\lambda_m}B^{[1\cdots m]}_{[\lambda_1\cdots\lambda_m]}\ .
\end{aligned}
$$

But the second term in the last member vanishes because of $\left\{ {\mu_1 \atop \mu_t \alpha} \right\} = \left\{ {\mu_1 \atop \alpha \mu_t} \right\}$ and we get

$$2m\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}\left\{ {\omega \atop \mu_1]\omega} \right\} h^{\mu_1\lambda_1}\cdots h^{\mu_m\lambda_m}B^{[1\cdots m]}_{[\lambda_1\cdots\lambda_m]}g$$

as the sum of the second and the fourth terms in the expression of F in (1.3). This cancels the last term. Moreover, it is easy to see that the third term cancels the fifth term.

Hence we have

$$F = -2m^2\eta^{[1}B^{2\cdots m]}_{[\mu_2\cdots\mu_m}H_{\mu_1][\lambda_1}{}^{[1}B^{2\cdots m]}_{\lambda_2\cdots\lambda_m]}h^{\mu_1\lambda_1}\cdots h^{\mu_m\lambda_m}g\ .$$

A critical mapping is characterized by the vanishing of F for all η^k. Thus μ is a critical mapping if and only if

$$\eta^{[j_1}B^{j_2\cdots j_m]}_{\mu_2\cdots\mu_m}H_{\mu_1\lambda_1}{}^{[i_1}B^{i_2\cdots i_m]}_{\lambda_2\cdots\lambda_m}h^{[\mu_1[\lambda_1}\cdots h^{\mu_m]\lambda_m]} = 0 \tag{1.4}$$

is satisfied by every vector field η^h of M.

Let us define H^{jih}, H^h, $'h^{ji}$ and $'h_{ji}$ by

$$H^{jih} = B^{ji}_{\mu\lambda}H^{\mu\lambda h} = B^{ji}_{\omega\nu}h^{\omega\mu}h^{\nu\lambda}H_{\mu\lambda}{}^h\ , \qquad H^h = H_{\mu\lambda}{}^h h^{\mu\lambda} = H^{\omega}{}_{\omega}{}^h\ ,$$
$$'h^{ji} = B^{ji}_{\mu\lambda}h^{\mu\lambda}\ , \qquad 'h_{ji}'h^{jh} = \delta^h_i\ .$$

As we have

$$m\eta^{[j_1}B^{j_2\cdots j_m]}_{\mu_2\cdots\mu_m}H_{\mu_1\lambda_1}{}^{[i_1}B^{i_2\cdots i_m]}_{\lambda_2\cdots\lambda_m}h^{[\mu_1[\lambda_1}\cdots h^{\mu_m]\lambda_m]} = \eta^{[j_1}B^{j_2\cdots j_m]}_{\mu_2\cdots\mu_m}H_{\mu_1\lambda_1}{}^{[i_1}B^{i_2\cdots i_m]}_{\lambda_2\cdots\lambda_m}$$
$$\times\left(h^{\mu_1\lambda_1}h^{\mu_2\lambda_2}\cdots h^{\mu_m\lambda_m} - \sum_{t=2}^{m}h^{\mu_1\lambda_t}h^{\mu_2\lambda_2}\cdots h^{\mu_t\lambda_1}\cdots h^{\mu_m\lambda_m}\right),$$

we can write (1.4) in the form

$$\eta^{[j_1}H^{[i_1}{'h}^{j_2i_2}\cdots {'h}^{j_m]i_m]}$$
$$-\sum_{t=2}^{m}\eta^{[j_1}H^{[i_tj_ti_1}{'h}^{j_2i_2}\cdots {'h}^{j_{t-1}i_{t-1}}{'h}^{j_{t+1}i_{t+1}}\cdots {'h}^{j_m]i_m]} = 0\ ,$$

hence

$$\begin{aligned}&\eta^{[j_1}H^{[i_1}{'h}^{j_2i_2}\cdots {'h}^{j_m]i_m]}\\ &\quad + (m-1)\eta^{[j_1}H^{[i_1j_2i_2}{'h}^{j_3i_3}\cdots {'h}^{j_m]i_m]} = 0\ .\end{aligned} \tag{1.5}$$

Since $i_1,\cdots,i_m$ and $j_1,\cdots,j_m$ run only the range $\{1,\cdots,m\}$, an equality of the form

$$A^{[j_1[i_1j_2i_2}{'h}^{j_3i_3}\cdots {'h}^{j_m]i_m]} = 0$$

is equivalent to

$$A^{j_1i_1j_2i_2}({'h}_{j_1i_1}{'h}_{j_2i_2} - {'h}_{j_2i_1}{'h}_{j_1i_2}) = 0\ .$$

Hence we get from (1.5)

$$\eta^j H^{i\prime} h_{ji} + \eta^j (H^{kli} - H^{ilk})' h_{jk}{}' h_{li} = 0 ,$$

and we can conclude that μ is a critical mapping if and only if μ satisfies

$$H^i - H^{kji\prime} h_{kj} + H^{ikj\prime} h_{kj} = 0 . \tag{1.6}$$

Thus we have obtained the following theorem.

THEOREM 1.1. *Let (M, g) and (N, h) be connected compact orientable Riemannian manifolds of dimension m and n respectively, where $n \geqq m$, and $\mu: N \to M$ be a differentiable mapping of rank m everywhere. Then a necessary and sufficient condition for μ to be a critical mapping is that μ satisfies the equations*

$$H_{\mu\lambda}{}^h h^{\mu\lambda} - H_{\omega\nu}{}^h h^{\omega\mu} h^{\nu\lambda} B_{\mu\lambda}^{ji}{}' h_{ji} + H_{\omega\nu}{}^j h^{\omega\mu} h^{\nu\lambda} B_{\mu\lambda}^{hk}{}' h_{kj} = 0 . \tag{1.7}$$

2. Some special cases. Let us consider the case where $\dim M = \dim N = m$. Then we can define B_h^κ by $B_\kappa^h B_i^\kappa = \delta_i^h$ and get

$$'h_{ji} = B_{ji}^{\mu\lambda} h_{\mu\lambda} , \qquad B_{\mu\lambda}^{ji}{}' h_{ji} = h_{\mu\lambda} .$$

Hence we have in this special case

$$H_{\omega\nu}{}^h h^{\omega\mu} h^{\nu\lambda} B_{\mu\lambda}^{ji}{}' h_{ji} = H_{\omega\nu}{}^h h^{\omega\nu} .$$

This proves the following theorem.

THEOREM 2.1. *Let (M, g) and (N, h) be connected compact orientable Riemannian manifolds of the same dimension m and $\mu: N \to M$ be a differentiable mapping of rank m everywhere. Then a necessary and sufficient condition for μ to be a critical mapping is that μ satisfies the equations*

$$H_{\mu\lambda}{}^h B_h^\lambda = 0 . \tag{2.1}$$

Let us assume that there exists the inverse $\mu^{-1}: M \to N$ of μ. Then B_h^κ plays the same role in μ^{-1} as B_κ^h does in μ and the connecting tensor

$$H_{ji}{}^\kappa = \partial_j B_i^\kappa + \left\{ {\kappa \atop \mu\lambda} \right\} B_{ji}^{\mu\lambda} - \left\{ {h \atop ji} \right\} B_h^\kappa$$

satisfies

$$B_{\mu\lambda\kappa}^{jih} H_{ji}{}^\kappa = -H_{\mu\lambda}{}^h .$$

This proves that $H_{ji}{}^\kappa B_\kappa^i$ vanishes if and only if $H_{\mu\lambda}{}^h B_h^\lambda$ vanishes. Thus we have the

COROLLARY 2.2. *If a differentiable mapping $\mu: N \to M$ admits the inverse mapping $\mu^{-1}: M \to N$, one is a critical mapping if and only if the other is a critical mapping.*

Let us consider connected compact orientable Riemannian manifolds (M_1, f), (M_2, g), (M_3, h) of the same dimension m and assume that there exist critical mappings $\mu_{12}: M_2 \to M_1$ and $\mu_{23}: M_3 \to M_2$. Let us use u^α, x^h and y^κ for the local coordinates in M_1, M_2, and M_3 respectively. For the mapping μ_{12} we have

$$B_h^\alpha = \frac{\partial u^\alpha}{\partial x^h}, \qquad H_{ji}{}^\alpha = \partial_j B_i^\alpha + \begin{Bmatrix} \alpha \\ \gamma\beta \end{Bmatrix} B_{ji}^{\gamma\beta} - \begin{Bmatrix} h \\ ji \end{Bmatrix} B_h^\alpha ,$$

and for the mapping μ_{23} we have

$$B_\kappa^h = \frac{\partial x^h}{\partial y^\kappa}, \qquad H_{\mu\lambda}{}^h = \partial_\mu B_\lambda^h + \begin{Bmatrix} h \\ ji \end{Bmatrix} B_{\mu\lambda}^{ji} - \begin{Bmatrix} \kappa \\ \mu\lambda \end{Bmatrix} B_\kappa^h ,$$

where $\begin{Bmatrix} \alpha \\ \gamma\beta \end{Bmatrix}$, $\begin{Bmatrix} h \\ ji \end{Bmatrix}$, and $\begin{Bmatrix} \kappa \\ \mu\lambda \end{Bmatrix}$ are the Christoffels derived from $f_{\gamma\beta}$, g_{ji}, and $h_{\mu\lambda}$ respectively. If we define B_α^h and B_h^κ by

$$B_\beta^h B_h^\alpha = \delta_\beta^\alpha , \qquad B_\lambda^h B_h^\kappa = \delta_\lambda^\kappa ,$$

we have

$$H_{ji}{}^\alpha B_\alpha^i = 0 , \qquad H_{\mu\lambda}{}^h B_h^\lambda = 0 . \tag{2.2}$$

Let us consider the mapping $\mu_{13} = \mu_{12}\mu_{23}$ of M_3 onto M_1. The connecting tensor of this mapping is

$$B_\kappa^\alpha = \frac{\partial u^\alpha}{\partial y^\kappa} = B_i^\alpha B_\kappa^i$$

and we get

$$\begin{aligned} H_{\mu\lambda}{}^\alpha &= \partial_\mu B_\lambda^\alpha + \begin{Bmatrix} \alpha \\ \gamma\beta \end{Bmatrix} B_{\mu\lambda}^{\gamma\beta} - \begin{Bmatrix} \kappa \\ \mu\lambda \end{Bmatrix} B_\kappa^\alpha \\ &= (B_\mu^j \partial_j B_i^\alpha) B_\lambda^i + B_h^\alpha \partial_\mu B_\lambda^h \\ &\quad + \begin{Bmatrix} \alpha \\ \gamma\beta \end{Bmatrix} B_{ji}^{\gamma\beta} B_{\mu\lambda}^{ji} - \begin{Bmatrix} \kappa \\ \mu\lambda \end{Bmatrix} B_h^\alpha B_\kappa^h \end{aligned}$$

hence

$$H_{\mu\lambda}{}^\alpha = H_{ji}{}^\alpha B_{\mu\lambda}^{ji} + H_{\mu\lambda}{}^h B_h^\alpha . \tag{2.3}$$

Then we immediately obtain

$$\begin{aligned} H_{\mu\lambda}{}^\alpha B_\alpha^\lambda &= H_{ji}{}^\alpha B_{\mu\lambda}^{ji} B_k^\lambda B_\alpha^k + H_{\mu\lambda}{}^h B_h^\alpha B_k^\lambda B_\alpha^k \\ &= B_\mu^j H_{ji}{}^\alpha B_\alpha^i + H_{\mu\lambda}{}^h B_h^\lambda \\ &= 0 \end{aligned}$$

by virtue of (2.2). This proves that, if μ_{12} and μ_{23} are critical mappings, then $\mu_{12}\mu_{23}$ is also a critical mapping.

Now let us consider a set of Riemannian manifolds $\{(M_\lambda, g_\lambda), \lambda \in \Lambda\}$ of the same dimension where each manifold is connected, compact and orientable. For any $\kappa, \lambda \in \Lambda$ we denote the set of homeomorphic critical mappings $M_\kappa \to M_\lambda$ by $G_{\lambda,\kappa}$. $G_{\lambda,\lambda}$ contains the identity mapping $e_\lambda: M_\lambda \to M_\lambda$, but e_κ and e_λ are distinguished if $\kappa \neq \lambda$. Then from the above results we see that the union of $G_{\lambda,\kappa}$ for all $\kappa, \lambda \in \Lambda$ forms a groupoid.
Thus we obtain the following theorem.

THEOREM 2.3. *When a set of Riemannian manifolds $\{(M_\lambda, g_\lambda), \lambda \in \Lambda\}$ of the same dimension is given where each manifold is connected, compact and orientable, the set of homeomorphic critical mappings forms a groupoid.*

In this theorem we have assumed that all Riemannian manifolds are of the same dimension. This assumption is essential. We consider now a case where $\mu_{12}: M_2 \to M_1$ and $\mu_{23}: M_3 \to M_2$ are critical mappings and $m_1 = \dim M_1$, $m_2 = \dim M_2$, $m_3 = \dim M_3$ satisfy $m_1 = m_2 < m_3$. Then we have (1.7) and $H_{ji}{}^\alpha B^i_\alpha = 0$. If we consider the mapping $\mu_{13} = \mu_{12}\mu_{23}$ we get (2.3) for this mapping too. Then we can prove that μ_{13} is also a critical mapping.

For this purpose we define $''h^{\gamma\beta}$ and $''h_{\gamma\beta}$ by

$$''h^{\gamma\beta} = B^{\gamma\beta}_{\mu\lambda} h^{\mu\lambda}, \qquad ''h_{\gamma\beta}{}''h^{\gamma\alpha} = \delta^\alpha_\beta .$$

Then we get

$$\begin{aligned} H_{\mu\lambda}{}^\alpha h^{\mu\lambda} &= H_{ji}{}^\alpha B^{ji}_{\mu\lambda} h^{\mu\lambda} + H_{\mu\lambda}{}^h h^{\mu\lambda} B^\alpha_h \\ &= H_{ji}{}^{\alpha}{}'h^{ji} + H_{\mu\lambda}{}^h h^{\mu\lambda} B^\alpha_h , \end{aligned}$$

$$\begin{aligned} &-H_{\omega\nu}{}^\alpha h^{\omega\mu} h^{\nu\lambda} B^{\gamma\beta}_{\mu\lambda}{}''h_{\gamma\beta} \\ &\quad = -(H_{ji}{}^\alpha B^{ji}_{\omega\nu} h^{\omega\mu} h^{\nu\lambda} + H_{\omega\nu}{}^h B^\alpha_h h^{\omega\mu} h^{\nu\lambda}) B^{\gamma\beta}_{lk} B^{lk}_{\mu\lambda}{}''h_{\gamma\beta} \\ &\quad = -H_{ji}{}^{\alpha}{}'h^{jl}{}'h^{ik}{}'h_{lk} - H_{\omega\nu}{}^h h^{\omega\mu} h^{\nu\lambda} B^{lk}_{\mu\lambda}{}'h_{lk} B^\alpha_h \\ &\quad = -H_{ji}{}^{\alpha}{}'h^{ji} - H_{\omega\nu}{}^h h^{\omega\mu} h^{\nu\lambda} B^{ji}_{\mu\lambda}{}'h_{ji} B^\alpha_h , \end{aligned}$$

$$\begin{aligned} &H_{\omega\nu}{}^\gamma h^{\omega\mu} h^{\nu\lambda} B^{\alpha\varepsilon}_{\mu\lambda}{}''h_{\varepsilon\gamma} \\ &\quad = H_{lk}{}^\gamma B^{lk}_{\omega\nu} h^{\omega\mu} h^{\nu\lambda} B^{hj}_{\mu\lambda} B^{\alpha\varepsilon}_{hj}{}''h_{\varepsilon\gamma} + H_{\omega\nu}{}^j B^\gamma_j h^{\omega\mu} h^{\nu\lambda} B^{hk}_{\mu\lambda} B^{\alpha\varepsilon}_{hk}{}''h_{\varepsilon\gamma} \\ &\quad = H_{lk}{}^{\gamma}{}'h^{lh}{}'h^{kj} B^\varepsilon_j{}''h_{\varepsilon\gamma} B^\alpha_h + H_{\omega\nu}{}^j h^{\omega\mu} h^{\nu\lambda} B^{hk}_{\mu\lambda}{}'h_{kj} B^\alpha_h \\ &\quad = H_{lk}{}^\gamma B^{k}_\gamma{}'h^{lk} B^\alpha_h + H_{\omega\nu}{}^j h^{\omega\mu} h^{\nu\lambda} B^{hk}_{\mu\lambda}{}'h_{kj} B^\alpha_h . \end{aligned}$$

As the first term in the last member vanishes, we get

$$H_{\mu\lambda}{}^\alpha h^{\mu\lambda} - H_{\omega\nu}{}^\alpha h^{\omega\mu} h^{\nu\lambda} B^{\gamma\beta}_{\mu\lambda}{}''h_{\gamma\beta} + H_{\omega\nu}{}^\gamma h^{\omega\mu} h^{\nu\lambda} B^{\alpha\varepsilon}_{\mu\lambda}{}''h_{\varepsilon\gamma} = 0$$

by virtue of (1.7).

Thus we have the following theorem.

THEOREM 2.4. *Let there be three connected compact orientable Riemannian manifolds* (M_1, f), (M_2, g), *and* (M_3, h) *admitting critical mappings* $\mu_{12}: M_2 \to M_1$ *and* $\mu_{23}: M_3 \to M_2$. *If* $\dim M_1 = \dim M_2$ *and* $\dim M_2 < \dim M_3$, *the mapping* $\mu_{13} = \mu_{12}\mu_{23}$ *is a critical mapping.*

If we want to prove only the Theorem 2.3, we can use the following property of a critical mapping.

If $\dim N = \dim M = m$, the connecting tensor B^h_κ of a critical mapping $\mu: (N, h) \to (M, g)$ satisfies

$$B^{[h_1\cdots h_m]}_{[\kappa_1\cdots\kappa_m]}\frac{\sqrt{g}}{\sqrt{h}} = \text{const}\,. \tag{2.4}$$

Conversely, if (2.4) is satisfied, μ is a critical mapping.

(2.4) is proved by taking the partial derivatives and using (2.1).

3. Infinitesimal transformations of a Riemannian manifold. Let us take a one-parameter group of transformations $\mu(t)$ of a connected compact orientable Riemannian manifold (M, g). Then we have a case of $N = M$, $h = g$. If $\mu(t)$ takes a point P into $Q = \mu(t)P$ and the local coordinates of P and Q are respectively denoted by $x^h(P)$ and $x^h(Q)$, $\partial x^h(Q)/\partial x^i(P)$ plays the role of B^h_κ and

$$\frac{\partial^2 x^h(Q)}{\partial x^j(P)\partial x^i(P)} + \left\{ {h \atop lk} \right\}_Q \frac{\partial x^l(Q)}{\partial x^j(P)}\frac{\partial x^k(Q)}{\partial x^i(P)} - \left\{ {k \atop ji} \right\}_P \frac{\partial x^h(Q)}{\partial x^k(P)}$$

plays the role of $H_{\mu\lambda}{}^h$. Hence the transformations $\mu(t)$ are critical mappings of (M, g) onto (M, g) if and only if

$$\frac{\partial x^i(P)}{\partial x^h(Q)}\left[\frac{\partial^2 x^h(Q)}{\partial x^j(P)\partial x^i(P)} + \left\{ {h \atop lk} \right\}_Q \frac{\partial x^l(Q)}{\partial x^j(P)}\frac{\partial x^k(Q)}{\partial x^i(P)} - \left\{ {k \atop ji} \right\}_P \frac{\partial x^h(Q)}{\partial x^k(P)}\right] = 0 \tag{3.1}$$

is satisfied.

Let v^h be a vector field on M generating the group $\mu(t)$. Then we get

$$\delta^i_h\left[\frac{\partial^2 v^h}{\partial x^j \partial x^i} + \partial_k\left\{ {h \atop ji} \right\}v^k + \left\{ {h \atop ki} \right\}\partial_j v^k + \left\{ {h \atop jk} \right\}\partial_i v^k - \left\{ {k \atop ji} \right\}\partial_k v^h\right] = 0$$

from (3.1). But this is equivalent to

$$\nabla_j \nabla_i v^i = 0\,,$$

hence $\nabla_i v^i = C$. On the other hand, we have always

$$\int_M \nabla_i v^i dV_M = 0 .$$

Hence we get

$$\nabla_i v^i = 0 .$$

Thus we obtain the following theorem.

THEOREM 3.1. *A one-parameter group of transformations of a connected compact orientable Riemannian manifold (M, g) generated by a vector field v^h is a group of critical mappings if and only if v^h satisfies $\nabla_i v^i = 0$. The set of all such vector fields forms a Lie algebra.*

4. Examples.

1°. Coclosed mappings. Let $\mu: (N, h) \to (M, g)$ be a coclosed mapping [1]. Then

$$-H_{\omega\nu}{}^h h^{\omega\mu} B^i_\mu + H_{\omega\nu}{}^i h^{\omega\mu} B^h_\mu = B^k_\nu P_k{}^{ih}, \; H_{\mu\lambda}{}^h h^{\mu\lambda} = -P_k{}^{kh} \tag{4.1}$$

are compatible. From (4.1) we obtain

$$-H_{\omega\nu}{}^h h^{\omega\mu} h^{\nu\lambda} B^{ij}_{\mu\lambda}{}' h_{ij} + H_{\omega\nu}{}^i h^{\omega\mu} h^{\nu\lambda} B^{hj}_{\mu\lambda}{}' h_{ji} = P_k{}^{ih} B^{kj}_{\nu\lambda} h^{\nu\lambda}{}' h_{ij} = P_k{}^{kh} = -H_{\mu\lambda}{}^h h^{\mu\lambda} ,$$

which proves that μ is then a critical mapping.

A geodesic mapping is a mapping where $H_{\mu\lambda}{}^h$ vanishes. Hence this is a coclosed mapping [1] and also a critical mapping.

2°. A critical mapping $\mu: (N, h) \to (M, g)$ where $\dim N - \dim M = 1$. In this case a vector field ξ^κ of (N, h) is determined by

$$B^h_\kappa \xi^\kappa = 1 , \qquad h_{\mu\lambda} \xi^\mu \xi^\lambda = 1 .$$

Let ξ_λ be defined by $\xi_\lambda = h_{\lambda\kappa} \xi^\kappa$ and let (B^κ_i, ξ^κ) be the inverse matrix of $(B^h_\lambda, \xi_\lambda)$, namely such that

$$B^\kappa_i \xi_\kappa = 0 , \qquad B^\kappa_i B^h_\kappa = \delta^h_i .$$

Then we have

$$'h_{ji} = B^{\mu\lambda}_{ji} h_{\mu\lambda} , \qquad B^\kappa_i B^i_\lambda = \delta^\kappa_\lambda - \xi^\kappa \xi_\lambda .$$

The condition that μ is a critical mapping is written in the form

$$H_{\mu\lambda}{}^h h^{\mu\lambda} - H_{\omega\nu}{}^h h^{\omega\mu} h^{\nu\lambda} B^{ji}_{\mu\lambda} B^{\rho\sigma}_{ji} h_{\rho\sigma} + H_{\omega\nu}{}^j h^{\omega\mu} h^{\nu\lambda} B^{hk}_{\mu\lambda} B^{\rho\sigma}_{kj} h_{\rho\sigma} = 0 .$$

As we have

$$H_{\omega\nu}{}^h h^{\omega\mu} h^{\nu\lambda} (\delta^\rho_\mu - \xi_\mu \xi^\rho)(\delta^\sigma_\lambda - \xi_\lambda \xi^\sigma) h_{\rho\sigma} = H_{\omega\nu}{}^h h^{\omega\nu} - H_{\omega\nu}{}^h \xi^\omega \xi^\nu$$

and

$$H_{\omega\nu}{}^j h^{\omega\mu} h^{\nu\lambda} B^h_\mu B^\sigma_j (\delta^\rho_\lambda - \xi_\lambda \xi^\rho) h_{\rho\sigma} = H_{\omega\nu}{}^j h^{\omega\mu} B^h_\mu B^\nu_j ,$$

we get

$$H_{\mu\lambda}{}^{h}\xi^{\mu}\xi^{\lambda} + H_{\mu\lambda}{}^{j}h^{\mu\kappa}B_{\kappa}^{h}B_{j}^{\lambda} = 0 \tag{4.2}$$

as a necessary and sufficient condition of a critical mapping.

3°. Projection of a fibred Riemannian manifold $\tilde{M}$ with an invariant Riemannian metric h onto the base manifold (M^*, g). Let $\tilde{M}$ be a fibred Riemannian manifold with S^1 as the type fibre and with an invariant Riemannian metric h [2]. The base manifold (M^*, g) is assumed to be a compact orientable Riemannian manifold. We denote the projection by π. For this mapping the vector field ξ^{κ} determined by

$$B_{\kappa}^{h}\xi^{\kappa} = 0\,, \qquad h_{\mu\lambda}\xi^{\mu}\xi^{\lambda} = 1$$

is a Killing vector field of $\tilde{M}$. If $\tilde{\nabla}$ denotes the covariant differentiation with respect to the metric h, we get $\tilde{\nabla}_{\mu}\xi_{\lambda} + \tilde{\nabla}_{\lambda}\xi_{\mu} = 0$ and $\xi^{\mu}\tilde{\nabla}_{\mu}\xi^{\lambda} = 0$.

As $h_{\mu\lambda}$ is an invariant metric, we have

$$B_{\mu\lambda}^{ji}h^{\mu\lambda} = g^{ji}\,. \tag{4.3}$$

We have also

$$H_{\mu\lambda}{}^{h}\xi^{\mu}\xi^{\lambda} = 0$$

by virtue of $B_{\kappa}^{h}\xi^{\kappa} = 0$ and $(\xi^{\mu}\tilde{\nabla}_{\mu}\xi^{\lambda})B_{\lambda}^{h} = 0$.

On the other hand, applying van der Waerden-Bortolotti differentiation to (4.3) we get

$$H_{\nu\mu}{}^{j}B_{\lambda}^{i}h^{\mu\lambda} + H_{\nu\lambda}{}^{i}B_{\mu}^{j}h^{\mu\lambda} = \nabla_{\nu}g^{ji} = 0$$

and consequently

$$0 = H_{\nu\mu}{}^{j}B_{\lambda}^{i}h^{\mu\lambda}B_{j}^{\rho}B_{i}^{\sigma}h_{\rho\sigma} = H_{\nu\mu}{}^{j}h^{\mu\lambda}B_{j}^{\rho}h_{\rho\sigma}(\delta_{\lambda}^{\sigma} - \xi_{\lambda}\xi^{\sigma}) = H_{\nu\mu}{}^{j}B_{j}^{\mu}\,.$$

Hence π satisfies (4.2) and is a critical mapping among all mappings $\mu\colon \tilde{M} \to M^*$.

4°. A critical mapping $\mu\colon (N, h) \to (M, g)$ where $\dim N = \dim M$. Let the local coordinates of N and M be chosen such that the point P of N and the point μP of M have the same coordinates x^h. Then we have

$$B_{\kappa}^{h} = \delta_{\kappa}^{h}\,, \qquad H_{ji}{}^{h} = \left\{ {h \atop ji} \right\}_g - \left\{ {h \atop ji} \right\}_h$$

where $\left\{ {h \atop ji} \right\}_g$ and $\left\{ {h \atop ji} \right\}_h$ are the Christoffels derived respectively from g and h. From (2.1) we get

$$\left\{ {i \atop ji} \right\}_g = \left\{ {i \atop ji} \right\}_h$$

as a necessary and sufficient condition for a critical mapping. This result also proves Theorem 2.3.

References

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