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GENERATORS AND DIRECT INTEGRAL DECOMPOSITIONS OF W^*-ALGEBRAS

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Let A be a W^*-algebra on separable Hilbert space H. A is *singly generated* if there is an operator $T \in A$ such that A is the smallest W^*-algebra containing T. It has long been conjectured that every A is singly generated. Recently, Douglas and Pearcy have shown that this conjecture can be reduced to the conjecture that if A is of type II_1 then A is *-isomorphic to $A \otimes I_2$ [3, Theorem 2].

In this paper we first show that if A has direct integral decomposition into factors $A = \int_\Lambda \oplus A(\lambda)\mu(d\lambda)$ then A is singly generated if and only if μ-a.e. $A(\lambda)$ are singly generated. We then make use of this fact together with several known results to give some other reductions of the conjecture concerning single generators.

In dealing with direct integrals, we follow the presentation of [6]. In particular, K denotes the underlying separable Hilbert space of H, and S denotes the unit ball in $B(K)$, taken with the strong*-topology. By B_n we denote a sequence of operators in A such that $\{B_n(\lambda)\}$ is strong*-dense in the unit ball of $A(\lambda)$ μ-a.e. and such that the $B_n(\lambda)$ are strong*-continuous. We denote by $M(R, T)$ a metric which defines the strong topology on bounded sets of S [6, Lemma I.4.9].

THEOREM 1. *A is singly generated if and only if $A(\lambda)$ is singly generated for μ-a.e. λ.*

PROOF. Let F be the countable collection of all polynomials in two non-commuting variables Z and Z^* with complex coefficients having rational real and imaginary parts. If $f \in F$, $f(T)$ denotes the operator obtained by replacing Z and Z^* by T and T^* respectively.

Suppose first that A is generated by $T = \int_\Lambda \oplus T(\lambda)\mu(d\lambda)$. For each B_n there is a sequence $f_j \in F$ such that $f_j(T) \to B_n$ strongly and (by the Kaplansky Density Theorem) such that $|f_j(T)| \leqq 1$. Hence by [8, Lemma 1] we may assume that $M(B_n(\lambda), f_j(T(\lambda))) \to 0$ μ-a.e.. Since $f_j(T(\lambda))$ belongs to the unit ball of $A(\lambda)$ μ-a.e. it is clear that $T(\lambda)$ generates $A(\lambda)$ μ-a.e..

Conversely, for any $f \in F$ and any natural numbers n and m define the subset $E(f, n, m)$ of $\Lambda \times S$ as the set of all pairs (λ, T) satisfying the following conditions.

(i) $T \in A(\lambda)$

(ii) $f(T) \in S \cap A(\lambda)$

(iii) $M(f(T), B_n(\lambda)) \leqq 1/m$.

Clearly each $E(f, n, m)$ is a Borel subset of $\Lambda \times S$, and hence so is $G = \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{f \in F} E(f, n, m)$. It is apparent that $(\lambda, T) \in G$ precisely when T generates $A(\lambda)$. Assume that $A(\lambda)$ is singly generated for all λ. Thus the projection of G onto Λ is all of Λ. By [6, Lemma I.4.7] there is a function $T(\lambda)$ from Λ to S such that $[\lambda, T(\lambda)] \in G$ for all λ. Let $T = \int_{\Lambda} \oplus T(\lambda)\mu(d\lambda)$. Then $T(\lambda)$ generates $A(\lambda)$ for every λ. Let C be an hermitian generator of the center of A, $Z = A \cap A'$. By [7, Lemma 4], A is singly generated, since it is clearly generated by the two commuting operators T and C. q.e.d.

We now draw some conclusions from this theorem.

COROLLARY 2. *If all factors of type* II_1 *are singly generated, then every* A *is singly generated.*

PROOF. This follows from the theorem and from the known result that type I factors are singly generated [5] and that properly infinite algebras are singly generated [9]. q.e.d.

Using an idea of Behncke, it is possible to reduce the conjecture still further. In [1, p. 47] Behncke defines the following property for a factor of type II_1. A of type II_1 has *property* A if there exist factors A_1 and A_2 of type II_1 such that $A \cong A_1 \otimes A_2$. In particular, by [4] every factor of type II_1 with central, non-hypercentral sequences has property A. Thus we have the following.

COROLLARY 3. *If every factor of type* II_1, *for which every central sequence is hypercentral, is singly generated, then every* A *is singly generated.*

Finally, we recall that Dixmier has conjectured that every non-hyperfinite type II_1 factor A is isomorphic to $A \otimes U(\Phi_2)$, where $U(\Phi_2)$ is the factor generated by the free group on two generators [2, p. 225]. Clearly this conjecture implies that all A are singly generated.

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