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ANALYTIC GENERATORS FOR ONE-PARAMETER GROUPS

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The purpose of generator theory of one-parameter semi-groups and groups of operators is to establish a correspondence between these families of operators and a single mathematical object, in general an unbounded operator. We propose in the case of groups such a correspondence, based on analytical extension.

It is proved that the "analytic generator" is closed (Theorem 2.4). We study the spectral properties of the "analytic generator" (Theorem 3.2 with its corollaries and Theorem 3.6) and give a representation formula for the group in terms of its "analytic generator" (Theorem 4.2). Spectral subspaces for the "analytic generator" are defined and it is shown (Corollary 5.7) that they coincide with the spectral subspaces associated by W. Arveson to an one-parameter group (see [1]). Finally, we examine two particular cases, obtaining also a new proof of Stone's representation theorem (Theorem 6.1).

We remark that our results can be used in Tomita's theory of standard von Neumann algebras (see the remarks about [5] and [15] in the last section).

1. Vector valued functions. In this section, with an introductory character, we are precising some facts about analyticity and integrability of vector valued functions.

We call *dual pair of Banach spaces* any pair $(X, \mathscr{F})$ of Banach spaces together with a bilinear functional

$$X \times \mathscr{F} \ni (x, \varphi) \mapsto \langle x, \varphi \rangle ,$$

such that

$$\|x\| = \sup_{\substack{\varphi \in \mathscr{F} \\ \|\varphi\| \leq 1}} |\langle x, \varphi \rangle| \quad \text{for any} \quad x \in X ,$$

$$\|\varphi\| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} |\langle x, \varphi \rangle| \quad \text{for any} \quad \varphi \in \mathscr{F} .$$

In all this paper we consider only complex Banach spaces.

We recall the following classical result ([8], Theorem 2.8.6):

PROPOSITION 1.1. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces and*

$S \subset X$ such that for every $\varphi \in \mathscr{F}$

$$\sup_{x \in S} |\langle x, \varphi \rangle| < +\infty .$$

Then

$$\sup_{x \in S} \|x\| < +\infty .$$

Proposition 1.1 implies the following result ([8], Theorem 3.10.1.)

PROPOSITION 1.2. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $D \subset \boldsymbol{C}^n$ an open set and $F: D \to X$ such that for every $\varphi \in \mathscr{F}$*

$$D \ni \alpha \mapsto \langle F(\alpha), \varphi \rangle$$

is analytic. Then F is analytic in the norm-topology of X.

By Proposition 1.2 the analyticity of an X-valued function depends not on the topology considered on X.

Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, Ω a topological space and $F: \Omega \to X$. Then F is called *$\mathscr{F}$-continuous* if for every $\varphi \in \mathscr{F}$ the function

$$\Omega \ni \alpha \mapsto \langle F(\alpha), \varphi \rangle$$

is continuous. If $\Omega \subset \boldsymbol{C}^n$ then F is called *$\mathscr{F}$-regular* if it is $\mathscr{F}$-continuous and its restriction to the interior of Ω is analytic.

We are interested especially in regular functions on vertical strip and half-planes. If $a < b$, f is regular on $\{\alpha \in \boldsymbol{C};\ a \leqq \operatorname{Re}\alpha \leqq b\}$ and $f(\alpha) = 0$ for $\operatorname{Re}\alpha = a$ then f vanishes identically. We recall also the following theorem of F. Carlson ([3] or [10], Part Three, Problem 328):

PROPOSITION 1.3. *Let $f: \{\alpha \in \boldsymbol{C};\ \operatorname{Re}\alpha \geqq 0\} \to \boldsymbol{C}$ be a regular function such that*

(i) $|f(\alpha)| \leqq c_1 e^{c_2|\alpha|}$ *for* $\operatorname{Re}\alpha \geqq 0$, *with* $c_1, c_2 \geqq 0$;
(ii) $|f(it)| \leqq c_3 e^{c_4|t|}$ *for* $t \in \boldsymbol{R}$, *with* $c_3 \geqq 0, \pi > c_4 \geqq 0$;
(iii) $0 = f(0) = f(1) = \cdots = f(n) = \cdots$.

Then f vanishes identically.

Let $(X, \mathscr{F})$ be a dual pair of Banach spaces.

An everywhere defined linear operator T on X is called *$\mathscr{F}$-continuous* if, considering on X the $\mathscr{F}$-topology, T is continuous. By Proposition 1.1 every $\mathscr{F}$-continuous linear operator is bounded. We denote the linear space of all $\mathscr{F}$-continuous linear operators on X by $B_{\mathscr{F}}(X)$. It is easy to see that any $T \in B_{\mathscr{F}}(X)$ defines a $T^* \in B_X(\mathscr{F})$ by

$$\langle T(x), \varphi \rangle = \langle x, T^*(\varphi) \rangle , \quad x \in X , \quad \varphi \in \mathscr{F} .$$

A linear operator T in X is called *$\mathscr{F}$-closed* if the graph of T is closed in the product of the $\mathscr{F}$-topologies. Every $T \in B_{\mathscr{F}}(X)$ is obviously $\mathscr{F}$-closed.

We consider for $(X, \mathscr{F})$ the following axioms:

(A_1) *the convex hull of every relatively $\mathscr{F}$-compact subset of X is relatively $\mathscr{F}$-compact*:

(A_2) *any everywhere defined $\mathscr{F}$-closed linear operator in X belongs to $B_{\mathscr{F}}(X)$.*

The following result is a slight extension of [13], Ch. IV., Exercise 39(a) and [1], Proposition 1.2:

PROPOSITION 1.4. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, satisfying (A_1), Ω a locally compact Hausdorff space, μ a complex regular Borel measure on Ω with variation $|\mu|$, let $F: \Omega \to X$ be such that for any compact $K \subset \Omega$ and $\varepsilon > 0$ there exists a compact $L \subset K$ such that $F|L$ is $\mathscr{F}$-continuous and $|\mu|(K|L) \leqq \varepsilon$ and such that*

$$\Omega \ni \alpha \mapsto ||F(\alpha)||$$

has a $|\mu|$-integrable majorant. Then there exists a unique $x_F \in X$ such that for any $\varphi \in \mathscr{F}$

$$\int_\Omega \langle F(\alpha), \varphi \rangle d\mu(\alpha) = \langle x_F, \varphi \rangle .$$

In the conditions of Proposition 1.4 we denote

$$x_F = \mathscr{F} - \int_\Omega F(\alpha) d\mu(\alpha) .$$

It is easy to verify that for every $T \in B_{\mathscr{F}}(X)$ the mapping $\alpha \mapsto T(F(\alpha))$ and the measure μ satisfy also the conditions of Proposition 1.4 and

$$\mathscr{F} - \int_\Omega T(F(\alpha)) d\mu(\alpha) = T\Big(\mathscr{F} - \int_\Omega F(\alpha) d\mu(\alpha)\Big) .$$

Let T be an $\mathscr{F}$-closed linear operator in X. The *resolvent set* $\rho(T)$ of T is the set of all $\lambda \in \boldsymbol{C}$ such that $\lambda - T: \mathscr{D}_T \to X$ is injective and surjective, that is $(\lambda - T)^{-1}$ is well and everywhere defined. $\rho(T)$ is open in $\boldsymbol{C}$. If $(X, \mathscr{F})$ satisfies (A_2) and $\lambda \in \rho(T)$ then $(\lambda - T)^{-1} \in B_{\mathscr{F}}(X)$. The *spectrum* of T is $\sigma(T) = \boldsymbol{C} \backslash \rho(T)$.

Let X be a Banach space and X^* its dual space. Then (X, X^*) endowed with the natural pairing is a dual pair of Banach spaces, satisfying (A_1) ([7], Theorem V.6.4) and (A_2) (the usual closed graph theorem). The pair (X^*, X) is also a dual pair of Banach spaces, satisfying (A_1)

(by the Alaoglu theorem) and (A_2) (by the usual closed graph theorem, [8], Theorem 2.10.4 and the Alaoglu theorem).

2. Groups of operators and analytical extensions. Let $(X, \mathscr{F})$ be a dual pair of Banach spaces.

An *one-parameter group* in $B_{\mathscr{F}}(X)$ is a family $\{U_t\}_{t\in \boldsymbol{R}}$ in $B_{\mathscr{F}}(X)$ such that

$$U_0 = \text{id}.,$$
$$U_{t+s} = U_t U_s , \qquad t, s \in \boldsymbol{R} .$$

The group $\{U_t\}_{t\in \boldsymbol{R}}$ is called

(i) *$\mathscr{F}$-continuous* if for every $x \in X$ the mapping $t \mapsto U_t x$ is $\mathscr{F}$-continuous;

(ii) *strongly continuous* if for every $x \in X$ the mapping $t \mapsto U_t x$ is norm-continuous. Obviously, (ii) $\Rightarrow$ (i).

If $\{U_t\}$ is $\mathscr{F}$-continuous, then using Proposition 1.1, we have for any $t_0 \in \boldsymbol{R}_+ = \{t \in \boldsymbol{R};\ t \geqq 0\}$

$$\sup_{-t_0 \leqq t \leqq t_0} \|U_t\| < +\infty .$$

$\{U_t\}$ is called

(iii) *bounded* if $\sup_{t\in \boldsymbol{R}} \|U_t\| < +\infty$.

The following known majorisation of $\|U_t\|$ is implied by [8], Theorem 7.6.1 and by the above remark:

LEMMA 2.1. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $\{U_t\}_{t\in \boldsymbol{R}}$ an $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$ and*

$$a > \max\left\{\inf_{t>0} \frac{\ln \|U_t\|}{t},\ \inf_{t<0} \frac{\ln \|U_t\|}{-t}\right\} .$$

Then there exists a constant $c > 0$ such that

$$\|U_t\| \leqq c e^{a|t|} , \qquad t \in \boldsymbol{R} .$$

Let $\{U_t\}_{t\in \boldsymbol{R}}$ be an one-parameter group in $B_{\mathscr{F}}(X)$. For every $\varepsilon_1, \varepsilon_2 \in \boldsymbol{R}$, $\varepsilon_1 \leqq 0$, $\varepsilon_2 \geqq 0$, we consider the following linear subspace of X:

$${}_{\varepsilon_1}\mathscr{D}_{\varepsilon_2} = \left\{x \in X;\ \begin{array}{l} it \mapsto U_t x \text{ has an } \mathscr{F}\text{-regular} \\ \text{extension on } \varepsilon_1 \leqq \operatorname{Re} \alpha \leqq \varepsilon_2 \end{array}\right\} .$$

If $x \in {}_{\varepsilon_1}\mathscr{D}_{\varepsilon_2}$ then the $\mathscr{F}$-regular extension of $it \mapsto U_t x$ on $\varepsilon_1 \leqq \operatorname{Re} \alpha \leqq \varepsilon_2$ is unique. We denote it by F_x.

Denote

$$_{-\infty}\mathscr{D}_{+\infty} = \bigcap_{\substack{\varepsilon_1, \varepsilon_2 \in \boldsymbol{R} \\ \varepsilon_1 \leqq 0 \\ \varepsilon_2 \geqq 0}} {}_{\varepsilon_1}\mathscr{D}_{\varepsilon_2} = \left\{ x \in X; \begin{array}{l} it \mapsto U_t x \text{ has an analytical} \\ \text{extension on } \boldsymbol{C} \end{array} \right\} .$$

The following density lemma is inspired from [9]:

LEMMA 2.2. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, satisfying* (A_1) *and $\{U_t\}_{t \in \boldsymbol{R}}$ an $\mathscr{F}$-continuous group in $B_{\mathscr{F}}(X)$. Then the sequential $\mathscr{F}$-closure of $_{-\infty}\mathscr{D}_{+\infty}$ is X.*

PROOF. Following Lemma 2.1, there exist positive constants a, c, such that

$$\|U_t\| \leqq c e^{a|t|} , \qquad t \in \boldsymbol{R} .$$

Let $x \in X$ be arbitrary. For every integer $n \geqq 1$ the mapping $t \mapsto e^{-nt^2} U_t x$ and the Lebesgue measure on $\boldsymbol{R}$ satisfy the conditions of Proposition 1.4, so we can consider

$$x_n = \sqrt{\frac{n}{\pi}} \mathscr{F} - \int_{-\infty}^{+\infty} e^{-nt^2} U_t x dt .$$

By the $\mathscr{F}$-continuity of $\{U_t\}$ in 0 and by the Lebesgue dominated convergence theorem the sequence $\{x_n\}$ converges to x in the $\mathscr{F}$-topology. On the other hand, for every $n \geqq 1$

$$\alpha \mapsto F_{x_n}(\alpha) = \sqrt{\frac{n}{\pi}} \mathscr{F} - \int_{-\infty}^{+\infty} e^{-n(t+i\alpha)^2} U_t x dt$$

is an analytical extension on $\boldsymbol{C}$ of $it \mapsto U_t x_n$. Hence $x_n \in {}_{-\infty}\mathscr{D}_{+\infty}$. q.e.d.

Using Lemma 2.2 and the remark before Lemma 2.1, it is easy to prove that if $\mathscr{F} = X^*$, then every $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$ is strongly continuous ([8], Corollary of Theorem 10.2.3).

Now we show how the group property of $\{U_t\}$ can be extended by analyticity:

LEMMA 2.3. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $\{U_t\}_{t \in \boldsymbol{R}}$ an $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$, $\varepsilon_1, \varepsilon_2 \in \boldsymbol{R}$, $\varepsilon_1 \leqq 0$, $\varepsilon_2 \geqq 0$, and $\beta \in \boldsymbol{C}$, $\varepsilon_1 \leqq \operatorname{Re} \beta \leqq \varepsilon_2$. Then for every $x \in {}_{\varepsilon_1}\mathscr{D}_{\varepsilon_2}$ we have:*

$$F_x(\beta) \in {}_{\varepsilon_1 - \operatorname{Re}\beta}\mathscr{D}_{\varepsilon_2 - \operatorname{Re}\beta} ,$$

$$F_{F_x(\beta)}(\alpha) = F_x(\alpha + \beta) , \quad \varepsilon_1 - \operatorname{Re}\beta \leqq \operatorname{Re}\alpha \leqq \varepsilon_2 - \operatorname{Re}\beta .$$

PROOF. Let $t \in \boldsymbol{R}$. The mappings

$$\gamma \mapsto U_t F_x(\gamma) ,$$
$$\gamma \mapsto F_x(it + \gamma)$$

are $\mathscr{F}$-regular extensions of $is \mapsto U_{t+s}x$ on $\varepsilon_1 \leqq \operatorname{Re} \gamma \leqq \varepsilon_2$, hence they coincide.

Consequently, for each $t \in \boldsymbol{R}$

$$U_t F_x(\beta) = F_x(it + \beta) ,$$

and this implies that $\alpha \mapsto F_x(\alpha + \beta)$ is an $\mathscr{F}$-regular extension of $it \mapsto U_t F_x(\beta)$ on $\varepsilon_1 - \operatorname{Re} \beta \leqq \operatorname{Re} \alpha \leqq \varepsilon_2 - \operatorname{Re} \beta$. From this the statement of the lemma follows. q.e.d.

Let $\{U_t\}_{t \in \boldsymbol{R}}$ be an $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$. For every $\alpha \in \boldsymbol{C}$ we define a linear operator B_α in X, called the *analytical extension of* $\{U_t\}$ in α, by the formulas:

$$\mathscr{D}_{B_\alpha} = \begin{cases} {}_0\mathscr{D}_{\operatorname{Re}\alpha} & \text{if } \operatorname{Re} \alpha \geqq 0 , \\ {}_{\operatorname{Re}\alpha}\mathscr{D}_0 & \text{if } \operatorname{Re} \alpha \leqq 0 , \end{cases}$$
$$B_\alpha x = F_x(\alpha) , \qquad x \in \mathscr{D}_{B_\alpha} .$$

We call B_1 the *analytic generator* of $\{U_t\}$ and denote it simply by B.

The following theorem is our basic result in this section:

THEOREM 2.4. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces such that $(X, \mathscr{F})$ and $(\mathscr{F}, X)$ satisfy* (A$_1$) *and $\{U_t\}_{t \in \boldsymbol{R}}$ an $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$. Then the sequential $\mathscr{F}$-closure of*

$$\bigcap_{\alpha \in \boldsymbol{C}} \mathscr{D}_{B_\alpha} = {}_{-\infty}\mathscr{D}_{+\infty}$$

is X, for any $\alpha \in \boldsymbol{C}$ the linear operator B_α is $\mathscr{F}$-closed and injective and:

$$B_{it} = U_t , \qquad t \in \boldsymbol{R} ,$$
$$B_{-\alpha} = B_\alpha^{-1} , \qquad \alpha \in \boldsymbol{C} ,$$
$$B_{\alpha_1 + \alpha_2} = B_{\alpha_1} B_{\alpha_2} , \qquad \alpha_1, \alpha_2 \in \boldsymbol{C},\ (\operatorname{Re} \alpha_1)(\operatorname{Re} \alpha_2) \geqq 0 .$$

PROOF. Following Lemma 2.2, the sequential $\mathscr{F}$-closure of $\bigcap_{\alpha \in \boldsymbol{C}} \mathscr{D}_{B_\alpha} = {}_{-\infty}\mathscr{D}_{+\infty}$ is X.

Let $\alpha \in \boldsymbol{C}$.

Suppose that (x, y) is in the closure of the graph of B_α in the product of the $\mathscr{F}$-topologies. Then there exists a net $\{x_\iota\} \subset \mathscr{D}_{B_\alpha}$ such that

$$x_\iota \longrightarrow x ,$$
$$B_\alpha x_\iota \longrightarrow y$$

in the Mackey topology associated to the $\mathscr{F}$-topology.

By Lemma 2.1 there exist $a, c > 0$ such that

$$\| U_t \| \leqq c e^{a|t|} , \qquad t \in \boldsymbol{R} .$$

Hence for any $\lambda \in \boldsymbol{C}$

$$\lim_{|t|\to\infty} e^{-t^2+\lambda t} \| U_t \| = 0 .$$

Let $\varphi \in \mathscr{F}$.

For every ι and for every $\gamma \in \boldsymbol{C}$ with $\mathrm{Re}\,\gamma$ between 0 and $\mathrm{Re}\,\alpha$, using Lemma 2.3, we have

$$\begin{aligned} &|e^{-(\gamma/i)^2}\langle F_{x_\iota}(\gamma), \varphi\rangle| \\ &\quad = |e^{-(\gamma/i)^2}\langle U_{\mathrm{Im}\gamma} F_{x_\iota}(\mathrm{Re}\,\gamma), \varphi\rangle| \\ &\quad \leqq e^{(\mathrm{Re}\gamma)^2-(\mathrm{Im}\gamma)^2} \| U_{\mathrm{Im}\gamma} \| \, \| F_{x_\iota}(\mathrm{Re}\,\gamma) \| \, \| \varphi \| \\ &\quad \leqq e^{(\mathrm{Re}\gamma)^2} c e^{-(\mathrm{Im}\gamma)^2+i|\mathrm{Im}\gamma|} \sup_{\substack{\varepsilon \text{ between} \\ 0 \text{ and } \mathrm{Re}\alpha}} \| F_{x_\iota}(\varepsilon) \| \, \| \varphi \| . \end{aligned}$$

Hence $\gamma \mapsto e^{-(\gamma/i)^2}\langle F_{x_\iota}(\gamma), \varphi\rangle$ is a bounded regular function on $\{\gamma \in \boldsymbol{C};\ \mathrm{Re}\,\gamma$ between 0 and $\mathrm{Re}\,\alpha\}$.

Since $\{U_t^*\}$ is an X-continuous one-parameter group in $B_X(\mathscr{F})$ and

$$\lim_{|t|\to\infty} e^{-t^2} \| U_t^* \| = \lim_{|t|\to\infty} e^{-t^2} \| U_t \| = 0 ,$$

the set

$$\{e^{-t^2} U_t^* \varphi;\ t \in \boldsymbol{R}\} \subset \mathscr{F}$$

is relatively X-compact. Since $(\mathscr{F}, X)$ satisfies (A_1), the convex hull of this set is also relatively X-compact, hence

$$e^{-t^2}\langle x_\iota, U_t^* \varphi\rangle \longrightarrow e^{-t^2}\langle x, U_t^* \varphi\rangle$$

uniformly for $t \in \boldsymbol{R}$.

Analogously,

$$e^{-((\alpha+it)/i)^2}\langle B_\alpha x_\iota, U_t^* \varphi\rangle \longrightarrow e^{-((\alpha+it)/i)^2}\langle y, U_t^* \varphi\rangle$$

uniformly for $t \in \boldsymbol{R}$.

In conclusion, the net $\{\gamma \mapsto e^{-(\gamma/i)^2}\langle F_{x_\iota}(\gamma), \varphi\rangle\}$ of bounded regular functions on $\{\gamma \in \boldsymbol{C};\ \mathrm{Re}\,\gamma$ between 0 and $\mathrm{Re}\,\alpha\}$ converges uniformly on the boundary. By the Phragmen-Lindelöf theorem it converges uniformly to a bounded regular function G. We have for any $t \in \boldsymbol{R}$

$$G_\varphi(it) = \lim_\iota e^{-t^2}\langle U_t x_\iota, \varphi\rangle = e^{-t^2}\langle U_t x, \varphi\rangle ,$$

$$G_\varphi(\alpha + it) = \lim_\iota e^{-((\alpha+it)/i)^2}\langle U_t B_\alpha x_\iota, \varphi\rangle = e^{-((\alpha+it)/i)^2}\langle U_t y, \varphi\rangle .$$

Now the $\mathscr{F}$-continuity of $\{U_t\}$ and the relations

$$\lim_{|t|\to\infty} e^{-t^2}\|U_t\| = 0 ,$$
$$\lim_{|t|\to\infty} e^{-((\alpha+it)/i)^2}\|U_t\| = 0$$

imply that the set

$$\{e^{-t^2}U_tx, e^{-((\alpha+it)/i)^2}U_ty; t \in \boldsymbol{R}\} \subset X$$

is relatively $\mathscr{F}$-compact. Since $(X, \mathscr{F})$ satisfies (A_1), the convex hull K of this set is also relatively $\mathscr{F}$-compact.

For every $\varphi \in \mathscr{F}$ and for every $t \in \boldsymbol{R}$

$$|G_\varphi(it)| = |\langle e^{-t^2}U_tx, \varphi\rangle| \leqq \sup_{z\in K} |\langle z, \varphi\rangle| ,$$
$$|G_\varphi(\alpha + it)| = |\langle e^{-((\alpha+it)/i)^2}U_ty, \varphi\rangle| \leqq \sup_{z\in K} |\langle z, \varphi\rangle| .$$

By the Phragmen-Lindelöf theorem, for any $\gamma \in \boldsymbol{C}$ with $\operatorname{Re}\gamma$ between 0 and $\operatorname{Re}\alpha$

$$|G_\varphi(\gamma)| \leqq \sup_{z\in K} |\langle z, \varphi\rangle| .$$

Let γ with $\operatorname{Re}\gamma$ between 0 and $\operatorname{Re}\alpha$. Then

$$\varphi \mapsto G_\varphi(\gamma)$$

is a linear functional on $\mathscr{F}$, continuous in the Mackey topology associated to the X-topology. There results that it is continuous in the X-topology, hence there exists $G(\gamma) \in X$ such that

$$\langle G(\gamma), \varphi\rangle = G_\varphi(\gamma) , \qquad \varphi \in \mathscr{F} .$$

Obviously G is an $\mathscr{F}$-regular mapping on $\{\gamma \in \boldsymbol{C}; \operatorname{Re}\gamma$ between 0 and $\operatorname{Re}\alpha\}$ and

$$G(it) = e^{-t^2}U_tx , \qquad t \in \boldsymbol{R} ,$$
$$G(\alpha) = e^{-(\alpha/i)^2}y .$$

Putting $F(\gamma) = e^{(\gamma/i)^2}G(\gamma)$, F is $\mathscr{F}$-regular on $\{\gamma \in \boldsymbol{C}; \operatorname{Re}\gamma$ between 0 and $\operatorname{Re}\alpha\}$ and

$$F(it) = U_tx , \qquad t \in \boldsymbol{R} ,$$
$$F(\alpha) = y .$$

Consequently $x \in \mathscr{D}_{B_\alpha}$ and $B_\alpha x = y$.

Hence we have proved that B_α is $\mathscr{F}$-closed.

Let $x \in \mathscr{D}_{B_\alpha}$, $B_\alpha x = 0$. Using Lemma 2.3, $F_x(\gamma) = 0$ for $\operatorname{Re}\gamma = \operatorname{Re}\alpha$, hence F_x is identically 0. In particular, $x = F_x(0) = 0$. Consequently B_α is injective.

Finally, the statement

$$B_{it} = U_t\ , \qquad t \in \boldsymbol{R}\ ,$$

is trivial and, using Lemma 2.3, it is easy to see that

$$B_{-\alpha} = B_\alpha^{-1}\ , \qquad \alpha \in \boldsymbol{C}\ ,$$
$$B_{\alpha_1+\alpha_2} = B_{\alpha_1}B_{\alpha_2}\ , \qquad \alpha_1, \alpha_2 \in \boldsymbol{C},\ (\mathrm{Re}\,\alpha_1)(\mathrm{Re}\,\alpha_2) \geqq 0\ .$$

q.e.d.

In particular, in the conditions of Theorem 2.4, the analytic generator of $\{U_t\}$ is $\mathscr{F}$-closed.

Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, satisfying (A_1), and $\{U_t\}$ an $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$. If $f\colon \boldsymbol{R} \longrightarrow \boldsymbol{C}$ is Lebesgue measurable and

$$\boldsymbol{R} \ni t \mapsto f(t)\|U_t\|$$

is Lebesgue integrable then for any $x \in X$

$$\mathscr{F} - \int_{-\infty}^{+\infty} f(t)U_t x dt$$

is well defined (Proposition 1.4). The next result shows that the operators B_α commute with the operator

$$x \mapsto \mathscr{F} - \int_{-\infty}^{+\infty} f(t)U_t x dt\ .$$

COROLLARY 2.5. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces such that $(X, \mathscr{F})$ and $(\mathscr{F}, X)$ satisfy (A_1), $\{U_t\}_{t\in\boldsymbol{R}}$ an $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$ and $f\colon \boldsymbol{R} \longrightarrow \boldsymbol{C}$ a Lebesgue measurable function such that*

$$\boldsymbol{R} \ni t \mapsto f(t)\|U_t\|$$

is Lebesgue integrable. Then for any $\alpha \in \boldsymbol{C}$ and for any $x \in \mathscr{D}_{B_\alpha}$

$$\mathscr{F} - \int_{-\infty}^{+\infty} f(t)U_t x dt \in \mathscr{D}_{B_\alpha}$$

and

$$B_\alpha\left(\mathscr{F} - \int_{-\infty}^{+\infty} f(t)U_t x dt\right) = \mathscr{F} - \int_{-\infty}^{+\infty} f(t)U_t B_\alpha x dt\ .$$

PROOF. If f is bounded and has compact support then it is easy to see that

$$\gamma \mapsto \mathscr{F} - \int_{-\infty}^{+\infty} f(t)F_x(it + \gamma)dt$$

is an $\mathscr{F}$-regular extension of

$$is \mapsto U_s\left(\mathscr{F} - \int_{-\infty}^{+\infty} f(t)U_t x dt\right)$$

on $\{\gamma \in \boldsymbol{C};\ \mathrm{Re}\,\gamma$ between 0 and $\mathrm{Re}\,\alpha\}$. In the general case we approximate f with the functions f_n defined by

$$f_n(t) = \begin{cases} f(t)\,, & |t| \leqq n \quad \text{and} \quad |f(t)| \leqq n\,, \\ 0\,, & |t| > n \quad \text{or} \quad |f(t)| > n\,, \end{cases}$$

and use the fact that B_α is $\mathscr{F}$-closed. q.e.d.

COROLLARY 2.6. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces such that $(X, \mathscr{F})$ and $(\mathscr{F}, X)$ satisfy* (A_1) *and $\{U_t\}_{t\in \boldsymbol{R}}$ an $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$. Then for any $\alpha \in \boldsymbol{C}$, B_α is the sequential $\mathscr{F}$-closure of the restriction of B_α to ${}_{-\infty}\mathscr{D}_{+\infty}$.*

PROOF. Let $x \in \mathscr{D}_{B_\alpha}$.

In the proof of Lemma 2.2 we have seen that for any integer $n \geqq 1$

$$x_n = \sqrt{\frac{n}{\pi}}\mathscr{F} - \int_{-\infty}^{+\infty} e^{-nt^2} U_t x dt \in {}_{-\infty}\mathscr{D}_{+\infty}\,.$$

Using Corollary 2.5,

$$B_\alpha x_n = \sqrt{\frac{n}{\pi}}\mathscr{F} - \int_{-\infty}^{+\infty} e^{-nt^2} U_t B_\alpha x dt\,.$$

Since $x_n \to x$ and $B_\alpha x_n \to Bx$ in the $\mathscr{F}$-topology, $(x, B_\alpha x)$ belongs to the sequential $\mathscr{F}$-closure of the graph of $B_\alpha|_{-\infty}\mathscr{D}_{+\infty}$. q.e.d.

We remark that by Theorem 2.4, in reasonable conditions, every continuous one-parameter group $\{U_t\}$ on X defines an analytic two-parameter group $\{B_{s+it}|_{-\infty}\mathscr{D}_{+\infty}\}$ on ${}_{-\infty}\mathscr{D}_{+\infty}$. We shall see in the fourth section that if $\{U_t\}$ is bounded then $\{B_{s+it}|_{-\infty}\mathscr{D}_{+\infty}\}$ is uniquely determined by $B_1|_{-\infty}\mathscr{D}_{+\infty}$.

3. Spectral properties of the analytic generator. In this section we study the spectral behaviour of the analytic generator B of a bounded one-parameter group. More precisely, we study the injectivity and the surjectivity of $\lambda + B$, $\lambda \in \boldsymbol{C}$, we give an integral formula for $(\lambda + B)^{-1}Bx$, $x \in \mathscr{D}_B$, in term of the group and we characterize the situation $\sigma(B) \subset \boldsymbol{R}_+$. In the particular cases considered in the last section we have always $\sigma(B) \subset \boldsymbol{R}_+$.

Denote $\boldsymbol{R}_- = \{t \in \boldsymbol{R};\ t \leqq 0\}$. For any $\lambda \in \boldsymbol{C}\backslash\boldsymbol{R}_-$ we define $\arg\lambda \in \boldsymbol{R}$ by

$$\lambda = |\lambda| e^{i\cdot\arg\lambda}\,, \qquad |\arg\lambda| < \pi\,.$$

Then the function

$$\boldsymbol{C} \ni \alpha \mapsto \lambda^{\alpha} = e^{\alpha(\ln|\lambda| + i \cdot \arg \lambda)}$$

is analytical.

LEMMA 3.1. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $\{U_t\}_{t\in \boldsymbol{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$, B_ε its analytical extension in $\varepsilon \in \boldsymbol{R}$ and $\lambda \in \boldsymbol{C}\backslash\boldsymbol{R}_-$. Then $\lambda + B_\varepsilon$ is injective.*

PROOF. Let $x \in \mathscr{D}_{B_\varepsilon}$ such that $(\lambda + B_\varepsilon)x = 0$. Then $it \mapsto U_t x$ has an analytical extension F_x on the whole complex plane and

$$F_x(\alpha + \varepsilon) = -\lambda F_x(\alpha)\,, \qquad \alpha \in \boldsymbol{C}\,.$$

If $\varepsilon = 0$ then $x = F_x(0) = -\lambda F_x(0) = -\lambda x$, so $x = 0$.

If $\varepsilon \neq 0$ then $\alpha \mapsto |\lambda|^{-\varepsilon^{-1}\alpha} F_x(\alpha)$ is a bounded analytic mapping on $\boldsymbol{C}$. By the Liouville theorem it is constant, hence

$$|\lambda|^{-1} F_x(\varepsilon) = F_x(0) = x\,,$$
$$F_x(\varepsilon) = |\lambda| x\,.$$

Since $F_x(\varepsilon) = -\lambda F_x(0) = -\lambda x$ and $\lambda \notin \boldsymbol{R}_-$, that is $\lambda \neq -|\lambda|$, we deduce $x = 0$. q.e.d.

The following theorem and its corollaries are our strongest results about the surjectivity of $\lambda + B$ in the general case:

THEOREM 3.2. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, satisfying (A_1), $\{U_t\}_{t\in \boldsymbol{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$, B_ε its analytical extension in $\varepsilon \in \boldsymbol{R}$, $\varepsilon > 0$, and $\lambda \in \boldsymbol{C}\backslash\boldsymbol{R}_-$. Then for any $x \in \mathscr{D}_{B_\varepsilon}$*

$$B_\varepsilon x \in \mathscr{D}_{(\lambda + B_\varepsilon)^{-1}}$$

and

$$(\lambda + B_\varepsilon)^{-1} B_\varepsilon x = (2\varepsilon i)^{-1} \mathscr{F} - \int_{c-i\infty}^{c+i\infty} \frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin \pi\varepsilon^{-1}\alpha} F_x(\alpha)\, d\alpha\,,$$

where $c \in \boldsymbol{R}$, $0 < c < \varepsilon$, is arbitrary.

PROOF. Using Lemma 2.3, Proposition 1.4 and the Cauchy integral theorem, it is easy to see that

$$\mathscr{F} - \int_{c-i\infty}^{c+i\infty} \frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin \pi\varepsilon^{-1}\alpha} F_x(\alpha)\, d\alpha$$

is well defined and depends not on c. Denote

$$\begin{aligned}
x_\lambda &= (2\varepsilon i)^{-1}\mathscr{F}-\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin\pi\varepsilon^{-1}\alpha}F_x(\alpha)d\alpha \\
&= (2\varepsilon i)^{-1}\mathscr{F}-\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin\pi\varepsilon^{-1}\alpha}(F_x(\alpha)-x)d\alpha \\
&\quad + (2\varepsilon i)^{-1}\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin\pi\varepsilon^{-1}\alpha}d\alpha x\,.
\end{aligned}$$

For each $t \in \boldsymbol{R}$

$$\begin{aligned}
U_t x_\lambda &= (2\varepsilon i)^{-1}\mathscr{F}-\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin\pi\varepsilon^{-1}\alpha}(F_x(\alpha+it)-F_x(it))d\alpha \\
&\quad + (2\varepsilon i)^{-1}\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin\pi\varepsilon^{-1}\alpha}d\alpha F_x(it) \\
&= (2\varepsilon i)^{-1}\mathscr{F}-\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}(\alpha-it)}}{\sin\pi\varepsilon^{-1}(\alpha-it)}(F_x(\alpha)-F_x(it))d\alpha \\
&\quad + (2\varepsilon i)^{-1}\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin\pi\varepsilon^{-1}\alpha}d\alpha F_x(it)\,.
\end{aligned}$$

Consider on $\{(\alpha, \beta) \in \boldsymbol{C}^2;\ 0 < \operatorname{Re}\alpha < \varepsilon,\ 0 \leqq \operatorname{Re}\beta \leqq \varepsilon\}$ the $\mathscr{F}$-regular mapping G defined by

$$G(\alpha, \beta) = \begin{cases} \dfrac{\lambda^{-\varepsilon^{-1}(\alpha-\beta)}}{\sin\pi\varepsilon^{-1}(\alpha-\beta)}(F_x(\alpha)-F_x(\beta)) & \text{if } \alpha \neq \beta\,, \\ \dfrac{1}{\pi\varepsilon^{-1}}F_x'(\alpha) & \text{if } \alpha = \beta\,. \end{cases}$$

Then

$$\begin{aligned}
U_t x_\lambda &= (2\varepsilon i)^{-1}\mathscr{F}-\int_{c-i\infty}^{c+i\infty}G(\alpha, it)d\alpha \\
&\quad + (2\varepsilon i)^{-1}\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin\pi\varepsilon^{-1}\alpha}d\alpha F_x(it)\,.
\end{aligned}$$

Hence $x_\lambda \in \mathscr{D}_{B_\varepsilon}$ and

$$\begin{aligned}
B_\varepsilon x_\lambda &= (2\varepsilon i)^{-1}\mathscr{F}-\int_{c-i\infty}^{c+i\infty}G(\alpha, \varepsilon)d\alpha \\
&\quad + (2\varepsilon i)^{-1}\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin\pi\varepsilon^{-1}\alpha}d\alpha B_\varepsilon x \\
&= (2\varepsilon i)^{-1}\mathscr{F}-\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha+1}}{\sin(\pi\varepsilon^{-1}\alpha-\pi)}(F_x(\alpha)-B_\varepsilon x)d\alpha \\
&\quad + (2\varepsilon i)^{-1}\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin\pi\varepsilon^{-1}\alpha}d\alpha B_\varepsilon x
\end{aligned}$$

$$= -\lambda(2\varepsilon i)^{-1}\mathscr{F}-\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin \pi\varepsilon^{-1}\alpha}F_x(\alpha)d\alpha$$

$$+ (1+\lambda)(2\varepsilon i)^{-1}\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin \pi\varepsilon^{-1}\alpha}d\alpha B_\varepsilon x .$$

By [5], Kap. 6, §8, we have

$$\begin{aligned}(2\varepsilon i)^{-1}\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin \pi\varepsilon^{-1}\alpha}d\alpha &= (2i)^{-1}\int_{\varepsilon^{-1}c-i\infty}^{\varepsilon^{-1}c+i\infty}\frac{\lambda^{-\gamma}}{\sin \pi\gamma}d\gamma \\ &= (2\pi i)^{-1}\int_{\varepsilon^{-1}c-i\infty}^{\varepsilon^{-1}c+i\infty}\lambda^{-\gamma}\Gamma(\gamma)\Gamma(1-\gamma)d\gamma \\ &= (1+\lambda)^{-1} ,\end{aligned}$$

so

$$\begin{aligned}B_\varepsilon x_\lambda &= -\lambda x_\lambda + B_\varepsilon x , \\ (\lambda + B_\varepsilon)x_\lambda &= B_\varepsilon x .\end{aligned}$$

Consequently, $B_\varepsilon x \in \mathscr{D}_{(\lambda+B_\varepsilon)^{-1}}$ and $(\lambda + B_\varepsilon)^{-1}B_\varepsilon x = x_\lambda$. q.e.d.

COROLLARY 3.3. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, satisfying (A_1), $\{U_t\}_{t\in \boldsymbol{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$, B_ε its analytical extension in $\varepsilon \in \boldsymbol{R}$, $\varepsilon > 0$, and $\lambda \in \boldsymbol{C}\backslash\boldsymbol{R}_-$. Then*

$$\mathscr{D}_{B_\varepsilon} \subset \mathscr{D}_{(\lambda+B_\varepsilon)^{-1}}$$

and for any $x \in \mathscr{D}_{B_\varepsilon}$

$$(\lambda + B_\varepsilon)^{-1}x = \lambda^{-1}x - (2\lambda\varepsilon i)^{-1}\mathscr{F}-\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin \pi\varepsilon^{-1}\alpha}F_x(\alpha)d\alpha ,$$

where $c \in \boldsymbol{R}$, $0 < c < \varepsilon$, is arbitrary.

PROOF. Let $x \in \mathscr{D}_{B_\varepsilon}$. Then by Theorem 3.2 $B_\varepsilon x \in \mathscr{D}_{(\lambda+B_\varepsilon)^{-1}}$ and

$$(\lambda + B_\varepsilon)^{-1}B_\varepsilon x = (2\varepsilon i)^{-1}\mathscr{F}-\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin \pi\varepsilon^{-1}\alpha}F_x(\alpha)d\alpha .$$

Since obviously $(\lambda + B_\varepsilon)x \in \mathscr{D}_{(\lambda+B_\varepsilon)^{-1}}$, there results that

$$x = \lambda^{-1}(\lambda + B_\varepsilon)x - \lambda^{-1}B_\varepsilon x \in \mathscr{D}_{(\lambda+B_\varepsilon)^{-1}}$$

and

$$\begin{aligned}(\lambda + B_\varepsilon)^{-1}x &= \lambda^{-1}x - \lambda^{-1}(\lambda + B_\varepsilon)^{-1}B_\varepsilon x \\ &= \lambda^{-1}x - (2\lambda\varepsilon i)^{-1}\mathscr{F}-\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin \pi\varepsilon^{-1}\alpha}F_x(\alpha)d\alpha .\end{aligned}$$

q.e.d.

We remark that Lemma 3.1, Corollary 3.3 and Corollary 2.5 imply that if $(X, \mathscr{F})$ satisfies (A_1) and $\{U_t\}$ is bounded then for every $\varepsilon \in \boldsymbol{R}$ and $\lambda \in \boldsymbol{C} \setminus \boldsymbol{R}_-$

$$(\lambda + B_\varepsilon)|_{-\infty}\mathscr{D}_{+\infty} : {}_{-\infty}\mathscr{D}_{+\infty} \longrightarrow {}_{-\infty}\mathscr{D}_{+\infty}$$

is injective and surjective. So the spectrum of $B_\varepsilon|_{-\infty}\mathscr{D}_{+\infty}$ is included in $\boldsymbol{R}_+$.

COROLLARY 3.4. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces such that $(X, \mathscr{F})$ and $(\mathscr{F}, X)$ satisfy (A_1), $\{U_t\}_{t \in \boldsymbol{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$, B_ε, B_δ its analytical extensions in $\varepsilon \in \boldsymbol{R}$, $\varepsilon > 0$, respectively in $\delta \in \boldsymbol{R}$, $\delta > 0$, and $\lambda \in \boldsymbol{C} \setminus \boldsymbol{R}_-$. Then*

$$\mathscr{D}_{B_\delta} \subset \mathscr{D}_{(\lambda + B_\varepsilon)^{-1}}$$

and for any $x \in \mathscr{D}_{B_\delta}$

$$(\lambda + B_\varepsilon)^{-1}x = \lambda^{-1}x - (2\lambda\varepsilon i)^{-1}\mathscr{F} - \int_{c-i\infty}^{c+i\infty} \frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin \pi\varepsilon^{-1}\alpha} F_x(\alpha)d\alpha ,$$

where $c \in \boldsymbol{R}$, $0 < c < \varepsilon$, $0 < c < \delta$, is arbitrary.

PROOF. If $\delta \geqq \varepsilon$ then $\mathscr{D}_{B_\delta} \supset \mathscr{D}_{B_\varepsilon}$, hence the statement of the corollary is a trivial consequence of Corollary 3.3.

Now suppose that $\delta < \varepsilon$. Let $x \in \mathscr{D}_{B_\delta}$. Using Corollary 2.6, there exists a sequence $\{x_n\} \subset \mathscr{D}_{B_\varepsilon}$ such that

$$x_n \longrightarrow x ,$$
$$B_\delta x_n \longrightarrow B_\delta x$$

in the $\mathscr{F}$-topology. We remark that by Proposition 1.1 every $\mathscr{F}$-convergent sequence is norm-bounded.

For every $t \in \boldsymbol{R}$

$$F_{x_n}(\delta + it) = U_t B_\delta x_n \longrightarrow U_t B_\delta x = F_x(\delta + it)$$

in the $\mathscr{F}$-topology, so, using the Lebesgue dominated convergence theorem,

$$\begin{aligned} y_n &= \lambda^{-1}x_n - (2\lambda\varepsilon i)^{-1}\mathscr{F} - \int_{c-i\infty}^{c+i\infty} \frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin \pi\varepsilon^{-1}\alpha} F_{x_n}(\alpha)d\alpha \\ &= \lambda^{-1}x_n - (2\lambda\varepsilon i)^{-1}\mathscr{F} - \int_{\delta-i\infty}^{\delta+i\infty} \frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin \pi\varepsilon^{-1}\alpha} F_{x_n}(\alpha)d\alpha \end{aligned}$$

converges in the $\mathscr{F}$-topology to

$$\begin{aligned} y &= \lambda^{-1}x - (2\lambda\varepsilon i)^{-1}\mathscr{F} - \int_{c-i\infty}^{c+i\infty} \frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin \pi\varepsilon^{-1}\alpha} F_x(\alpha)d\alpha \\ &= \lambda^{-1}x - (2\lambda\varepsilon i)^{-1}\mathscr{F} - \int_{\delta-i\infty}^{\delta+i\infty} \frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin \pi\varepsilon^{-1}\alpha} F_x(\alpha)d\alpha . \end{aligned}$$

On the other hand, following Corollary 3.3, for every n, $y_n \in \mathscr{D}_{\lambda+B_\varepsilon}$ and $(\lambda + B_\varepsilon)y_n = x_n$. So

$$(\lambda + B_\varepsilon)y_n \to x$$

in the $\mathscr{F}$-topology.

By Theorem 2.4 $\lambda + B_\varepsilon$ is $\mathscr{F}$-closed and we deduce that $y \in \mathscr{D}_{\lambda+B_\varepsilon}$ and $(\lambda + B_\varepsilon)y = x$. q.e.d.

In particular, in the conditions of Corollary 3.4,

$$\mathscr{D}_{(\lambda+B_\varepsilon)^{-1}} \supset \bigcup_{\delta>0} \mathscr{D}_{B_\delta} .$$

We remark also the following consequence of Theorem 3.2 which extends [5], Lemmas 4.2 and 4.3.

COROLLARY 3.5. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, satisfying (A_1), $\{U_t\}_{t\in R}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$, B_ε, $B_{\varepsilon/2}$, $B_{-\varepsilon/2}$ its analytical extensions respectively in ε, $\varepsilon/2$, $-\varepsilon/2$, where $\varepsilon \in \boldsymbol{R}$, $\varepsilon > 0$, and $\lambda \in \boldsymbol{C}\backslash\boldsymbol{R}_-$. Then for any $x \in \mathscr{D}_{B_{\varepsilon/2}} \cap \mathscr{D}_{B_{-\varepsilon/2}}$*

$$B_{\varepsilon/2}x \in \mathscr{D}_{(\lambda+B_\varepsilon)^{-1}}$$

and

$$(\lambda + B_\varepsilon)^{-1}B_{\varepsilon/2}x = \varepsilon^{-1}\mathscr{F}-\int_{-\infty}^{+\infty}\frac{\lambda^{-1/2-i\varepsilon^{-1}t}}{e^{\pi\varepsilon^{-1}t} + e^{-\pi\varepsilon^{-1}t}}U_t x dt .$$

PROOF. Following Lemma 2.3, $y = B_{-\varepsilon/2}x \in \mathscr{D}_{B_\varepsilon}$. Using Theorem 3.2, $B_{\varepsilon/2}x = B_\varepsilon y \in \mathscr{D}_{(\lambda+B_\varepsilon)^{-1}}$ and

$$\begin{aligned}
(\lambda + B_\varepsilon)^{-1}B_{\varepsilon/2}x &= (\lambda + B_\varepsilon)^{-1}B_\varepsilon y \\
&= (2\varepsilon i)^{-1}\mathscr{F}-\int_{\varepsilon/2-i\infty}^{\varepsilon/2+i\infty}\frac{\lambda^{-\varepsilon^{-1}\alpha}}{\sin \pi\varepsilon^{-1}\alpha}F_y(\alpha)d\alpha \\
&= (2\varepsilon)^{-1}\mathscr{F}-\int_{-\infty}^{+\infty}\frac{\lambda^{-(1/2)-i\varepsilon^{-1}t}}{\sin\left(\frac{\pi}{2} + i\pi\varepsilon^{-1}t\right)}F_y(\varepsilon/2 + it)dt \\
&= (2\varepsilon)^{-1}\mathscr{F}-\int_{-\infty}^{+\infty}\frac{\lambda^{-1/2-i\varepsilon^{-1}t}}{\cos i\pi\varepsilon^{-1}t}U_t B_{\varepsilon/2}y dt \\
&= \varepsilon^{-1}\mathscr{F}-\int_{-\infty}^{+\infty}\frac{\lambda^{-1/2-i\varepsilon^{-1}t}}{e^{\pi\varepsilon^{-1}t} + e^{-\pi\varepsilon^{-1}t}}U_t x dt .
\end{aligned}$$

q.e.d.

Now we can characterize the situation $\sigma(B) \subset \boldsymbol{R}_+$:

THEOREM 3.6. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces such* *that* $(X, \mathscr{F})$ *satisfies* (A_1), (A_2) *and* $(\mathscr{F}, X)$ *satisfies* (A_1). *Then for any bounded* $\mathscr{F}$*-continuous one-parameter group* $\{U_t\}_{t\in \boldsymbol{R}}$ *in* $B_{\mathscr{F}}(X)$ *the following statements are equivalent*:

(i) *there exist* $\varepsilon_0 \in \boldsymbol{R}$, $\varepsilon_0 > 0$, *and* $\lambda_0 \in \boldsymbol{C}\backslash\boldsymbol{R}_-$ *such that the image of* $\lambda_0 + B_{\varepsilon_0}$ *is sequentially* $\mathscr{F}$*-closed;*

(ii) *there exists* $\varepsilon_0 \in \boldsymbol{R}$, $\varepsilon_0 > 0$, *such that* $\rho(B_{\varepsilon_0}) \neq \emptyset$;

(iii) *for every* $\varepsilon \in \boldsymbol{R}$ *we have* $\sigma(B_\varepsilon) \subset \boldsymbol{R}_+$.

PROOF. Obviously, (iii) $\Rightarrow$ (ii) and, since $\rho(B_{\varepsilon_0})$ is open, (ii) $\Rightarrow$ (i). It remains to show that (i) $\Rightarrow$ (iii).

By Lemma 3.1 $\lambda_0 + B_{\varepsilon_0}$ is injective and by Corollary 3.3 and Lemma 2.2 the sequential $\mathscr{F}$-closure of the image of $\lambda_0 + B_{\varepsilon_0}$ is X. So $\lambda_0 + B_{\varepsilon_0}$ is injective and surjective. Since $(X, \mathscr{F})$ satisfies (A_2), $(\lambda_0 + B_{\varepsilon_0})^{-1} \in B_{\mathscr{F}}(X)$.

Let $\varepsilon \in \boldsymbol{R}$, $\varepsilon > 0$, and $\lambda \in \boldsymbol{C}\backslash\boldsymbol{R}_-$ arbitrary. By Lemma 3.1 $\lambda + B_\varepsilon$ is injective. We finish the proof by showing that $(\lambda + B_\varepsilon)\mathscr{D}_{\lambda+B_\varepsilon} = X$.

Let $x \in {}_{-\infty}\mathscr{D}_{+\infty}$. Following Corollary 3.3,

$$\begin{aligned}&\lambda(\lambda + B_\varepsilon)^{-1}x - \lambda_0(\lambda_0 + B_{\varepsilon_0})^{-1}x \\ &\quad = -(2i)^{-1}\mathscr{F}-\int_{c-i\infty}^{c+i\infty}\left(\frac{\varepsilon^{-1}\lambda^{-\varepsilon^{-1}\alpha}}{\sin \pi\varepsilon^{-1}\alpha} - \frac{\varepsilon_0^{-1}\lambda_0^{-\varepsilon_0^{-1}\alpha}}{\sin \pi\varepsilon_0^{-1}\alpha}\right)F_x(\alpha)d\alpha\,,\end{aligned}$$

where $c \in \boldsymbol{R}$, $0 < c < \delta = \min\{\varepsilon, \varepsilon_0\}$. It is easy to see that the analytic function

$$\{\alpha \in \boldsymbol{C};\ 0 < \operatorname{Re}\alpha < \delta\} \ni \alpha \mapsto \frac{\varepsilon^{-1}\lambda^{-\varepsilon^{-1}\alpha}}{\sin \pi\varepsilon^{-1}\alpha} - \frac{\varepsilon_0^{-1}\lambda_0^{-\varepsilon_0^{-1}\alpha}}{\sin \pi\varepsilon_0^{-1}\alpha}$$

has a continuous extension f on $\{\alpha \in \boldsymbol{C};\ 0 \leqq \operatorname{Re}\alpha < \delta\}$. Using the Cauchy integral theorem, there results:

$$\begin{aligned}&\lambda(\lambda + B_\varepsilon)^{-1}x - \lambda_0(\lambda_0 + B_{\varepsilon_0})^{-1}x \\ &\quad = -(1/2)\mathscr{F}-\int_{-\infty}^{+\infty} f(it)U_t x dt\,.\end{aligned}$$

Define for every $x \in X$

$$T_f(x) = -(1/2)\mathscr{F}-\int_{-\infty}^{+\infty} f(it)U_t x dt\,.$$

Since $(\mathscr{F}, X)$ satisfies (A_1), for every $\varphi \in \mathscr{F}$

$$\varphi_f = -(1/2)X-\int_{-\infty}^{+\infty} f(it)U_t^* \varphi dt$$

is well defined and belongs to $\mathscr{F}$. For every $x \in X$ and for every $\varphi \in \mathscr{F}$

$$\langle T_f(x), \varphi\rangle = \langle x, \varphi_f\rangle\,,$$

hence $T_f \in B_{\mathscr{F}}(X)$.

Consequently,

$$T = \lambda_0(\lambda_0 + B_{\varepsilon_0})^{-1} + T_f$$

belongs to $B_{\mathscr{F}}(X)$ and for $x \in {}_{-\infty}\mathscr{D}_{+\infty}$

$$Tx = \lambda(\lambda + B_\varepsilon)^{-1}x ,$$

that is

$$Tx \in \mathscr{D}_{\lambda+B_\varepsilon} ,$$
$$(\lambda + B_\varepsilon)Tx = \lambda x .$$

Using the $\mathscr{F}$-density of ${}_{-\infty}\mathscr{D}_{+\infty}$ and the fact that $\lambda + B_\varepsilon$ is $\mathscr{F}$-closed, we deduce that the above relations hold for every $x \in X$. Hence

$$(\lambda + B_\varepsilon)\mathscr{D}_{\lambda+B_\varepsilon} \supset (\lambda + B_\varepsilon)TX = X .$$

Now the proof is finished, because $\sigma(B_\varepsilon) \subset \boldsymbol{R}_+$ implies that $\sigma(B_{-\varepsilon}) = \sigma(B_\varepsilon^{-1}) \subset \boldsymbol{R}_+$. q.e.d.

In particular, if B is bounded then $\sigma(B) \subset \boldsymbol{R}_+$.

We remark that the results of this section can be extended for groups $\{U_t\}$ for which

$$\|U_t\| \leqq c_1 e^{c_2|t|},\ t \in \boldsymbol{R} , \qquad \text{where } c_2 < \pi ,$$

replacing in the statements $\boldsymbol{C}\backslash\boldsymbol{R}_-$ by $\{\lambda \in \boldsymbol{C}\backslash\boldsymbol{R}_-;\ |\arg\lambda| < \pi - c_2\}$ and $\boldsymbol{R}_+$ by $\{\lambda \in \boldsymbol{C}\backslash\boldsymbol{R}_-;\ |\arg\lambda| < c_2\}$.

4. **An inversion formula and the unicity theorem.** In this section we represent the bounded one-parameter groups in terms of their analytic generators. In particular, the analytic generators uniquely determine the groups.

LEMMA 4.1. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, satisfying (A_1), $\{U_t\}_{t\in\boldsymbol{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$, B its analytic generator and $x \in \mathscr{D}_B$. Then for every $\lambda \in \boldsymbol{R}$, $\lambda > 0$,*

$$Bx \in \mathscr{D}_{(\lambda+B)^{-1}}$$

and

$$\|(\lambda + B)^{-1}Bx\| \leqq \sup_{t\in\boldsymbol{R}} \|U_t\| \max\{\|x\|, \|Bx\|\}\frac{2}{\pi \sin^2 \pi c}\,\frac{1}{\lambda^c} ,$$

where $c \in \boldsymbol{R}$, $0 < c < 1$, is arbitrary.

PROOF. By Theorem 3.2 for every $\lambda \in \boldsymbol{R}$, $\lambda > 0$,

$$Bx \in \mathscr{D}_{(\lambda+B)^{-1}}$$

and

$$(\lambda + B)^{-1}Bx = (2i)^{-1}\mathscr{F}-\int_{c-i\infty}^{c+i\infty}\frac{\lambda^{-\alpha}}{\sin \pi\alpha}F_x(\alpha)d\alpha \ ,$$

where $c \in \boldsymbol{R}$, $0 < c < 1$, is arbitrary. Hence

$$\begin{aligned}||(\lambda + B)^{-1}Bx|| &\leqq \sup_{0 \leqq \mathrm{Re}\,\alpha \leqq 1} ||F_x(\alpha)||2^{-1}\int_{-\infty}^{+\infty}\frac{|\lambda^{-c-it}|}{|\sin \pi(c + it)|}dt \\ &= \sup_{0 \leqq \mathrm{Re}\,\alpha \leqq 1} ||F_x(\alpha)|| \int_{-\infty}^{+\infty}\frac{dt}{|e^{i\pi c-\pi t} - e^{-i\pi c+\pi t}|}\frac{1}{\lambda^c} \ .\end{aligned}$$

Using Lemma 2.3 and the Phragmen-Lindelöf theorem, we have

$$\sup_{0 \leqq \mathrm{Re}\,\alpha \leqq 1} ||F_x(\alpha)|| \leqq \sup_{t \in \boldsymbol{R}} ||U_t|| \max\{||x||, ||Bx||\} \ .$$

On the other hand,

$$\begin{aligned}&\int_{-\infty}^{+\infty}\frac{dt}{|e^{i\pi c-\pi t} - e^{-i\pi c+\pi t}|} \\ &\quad = \int_{-\infty}^{+\infty}\frac{dt}{(e^{2\pi t} + e^{-2\pi t} - 2\cos 2\pi c)^{1/2}} \\ &\quad = 2\int_{0}^{+\infty}\frac{dt}{(e^{2\pi t} + e^{-2\pi t} - 2\cos 2\pi c)^{1/2}} \\ &\quad \leqq 2\int_{0}^{+\infty}\frac{dt}{e^{\pi t} - e^{-\pi t}\cos 2\pi c} \\ &\quad = 2\int_{0}^{+\infty}\frac{e^{\pi t}dt}{e^{2\pi t} - \cos 2\pi c} \\ &\quad = \frac{2}{\pi}\int_{1}^{+\infty}\frac{ds}{s^2 - \cos 2\pi c} \ .\end{aligned}$$

If $\cos 2\pi c \leqq 0$, then

$$\int_{1}^{+\infty}\frac{ds}{s^2 - \cos 2\pi c} \leqq \int_{1}^{+\infty}\frac{ds}{s^2} = 1 \ ,$$

and if $\cos 2\pi c \geqq 0$, then

$$\int_{1}^{+\infty}\frac{ds}{s^2 - \cos 2\pi c} \leqq \int_{1}^{+\infty}\frac{ds}{(1 - \cos 2\pi c)s^2} = \frac{1}{1 - \cos 2\pi c} = \frac{1}{2\sin^2 \pi c} \ .$$

Hence for every $0 < c < 1$

$$\int_{-\infty}^{+\infty}\frac{dt}{|e^{i\pi c-\pi t} - e^{-i\pi c+\pi t}|} \leqq \frac{2}{\pi \sin^2 \pi c} \ .$$

q.e.d.

Lemma 4.1 implies that for every $x \in \mathscr{D}_B$ and for every $0 < c < 1$ there exists a constant $d(x, c)$ such that

$$\|(\lambda + B)^{-1}Bx\| \leqq \frac{d(x, c)}{\lambda^c}, \qquad \lambda > 0 .$$

Consequently, for any $\alpha \in \boldsymbol{C}$, $0 < \operatorname{Re} \alpha < 1$, the integral

$$\mathscr{F} - \int_0^{+\infty} \lambda^{\alpha-1}(\lambda + B)^{-1}Bxd\lambda$$

converges. Indeed, applying the above remark with $0 < c < \operatorname{Re} \alpha$, we deduce the convergence of the integral in 0 and, applying it with $\operatorname{Re} \alpha < c < 1$, we deduce the convergence in $+\infty$.

Hence for $0 < \operatorname{Re} \alpha < 1$ the "Balakrishnan fractional powers" B^α (see [2]) can be defined. Now we show that B^α is the analytical extension B_α of $\{U_t\}$ in α:

THEOREM 4.2. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, satisfying* (A_1), *$\{U_t\}_{t\in\boldsymbol{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$, B its analytic generator and $x \in \mathscr{D}_B$. Then for every $\alpha \in \boldsymbol{C}$, $0 < \operatorname{Re} \alpha < 1$,*

$$F_x(\alpha) = \frac{\sin \pi\alpha}{\pi} \mathscr{F} - \int_0^{+\infty} \lambda^{\alpha-1}(\lambda + B)^{-1}Bxd\lambda .$$

In particular, for every $t \in \boldsymbol{R}$

$$U_t x = \mathscr{F} - \lim_{\substack{\alpha \to it \\ 0 < \operatorname{Re}\alpha < 1}} \frac{\sin \pi\alpha}{\pi} \mathscr{F} - \int_0^{+\infty} \lambda^{\alpha-1}(\lambda + B)^{-1}Bxd\lambda .$$

PROOF. Let $\varphi \in \mathscr{F}$ arbitrary.
By Theorem 3.2, for every $\lambda > 0$

$$\langle(\lambda + B)^{-1}Bx, \varphi\rangle = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \lambda^{-\alpha} \frac{\pi}{\sin \pi\alpha} \langle F_x(\alpha), \varphi\rangle d\alpha,\ 0 < c < 1 ,$$

that is

$$\{\lambda \in \boldsymbol{R};\ \lambda > 0\} \ni \lambda \mapsto \langle(\lambda + B)^{-1}Bx, \varphi\rangle$$

is the inverse Mellin transform of

$$\{\alpha \in \boldsymbol{C};\ 0 < \operatorname{Re} \alpha < 1\} \ni \alpha \mapsto \frac{\pi}{\sin \pi\alpha} \langle(F_x\alpha), \varphi\rangle .$$

Since the conditions of [6], Kap. 6, §8, Satz 3, are satisfied, it follows that

$$\{\alpha \in \boldsymbol{C};\ 0 < \operatorname{Re}\alpha < 1\} \ni \alpha \mapsto \frac{\pi}{\sin \pi\alpha}\langle F_x(\alpha), \varphi\rangle$$

is the Mellin transform of

$$\{\lambda \in \boldsymbol{R};\ \lambda > 0\} \ni \lambda \mapsto \langle(\lambda + B)^{-1}Bx, \varphi\rangle\ .$$

Consequently, for every $\alpha \in \boldsymbol{C}$, $0 < \operatorname{Re}\alpha < 1$,

$$F_x(\alpha) = \frac{\sin \pi\alpha}{\pi}\mathscr{F}-\int_0^{+\infty}\lambda^{\alpha-1}(\lambda + B)^{-1}Bxd\lambda\ .$$

q.e.d.

Theorem 4.2 implies the following converse of Corollary 3.5:

COROLLARY 4.3. *Let* $(X, \mathscr{F})$ *be a dual pair of Banach spaces, satisfying* (A_1), $\{U_t\}_{t\in \boldsymbol{R}}$ *a bounded* $\mathscr{F}$*-continuous one-parameter group in* $B_{\mathscr{F}}(X)$, B, $B_{1/2}$, $B_{-1/2}$ *its analytical extensions respectively in* 1, 1/2, $-1/2$ *and* $x \in \mathscr{D}_{B_{1/2}} \cap \mathscr{D}_{B_{-1/2}}$. *Then for every* $t \in \boldsymbol{R}$

$$U_t x = \frac{e^{\pi t} + e^{-\pi t}}{2\pi}\mathscr{F}-\int_0^{+\infty}\lambda^{it-1/2}(\lambda + B)^{-1}B_{1/2}xd\lambda\ .$$

PROOF. We apply Theorem 4.2 with $y = B_{-1/2}x \in \mathscr{D}_B$ and $\alpha = 1/2 + it$.
q.e.d.

An other consequence of Theorem 4.2 is the following unicity theorem:

THEOREM 4.4. *Let* $(X, \mathscr{F})$ *be a dual pair of Banach spaces, satisfying* (A_1), $\{U_t\}_{t\in \boldsymbol{R}}$, $\{V_t\}_{t\in \boldsymbol{R}}$ *bounded* $\mathscr{F}$*-continuous one-parameter groups in* $B_{\mathscr{F}}(X)$, B, D *their analytic generators and* Y, Z $\mathscr{F}$*-closed linear subspaces of* X. *If*

$$\{y \in Y \cap \mathscr{D}_B \cap \mathscr{D}_D;\ (\lambda + B)^{-1}y - (\lambda + D)^{-1}y \in Z \quad \textit{for all} \quad \lambda > 0\}$$

is $\mathscr{F}$*-dense in* Y, *then*

$$(U_t - V_t)Y \subset Z\ , \qquad t \in \boldsymbol{R}\ .$$

In particular, if $B = D$ *then* $U_t = V_t$ *for all* $t \in \boldsymbol{R}$.

PROOF. Following Theorem 4.2, for any $y \in \mathscr{D}_B \cap \mathscr{D}_D$

$$(U_t - V_t)y$$

$$= \mathscr{F} - \lim_{\substack{\alpha\to it \\ 0<\operatorname{Re}\alpha<1}} \frac{\sin \pi\alpha}{\pi}\mathscr{F}-\int_0^{+\infty}\lambda^{\alpha-1}[(\lambda + B)^{-1}y - (\lambda + D)^{-1}y]d\lambda\ ,$$

hence, for y in an $\mathscr{F}$-dense subset of Y, $(U_t - V_t)y \in Z$. q.e.d.

This result implies our statement at the end of Section 2.

We remark that the above results can be extended to the case

$$\|U_t\| \leqq c_1 e^{c_2|t|}\,, \qquad t \in \boldsymbol{R}, \text{ where } c_2 < \pi\,.$$

Now we prove another unicity theorem:

THEOREM 4.5. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $\{U_t\}_{t\in\boldsymbol{R}}$, $\{V_t\}_{t\in\boldsymbol{R}}$ $\mathscr{F}$-continuous one-parameter groups in $B_{\mathscr{F}}(X)$ such that for some constants $c_1 > 0$, $\pi > c_2 > 0$*

$$\|U_t\| \leqq c_1 e^{c_2|t|}\,, \quad \|V_t\| \leqq c_1 e^{c_2|t|}\,, \qquad t \in \boldsymbol{R}\,,$$

B, D their analytic generators and Y, Z $\mathscr{F}$-closed linear subspaces of X. If

$$\left\{ y \in Y \cap \bigcap_{n=1}^{\infty} \mathscr{D}_{B^n} \cap \bigcap_{n=1}^{\infty} \mathscr{D}_{D^n};\ \begin{array}{l} \overline{\lim\limits_{n\to+\infty}} \|B^n y\|^{1/n} < +\infty,\ \overline{\lim\limits_{n\to+\infty}} \|D^n y\|^{1/n} < +\infty, \\ B^n y - D^n y \in Z \quad \text{for all} \quad n \geqq 1 \end{array} \right\}$$

is $\mathscr{F}$-dense in Y then

$$(U_t - V_t) Y \subset Z\,, \qquad t \in \boldsymbol{R}\,.$$

PROOF. Let B_α, D_α, be the analytical extensions of $\{U_t\}$ respectively of $\{V_t\}$ in $\alpha \in \boldsymbol{C}$ and

$$y \in Y \cap \bigcap_{n=1}^{\infty} \mathscr{D}_{B^n} \cap \bigcap_{n=1}^{\infty} \mathscr{D}_{D^n}$$

such that

$$\overline{\lim_{n\to+\infty}} \|B^n y\|^{1/n} < +\infty\,, \qquad \overline{\lim_{n\to+\infty}} \|D^n y\|^{1/n} < +\infty\,,$$
$$B^n y - D^n y \in Z \qquad \text{for all } n \geqq 1\,.$$

Then for every $\varphi \in Z^\perp$ in $\mathscr{F}$ the formula

$$f(\alpha) = \langle B_\alpha y - D_\alpha y, \varphi \rangle\,, \qquad \operatorname{Re} \alpha \geqq 0\,,$$

defines a regular function on $\{\alpha \in \boldsymbol{C};\ \operatorname{Re} \alpha \geqq 0\}$ which verifies the conditions (i), (ii), (iii) of Proposition 1.3. Consequently f vanishes identically, hence

$$\langle U_t y - V_t y, \varphi \rangle = 0\,, \qquad t \in \boldsymbol{R}\,.$$

Since $\varphi \in Z^\perp$ is arbitrary, it follows

$$U_t y - V_t y \in Z\,, \qquad t \in \boldsymbol{R}\,.$$

Now, using the $\mathscr{F}$-density hypothesis of the theorem, we deduce

$$(U_t - V_t) Y \subset Z\,, \qquad t \in \boldsymbol{R}\,.$$

q.e.d.

Theorem 2.4 and Lemma 3.1 imply that the eigenvalues of the analytic generator of a bounded one-parameter group are included in $\boldsymbol{R}_+\backslash\{0\}$. Now we characterize these eigenvalues in terms of the group:

COROLLARY 4.6. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $\{U_t\}_{t\in\boldsymbol{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$ and B its analytic generator. Then for $x \in X$ and $\lambda \in \boldsymbol{R}_+\backslash\{0\}$ the following statements are equivalent:*

(i) $$x \in \mathscr{D}_B\,, \quad Bx = \lambda x\,;$$

(ii) $$U_t x = \lambda^{it} x\,, \quad t \in \boldsymbol{R}\,.$$

PROOF. Obviously, (ii) implies (i). For the converse implication we apply Theorem 4.5 with $V_t v = \lambda^{it} v$, $t \in \boldsymbol{R}$, $v \in X$, $Y = \boldsymbol{C}x$ and $Z = \{0\}$.

q.e.d.

5. Spectral subspaces. In this section we analyse sets of elements $x \in X$ for which $n \longrightarrow B^n x$, where B is the analytic generator of a bounded $\mathscr{F}$-continuous one-parameter group $\{U_t\}$, has exponential increasing. The importance of these elements is justified by Theorem 4.5 and we shall show that they are $\mathscr{F}$-dense in X. We associate to every closed subinterval $[\lambda_1, \lambda_2]$ of $(0, +\infty)$ a closed linear subspace of X which is invariated by $\{U_t\}$ and on which the spectrum of B is included in $[\lambda_1, \lambda_2]$. We show that this subspace coincide with the spectral subspace $M^U([\ln \lambda_1, \ln \lambda_2])$ defined in [1], Section 2.

LEMMA 5.1. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $\{U_t\}_{t\in\boldsymbol{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$ and B its analytic generator. If $x \in \bigcap_{n=1}^{\infty} \mathscr{D}_{B^n}$ and $\lambda = \overline{\lim}_{n\to\infty} \|B^n x\|^{1/n}$ then for every integer $m \geqq 1$*

$$\|B^m x\| \leqq \sup_{t\in\boldsymbol{R}} \|U_t B^m x\| \leqq \lambda^m \sup_{t\in\boldsymbol{R}} \|U_t\|\, \|x\|\,.$$

PROOF. Let F_x be the $\mathscr{F}$-regular extension of $it \mapsto U_t x$ on $\{\alpha \in \boldsymbol{C};\ \operatorname{Re}\alpha \geqq 0\}$. If $n \geqq m \geqq 1$ are arbitrary integers then, using the "three line theorem" ([7], Theorem VI. 10. 3),

$$\sup_{t\in\boldsymbol{R}} \|U_t B^m x\| \leqq \Big(\sup_{t\in\boldsymbol{R}} \|U_t\|\, B^n x\|\Big)^{m/n} \Big(\sup_{t\in\boldsymbol{R}} \|U_t\|\, \|x\|\Big)^{1-m/n}\,.$$

Taking $n \longrightarrow +\infty$, there results

$$\sup_{t\in\boldsymbol{R}} \|U_t B^m x\| \leqq \lambda^m \sup_{t\in\boldsymbol{R}} \|U_t\|\, \|x\|\,.$$

q.e.d.

In particular, if $\overline{\lim}_{n\to\infty} \|B^n x\|^{1/n} = 0$, then $Bx = 0$, so $x = 0$.

Now we are able to give the structure of $\{U_t\}$ if $\sigma(B)$ is a compact subset of $(0, +\infty)$:

THEOREM 5.2. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces, $\{U_t\}_{t\in\boldsymbol{R}}$ a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$ and B its analytic generator. If*

$$m = \inf_{\lambda\in\sigma(B)} |\lambda| > 0 , \qquad M = \sup_{\lambda\in\sigma(B)} |\lambda| < +\infty ,$$

then there exists $H\in B_{\mathscr{F}}(X)$, $\sigma(H)\subset[\ln(m), \ln(M)]$, such that

$$U_t = \exp(itH) , \quad t\in\boldsymbol{R} .$$

PROOF. Lemma 5.1 implies that B and B^{-1} are bounded:

$$\|B\| \leqq \sup_{t\in\boldsymbol{R}} \|U_t\| M ,$$

$$\|B^{-1}\| \leqq \sup_{t\in\boldsymbol{R}} \|U_t\| \frac{1}{m} .$$

So, using Proposition 1.2, there results that $\{U_t\}$ is strongly continuous. Following Theorem 3.6, $\sigma(B)\subset[m, M]$.

Let $\Gamma: [0, 1]\to\{\lambda\in\boldsymbol{C}\backslash\boldsymbol{R}_-; |\arg\lambda|\leqq 1\}$ be a positively oriented rectifiable closed Jordan curve around $[m, M]$. Consider the one-parameter group $\{V_t\}$ defined by analytical functional calculus (see [7], Chap. VII):

$$V_t = B^{it} = \frac{1}{2\pi i}\int_\Gamma \lambda^{it}(\lambda - B)^{-1}d\lambda .$$

Then for any $t\in\boldsymbol{R}$

$$\|V_t\| \leqq \frac{1}{2\pi}\,\text{length}\,(\Gamma) \sup_{\lambda\in\Gamma([0,1])} \|(\lambda - B)^{-1}\| e^{|t|}$$

and the analytic generator of $\{V_t\}$ is B. Using Theorem 4.6 with $Y = X$ and $Z = \{0\}$, we deduce

$$U_t = B^{it} , \qquad t\in\boldsymbol{R} .$$

Now let f be the analytic function defined on $\boldsymbol{C}\backslash\boldsymbol{R}_-$ by the formula

$$f(\lambda) = \ln\lambda = \ln|\lambda| + i\arg\lambda .$$

Define

$$H = f(B) .$$

By the spectral mapping theorem ([7], Theorem VII. 3.11) $\sigma(H) = \ln\sigma(B)\subset[\ln(m), \ln(M)]$ and by [7], Theorem VII. 3.12

$$U_t = \exp(itH) , \qquad t\in\boldsymbol{R} .$$

q.e.d.

In particular, $\{U_t\}$ is uniformly continuous if and only if $\sigma(B)$ is a compact subset of $(0, +\infty)$.

LEMMA 5.3. *Let* $(X, \mathscr{F})$ *be a dual pair of Banach spaces,* $\{U_t\}_{t\in R}$ *a bounded* $\mathscr{F}$*-continuous one-parameter group in* $B_{\mathscr{F}}(X)$ *and* B *its analytic generator. If* $0 \neq x \in \bigcap_{n=-\infty}^{\infty} \mathscr{D}_{B^n}$ *then*

$$\overline{\lim_{n\to\infty}} \|B^n x\|^{1/n} \overline{\lim_{n\to\infty}} \|B^{-n} x\|^{1/n} \geqq 1 .$$

The equality holds if and only if x *is an eigenvector of* B *(with eigenvalue* $\overline{\lim}_{n\to\infty} \|B^n x\|^{1/n}$*).*

PROOF. The case $\overline{\lim}_{n\to\infty} \|B^n x\|^{1/n} = +\infty$ is trivial, so we suppose that $\lambda = \overline{\lim}_{n\to\infty} \|B^n x\|^{1/n} < +\infty$.

Let $m \geqq 1$ be an arbitrary integer. Since $\overline{\lim}_{n\to\infty} \|B^n(B^{-m}x)\|^{1/n} = \lambda$, using Lemma 5.1 we have

$$\|x\| = \|B^m(B^{-m}x)\| \leqq \lambda^m \sup_{t\in R} \|U_t\| \, \|B^{-m}x\| ,$$

$$\|x\|^{1/m} \leqq \left(\sup_{t\in R} \|U_t\|\right)^{1/m} \|B^{-m}x\|^{1/m} .$$

Taking $m \to \infty$, there results

$$1 \leqq \lambda \overline{\lim_{m\to\infty}} \|B^{-m}x\|^{1/m} .$$

Now we suppose that

$$\overline{\lim_{n\to\infty}} \|B^{-n}x\|^{1/n} = \lambda^{-1} .$$

Then $\alpha \mapsto \lambda^{-\alpha} F_z(\alpha)$ is a bounded entire mapping, so, by the Liouville theorem, it is constant. Hence

$$\lambda^{-1}Bx = x ,$$

$$Bx = \lambda x .$$

q.e.d.

Let $\{U_t\}$ be a bounded $\mathscr{F}$-continuous one-parameter group in $B_{\mathscr{F}}(X)$ and B its analytic generator. For $0 < \lambda_1 \leqq \lambda_2 < +\infty$ we define the *spectral subspace*

$$X^B([\lambda_1, \lambda_2]) = \left\{x \in \bigcap_{n=-\infty}^{\infty} \mathscr{D}_{B^n};\ \overline{\lim_{n\to\infty}} \|B^n x\|^{1/n} \leqq \lambda_2,\ \overline{\lim_{n\to\infty}} \|B^{-n}x\|^{1/n} \leqq \frac{1}{\lambda_1}\right\} .$$

Using Lemmas 5.1 and 5.3, it is easy to see:

(i) *if* $[\lambda_1^1, \lambda_2^1] \subset [\lambda_1^2, \lambda_2^2]$ *then*

$$X^B([\lambda_1^1, \lambda_2^1]) \subset X^B([\lambda_1^2, \lambda_2^2]) ;$$

(ii) *if* $\{[\lambda_1^\iota, \lambda_2^\iota]\}_{\iota\in I}$ *has non-empty intersection then*

$$X^B\Big(\bigcap_{\iota\in I}[\lambda_1^\iota, \lambda_2^\iota]\Big) = \bigcap_{\iota\in I} X^B([\lambda_1^\iota, \lambda_2^\iota]) ;$$

(iii) *every* $X^B([\lambda_1, \lambda_2])$ *is norm-closed and invariant under the action of* $\{U_t\}$;

(iv) $X^B([\lambda, \lambda]) = \{x \in \mathscr{D}_B;\ Bx = \lambda x\}$, $0 < \lambda < +\infty$;

(v) *for every* $x \in X^B([\lambda_1, \lambda_2])$ *and for every integer* $n \geqq 1$

$$\lambda_1^n\Big(\sup_{t\in\boldsymbol{R}} \|U_t\|\Big)^{-1}\|x\| \leqq \|B^n x\| \leqq \lambda_2^n \sup_{t\in\boldsymbol{R}} \|U_t\|\,\|x\| ;$$

in particular,

$$\sigma(B\,|\,X^B([\lambda_1, \lambda_2])) \subset [\lambda_1, \lambda_2] .$$

For every $f \in L^1(\boldsymbol{R})$ we define its inverse Fourier transform $\hat{f}$ by

$$\hat{f}(s) = \int_{-\infty}^{+\infty} f(t)e^{its}dt , \qquad s \in \boldsymbol{R} .$$

If $\hat{f} \in L^1(\boldsymbol{R})$ then the following inversion formula holds:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(s)e^{-its}ds , \qquad t \in \boldsymbol{R} .$$

Now we prove our main technical lemma:

LEMMA 5.4. *Let* $(X, \mathscr{F})$ *be a dual pair of Banach spaces, satisfying* (A_1), $\{U_t\}_{t\in\boldsymbol{R}}$ *a bounded* $\mathscr{F}$*-continuous one-parameter group in* $B_{\mathscr{F}}(X)$, B *its analytic generator and* $X^B([\lambda_1, \lambda_2])$ *the spectral subspace associated to* $[\lambda_1, \lambda_2] \subset (0, +\infty)$. *Then for every* $f \in L^1(\boldsymbol{R})$, $\hat{f} \in C^2(\boldsymbol{R})$, $\operatorname{supp}\hat{f} \subset [\ln\lambda_1, \ln\lambda_2]$, *and for every* $x \in X$

$$\mathscr{F} - \int_{-\infty}^{+\infty} f(t)U_t x dt \in X^B([\lambda_1, \lambda_2]) .$$

On the other hand, for every $f \in L^1(\boldsymbol{R})$, $\hat{f} \in C^2(\boldsymbol{R})$, $\operatorname{supp}\hat{f}$ *compact,* $\operatorname{supp}\hat{f} \cap [\ln\lambda_1, \ln\lambda_2] = \varnothing$, *and for every* $x \in X^B([\lambda_1, \lambda_2])$

$$\mathscr{F} - \int_{-\infty}^{+\infty} f(t)U_t x dt = 0 .$$

PROOF. Let $f \in L^1(\boldsymbol{R})$, $\hat{f} \in C^2(\boldsymbol{R})$, $\operatorname{supp}\hat{f} \subset [\ln\lambda_1, \ln\lambda_2]$. Then f has an entire extension defined by

$$f(\alpha) = \frac{1}{2\pi}\int_{\ln\lambda_1}^{\ln\lambda_2}\hat{f}(s)e^{-i\alpha s}ds .$$

Since

$$(i\alpha)^2 f(\alpha) = \frac{1}{2\pi}\int_{\ln\lambda_1}^{\ln\lambda_2} \hat{f}''(s)e^{-i\alpha s}ds\ ,$$

there exists a constant $c > 0$ such that

$$|f(\alpha)| \leqq \frac{c}{1+|\alpha|^2} e^{\mathrm{Im}\,\alpha \ln\lambda_2},\ \mathrm{Im}\,\alpha \geqq 0\ ,$$

$$|f(\alpha)| \leqq \frac{c}{1+|\alpha|^2} e^{\mathrm{Im}\,\alpha \ln\lambda_1},\ \mathrm{Im}\,\alpha \leqq 0\ .$$

Let $x \in X$ be arbitrary. It is easy to see that

$$\alpha \mapsto \mathscr{F} - \int_{-\infty}^{+\infty} f(t+i\alpha)U_t x dt$$

is an entire extension of

$$is \mapsto U_s\left(\mathscr{F} - \int_{-\infty}^{+\infty} f(t)U_t x dt\right),$$

hence

$$y = \mathscr{F} - \int_{-\infty}^{+\infty} f(t)U_t x dt \in \bigcap_{n=-\infty}^{\infty} \mathscr{D}_{B^n}$$

and

$$B^n y = \mathscr{F} - \int_{-\infty}^{+\infty} f(t+in)U_t x dt\ , \quad -\infty < n < +\infty\ .$$

For $n \geqq 0$

$$||B^n y|| \leqq c\int_{-\infty}^{+\infty} \frac{1}{1+t^2} dt e^{n\ln\lambda_2} \sup_{t\in \boldsymbol{R}} ||U_t||\ ||x||\ ,$$

so

$$\overline{\lim_{n\to\infty}}\, ||B^n y||^{1/n} \leqq e^{\ln\lambda_2} = \lambda_2\ .$$

Analogously,

$$\overline{\lim_{n\to\infty}}\, ||B^{-n} y||^{1/n} \leqq e^{-\ln\lambda_1} = \frac{1}{\lambda_1}\ .$$

In conclusion, $y \in X^B([\lambda_1, \lambda_2])$.

To prove the second statement, we can suppose either

$$\mathrm{supp}\,\hat{f} \subset (\ln\lambda_2, +\infty)\ ,$$

or

$$\mathrm{supp}\,\hat{f} \subset (-\infty, \ln\lambda_1)\ .$$

In the first situation, denoting by $a > \ln \lambda_2$ the greatest lower bound of $\operatorname{supp} \hat{f}$, f has an entire extension such that for some $c > 0$

$$|f(\alpha)| \leqq \frac{c}{1 + |\alpha|^2} e^{a \operatorname{Im} \alpha}, \qquad \operatorname{Im} \alpha \leqq 0 .$$

Using the Cauchy integral theorem and Lemma 5.1, for every $x \in X^B([\lambda_1, \lambda_2])$ and for every integer $n \geqq 1$

$$\begin{aligned} &\left\| \mathscr{F} - \int_{-\infty}^{+\infty} f(t) U_t x dt \right\| \\ &= \left\| \mathscr{F} - \int_{-\infty}^{+\infty} f(t - in) U_t B^n x dt \right\| \\ &\leqq c \int_{-\infty}^{+\infty} \frac{1}{1 + t^2} dt e^{-an} \lambda_2^n \sup_{t \in R} \| U_t \| \, \| x \| . \end{aligned}$$

Since $\lim_{n \to \infty} e^{-an} \lambda_2^n = 0$, we deduce

$$\mathscr{F} - \int_{-\infty}^{+\infty} f(t) U_t x dt = 0 .$$

In the second situation we follow a similar reasoning. q.e.d.

To exploit Lemma 5.4, we need the following "approximate unit lemma", which is undoubtly known but for which we found not reference:

LEMMA 5.5. *There exists a sequence* $\{f_n\}$ *in* $L^1(\boldsymbol{R})$ *such that for any* $n \geqq 1$

$$\int_{-\infty}^{+\infty} |f_n(t)| dt \leqq \frac{8}{\pi} ,$$

$$|f_n(t)| \leqq \frac{2}{\pi n} \frac{1}{t^2} , \qquad t \neq 0 ,$$

$$\hat{f}_n \in C^2(\boldsymbol{R}) ,$$

$$0 \leqq \hat{f}_n \leqq 1, \hat{f}_n = 1 \text{ on } [-n, n], \hat{f}_n = 0 \text{ on } (-\infty, -3n] \cup [3n, +\infty) .$$

Let $(X, \mathscr{F})$ *be a dual pair of Banach spaces, satisfying* (A_1), *and* $\{U_t\}_{t \in R}$ *a bounded* $\mathscr{F}$*-continuous one-parameter group in* $B_{\mathscr{F}}(X)$. *Then for every* $x \in X$

$$\mathscr{F} - \int_{-\infty}^{+\infty} f_n(t) U_t x dt \to x$$

in the $\mathscr{F}$*-topology.*

PROOF. For every $n \geqq 1$ we define the continuous function ψ_n on $\boldsymbol{R}$ by the relations:

$$\psi_n(s) = \begin{cases} 0 & s \in (-\infty, -3n] , \\ \dfrac{2}{n^2} & s = -\dfrac{5n}{2} , \\ -\dfrac{2}{n^2} & s = -\dfrac{3n}{2} , \\ 0 & s \in [-n, 0] , \end{cases}$$

ψ_n is linear on the intervals

$$\left[-3n, -\frac{5n}{2}\right], \ \left[-\frac{5n}{2}, -\frac{3n}{2}\right], \ \left[-\frac{3n}{2}, -n\right],$$

$$\psi_n(s) = \psi_n(-s), \ s \in \boldsymbol{R} .$$

Let $\varphi_n \in C^2(\boldsymbol{R})$ be defined by:

$$\varphi_n(s) = \int_{-\infty}^{s}\left(\int_{-\infty}^{u} \psi_n(v)dv\right)du , \qquad s \in \boldsymbol{R} .$$

It is easy to verify that

$$0 \leqq \varphi_n \leqq 1, \ \varphi_n = 1 \text{ on } [-n, n], \ \varphi_n = 0 \text{ on } (-\infty, -3n] \cup [3n, +\infty) ,$$

$$\int_{-\infty}^{+\infty} \varphi_n(s)ds = 4n ,$$

$$\int_{-\infty}^{+\infty} |\varphi_n''(s)|ds = \int_{-\infty}^{+\infty} |\psi_n(s)|ds = \frac{4}{n} .$$

Now we define for every $n \geqq 1$ the function f_n as being the Fourier transform of φ_n:

$$f_n(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi_n(s)e^{-its}ds , \qquad t \in \boldsymbol{R} .$$

So, for every t,

$$|f_n(t)| \leqq \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi_n(s)ds = \frac{2n}{\pi} .$$

Since

$$(it)^2 f_n(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi_n''(s)e^{-its}ds , \qquad t \in \boldsymbol{R} ,$$

we have for every $t \neq 0$

$$|f_n(t)| \leqq \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\varphi_n''(s)|ds \frac{1}{t^2} = \frac{2}{\pi n}\frac{1}{t^2} .$$

Consequently,

$$\int_{-\infty}^{+\infty}|f_n(t)|dt \leqq \int_{-\infty}^{-n^{-1}}\frac{2}{\pi n}\frac{1}{t^2}dt + \int_{-n^{-1}}^{n^{-1}}\frac{2n}{\pi}dt + \int_{n^{-1}}^{+\infty}\frac{2}{\pi n}\frac{1}{t^2}dt$$
$$= \frac{2}{\pi} + \frac{4}{\pi} + \frac{2}{\pi}$$
$$= \frac{8}{\pi}.$$

If $(X, \mathscr{F})$ and $\{U_t\}$ are as in the statement of the lemma, then for every $x \in X$, $\varphi \in \mathscr{F}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for $|t| \leqq \delta$ we have $|\langle U_t x - x, \varphi\rangle| \leqq \pi\varepsilon/16$. Since

$$\int_{-\infty}^{+\infty} f_n(t)dt = \hat{f}_n(0) = \varphi_n(0) = 1,$$

there results

$$\left|\left\langle \mathscr{F} - \int_{-\infty}^{+\infty} f_n(t)U_t x dt - x, \varphi\right\rangle\right|$$
$$= \left|\int_{-\infty}^{+\infty} f_n(t)\langle U_t x - x, \varphi\rangle dt\right|$$
$$\leqq \int_{|t|\leqq\delta}|f_n(t)|dt\frac{\pi\varepsilon}{16} + \int_{|t|>\delta}\frac{2}{\pi n}\frac{1}{t^2}dt\Big(1 + \sup_{t\in R}\|U_t\|\Big)\|x\|\,\|\varphi\|$$
$$\leqq \frac{\varepsilon}{2} + \frac{4}{\pi\delta n}\Big(1 + \sup_{t\in R}\|U_t\|\Big)\|x\|\,\|\varphi\|.$$

Hence there exists n_ε such that for $n \geqq n_\varepsilon$

$$\left|\left\langle \mathscr{F} - \int_{-\infty}^{+\infty} f_n(t)U_t x dt - x, \varphi\right\rangle\right| \leqq \varepsilon.$$

q.e.d.

Suppose that $(X, \mathscr{F})$ and $(\mathscr{F}, X)$ satisfy (A_1) and $x \in \mathscr{D}_B$. If $\{f_n\}$ are as in Lemma 5.5, then by Lemma 5.4

$$x_n = \mathscr{F} - \int_{-\infty}^{+\infty} f_n(t)U_t x dt \in X^B([e^{-3n}, e^{3n}]),$$

by Corollary 2.5

$$Bx_n = \mathscr{F} - \int_{-\infty}^{+\infty} f_n(t)U_t Bx dt$$

and by Lemma 5.5

$$x_n \to x, \quad Bx_n \to Bx$$

in the $\mathscr{F}$-topology. Consequently, for bounded groups Corollary 2.6 can be improved:

(vi) *if* $(X, \mathscr{F})$ *and* $(\mathscr{F}, X)$ *satisfy* (A_1) *then the sequential* $\mathscr{F}$-*closure of* $B \mid \bigcup_{0<\lambda_1 \leqq \lambda_2 < +\infty} X^B([\lambda_1, \lambda_2])$ *is* B.

Supposing that $(X, \mathscr{F})$ satisfies (A_1), we define for $0 < \lambda_1 < \lambda_2 < +\infty$

$$X^B((\lambda_1, \lambda_2)) = \left\{ \mathscr{F} - \int_{-\infty}^{+\infty} f(t) U_t x dt;\ \begin{array}{l} f \in L^1(R), \hat{f} \in C^2(\boldsymbol{R}), \\ \operatorname{supp} \hat{f} \subset [\ln \lambda_1, \ln \lambda_2],\ x \in X \end{array} \right\}.$$

By Lemma 5.4, $X^B((\lambda_1, \lambda_2)) \subset X^B([\lambda_1, \lambda_2])$.

We prove a "regularity property" which improves [1], Proposition 2.2 in the case of the additive group $\boldsymbol{R}$:

THEOREM 5.6. *Let* $(X, \mathscr{F})$ *be a dual pair of Banach spaces, satisfying* (A_1), $\{U_t\}_{t \in \boldsymbol{R}}$ *a bounded* $\mathscr{F}$-*continuous one-parameter group in* $B_{\mathscr{F}}(X)$, B *its analytic generator and* $0 < \lambda_1 \leqq \lambda_2 < +\infty$. *Then*

$$\begin{aligned} X^B([\lambda_1, \lambda_2]) &= \bigcap_{\delta > 1} X^B((\lambda_1 \delta^{-1}, \lambda_2 \delta)) \\ &= \left\{ x \in X;\ \begin{array}{l} \textit{for every } f \in L^1(\boldsymbol{R}), \hat{f} \in C^2(\boldsymbol{R}), \operatorname{supp} \hat{f} \\ \textit{compact}, \quad \operatorname{supp} \hat{f} \cap [\ln \lambda_1, \ln \lambda_2] = \varnothing, \\ \textit{we have } \mathscr{F} - \int_{-\infty}^{+\infty} f(t) U_t x dt = 0 \end{array} \right\}. \end{aligned}$$

PROOF. We denote the last set in the above equalities by S.

Let $x \in S$, $\varepsilon > 0$ and $f \in L^1(\boldsymbol{R})$ such that $\hat{f} \in C^2(\boldsymbol{R})$, $\hat{f} = 1$ on $[\ln \lambda_1 - \varepsilon, \ln \lambda_2 + \varepsilon]$ and $\operatorname{supp} \hat{f} \subset [\ln \lambda_1 - 2\varepsilon, \ln \lambda_2 + 2\varepsilon]$. Consider a sequence $\{f_n\}$ in $L^1(\boldsymbol{R})$ as in Lemma 5.5. Since $\widehat{f_n - f * f_n} = \hat{f}_n - \hat{f}\hat{f}_n$ belongs to $C^2(\boldsymbol{R})$, has compact support and vanishes on $[\ln \lambda_1 - \varepsilon, \ln \lambda_2 + \varepsilon]$, there results

$$\begin{aligned} &\mathscr{F} - \int_{-\infty}^{+\infty} f_n(t) U_t x dt \\ &= \mathscr{F} - \int_{-\infty}^{+\infty} (f * f_n)(t) U_t x dt + \mathscr{F} - \int_{-\infty}^{+\infty} (f_n - f * f_n)(t) U_t x dt \\ &= \mathscr{F} - \int_{-\infty}^{+\infty} (f * f_n)(t) U_t x dt . \end{aligned}$$

If $n \geqq n_0 \geqq \max \{|\ln \lambda_1 - 2\varepsilon|, |\ln \lambda_2 + 2\varepsilon|\}$ then $\widehat{f * f_n}(s) = \hat{f}(s) \hat{f}_n(s) = \hat{f}(s)$, $s \in \boldsymbol{R}$, hence $f * f_n = f$. Consequently, for $n \geqq n_0$

$$\mathscr{F} - \int_{-\infty}^{+\infty} f_n(t) U_t x dt = \mathscr{F} - \int_{-\infty}^{+\infty} f(t) U_t x dt .$$

By Lemma 5.5

$$\mathscr{F} - \int_{-\infty}^{+\infty} f_n(t) U_t x dt \rightarrow x$$

in the $\mathscr{F}$-topology, so we deduce

$$x = \mathscr{F} - \int_{-\infty}^{+\infty} f(t) U_t x dt \in X^B((\lambda_1 e^{-2\varepsilon}, \lambda_2 e^{2\varepsilon})) .$$

In conclusion, for every $\delta > 1$, $S \subset X^B((\lambda_1\delta^{-1}, \lambda_2\delta))$.

Using Lemma 5.4, the above inclusion and property (ii) of the spectral subspaces, we deduce:

$$\begin{aligned} X^B([\lambda_1, \lambda_2]) &\subset S \\ &\subset \bigcap_{\delta>1} X^B((\lambda_1\delta^{-1}, \lambda_2\delta)) \subset \bigcap_{\delta>1} X^B([\lambda_1\delta^{-1}, \lambda_2\delta]) \\ &= X^B([\lambda_1, \lambda_2]) . \end{aligned}$$

q.e.d.

If $(X, \mathscr{F})$ and $(\mathscr{F}, X)$ satisfy (A_1) then for every $f \in L^1(\boldsymbol{R})$ the operator

$$X \ni x \mapsto \mathscr{F} - \int_{-\infty}^{+\infty} f(t) U_t x dt$$

is $\mathscr{F}$-continuous ([1], Proposition 1.4). Hence Theorem 5.6 implies:

(vii) *if* $(X, \mathscr{F})$ *and* $(\mathscr{F}, X)$ *satisfy* (A_1) *then every* $X^B([\lambda_1, \lambda_2])$ *is* $\mathscr{F}$*-closed.*

COROLLARY 5.7. *Let* $(X, \mathscr{F})$ *be a dual pair of Banach spaces such that* $(X, \mathscr{F})$ *and* $(\mathscr{F}, X)$ *satisfy* (A_1), $\{U_t\}_{t\in\boldsymbol{R}}$ *a bounded* $\mathscr{F}$*-continuous one-parameter group in* $B_{\mathscr{F}}(X)$, B *its analytic generator and* $0 < \lambda_1 \leqq \lambda_2 < +\infty$. *Then*

$$X^B([\lambda_1, \lambda_2]) = \left\{ x \in X; \begin{array}{l} \text{for every } f \in L^1(\boldsymbol{R}), \operatorname{supp} \hat{f} \cap [\ln \lambda_1, \ln \lambda_2] = \emptyset, \\ \text{we have } \mathscr{F} - \int_{-\infty}^{+\infty} f(t) U_t x dt = 0 \end{array} \right\} .$$

PROOF. We denote the right hand side of the above equality by M. By Theorem 5.6 $M \subset X^B([\lambda_1, \lambda_2])$.

Now let $x \in X^B([\lambda_1, \lambda_2])$ and $f \in L^1(\boldsymbol{R})$, $\operatorname{supp} \hat{f} \cap [\ln \lambda_1, \ln \lambda_2] = \emptyset$. Then there exists $\varepsilon > 0$ such that $\hat{f}$ vanishes on $[\ln \lambda_1 - \varepsilon, \ln \lambda_2 + \varepsilon]$. Following Theorem 5.6 $x \in X^B((\lambda_1 e^{-\varepsilon}, \lambda_2 e^{\varepsilon}))$, so for some $g \in L^1(R)$, $\operatorname{supp} \hat{g} \subset [\ln \lambda_1 - \varepsilon, \ln \lambda_2 + \varepsilon]$, and for some $y \in X$

$$x = \mathscr{F} - \int_{-\infty}^{+\infty} g(s) U_s y ds .$$

Using the fact that

$$X \ni z \mapsto \mathscr{F} - \int_{-\infty}^{+\infty} f(t) U_t z dt$$

is $\mathscr{F}$-continuous, for every $\varphi \in \mathscr{F}$

$$\left\langle \mathscr{F} - \int_{-\infty}^{+\infty} f(t) U_t x dt,\ \varphi \right\rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t)g(s)\langle U_{t+s}y,\ \varphi\rangle dtds$$
$$= \int_{-\infty}^{+\infty} (f*g)(t)\langle U_t y,\ \varphi\rangle dt\ .$$

Since $\widehat{f*g} = \hat{f}\hat{g} = 0$, there results $f*g = 0$, so

$$\mathscr{F} - \int_{-\infty}^{+\infty} f(t) U_t x dt = 0\ .$$

In conclusion, $X^B([\lambda_1, \lambda_2]) \subset M$.

q.e.d.

The right hand side of the equality from Corollary 5.7 was defined by W. Arveson in [1], where he denotes it by $M^U([\ln \lambda_1, \ln \lambda_2])$. Hence the spectral subspaces of B considered by us coincide with the spectral subspaces of $\{U_t\}$ defined by Arveson.

The use of spectral subspaces reduces the study of bounded $\mathscr{F}$-continuous one-parameter groups to computations with bounded uniformly continuous one-parameter groups.

We remark that further progresses may be expected using techniques of generalized scalar operators (see [4]).

6. **Some examples.** In this section we examine two important particular cases.

Firstly we consider strongly continuous groups of unitary operators on a Hilbert space. For functional calculus with self-adjoint operators we send to [11].

THEOREM 6.1. *Let H be a Hilbert space, $\{u_t\}_{t\in \boldsymbol{R}}$ a strongly continuous one-parameter group of unitary operators on H, b_α its analytical extension in $\alpha \in \boldsymbol{C}$ and $b = b_1$ its analytic generator. Then b is self-adjoint, positive, injective and for any $\alpha \in \boldsymbol{C}$*

$$b_\alpha = b^\alpha\ .$$

PROOF. Let $\varepsilon > 0$.

For each $\xi, \eta \in \mathscr{D}_{b_\varepsilon}$ the mappings

$$\alpha \mapsto (F_\xi(\alpha) \mid \eta)\ ,$$
$$\alpha \mapsto (\xi \mid F_\eta(\bar{\alpha})) = \overline{(F_\eta(\bar{\alpha}) \mid \xi)}$$

are regular extensions of $it \mapsto (u_t\xi \mid \eta)$ on $\{\alpha \in \boldsymbol{C};\ 0 \leqq \operatorname{Re}\alpha \leqq \varepsilon\}$, hence they coincide. Consequently

$$(b_\varepsilon \xi \mid \eta) = (\xi \mid b_\varepsilon \eta)\ ,$$

that is b_ε is symmetric.

If $\xi \in \mathscr{D}_{b_\varepsilon}$ and $\eta \in \mathscr{D}_{b_\varepsilon^*}$ then the mapping

$$\alpha \mapsto (F_\xi(\alpha)\,|\,\eta)$$

is bounded and regular on $\{\alpha \in \boldsymbol{C};\, 0 \leqq \operatorname{Re}\alpha \leqq \varepsilon\}$. We have the following estimations on the boundary:

$$|(F_\xi(it)\,|\,\eta)| = |(u_t\xi\,|\,\eta)| \leqq \|\xi\|\,\|\eta\|\,,$$
$$|(F_\xi(\varepsilon + it)\,|\,\eta)| = |(u_t\xi\,|\,b_\varepsilon^*\eta)| \leqq \|\xi\|\,\|b_\varepsilon^*\eta\|\,.$$

Using the Phragmen-Lindelöf theorem,

$$|(F_\xi(\alpha)\,|\,\eta)| \leqq \|\xi\| \max\{\|\eta\|, \|b_\varepsilon^*\eta\|\}\,, \qquad 0 \leqq \operatorname{Re}\alpha \leqq \varepsilon\,.$$

Thus there exists $G(\bar{\alpha}) \in H$ such that

$$(\xi\,|\,G(\bar{\alpha})) = (F_\xi(\alpha)\,|\,\eta)\,, \qquad \xi \in \mathscr{D}_{b_\varepsilon}\,.$$

It is easy to see that $\alpha \mapsto G(\alpha)$ is an H-regular extension of $it \mapsto u_t\eta$ on $\{\alpha \in \boldsymbol{C};\, 0 \leqq \operatorname{Re}\alpha \leqq \varepsilon\}$. Hence $\eta \in \mathscr{D}_{b_\varepsilon}$, so b_ε is self-adjoint.

Since by Theorem 2.4 $b = b_{1/2}b_{1/2}$, it follows that b is self-adjoint, positive and injective.

By Theorem 2.4 and by the unicity of the self-adjoint positive square root of a self-adjoint positive operator, we deduce that for all dyadic rationals $r \in \boldsymbol{R}$

$$b_r = b^r\,.$$

Hence for each $\xi \in \bigcap_{n=-\infty}^{+\infty} \mathscr{D}_{b^n}$ the integer mappings

$$\alpha \mapsto F_\xi(\alpha)\,,$$
$$\alpha \mapsto b^\alpha\xi$$

coincide on the dyadic rationals; therefore they coincide on $\boldsymbol{C}$. Consequently, for any $\alpha \in \boldsymbol{C}$, b_α and b^α coincide on $\bigcap_{n=-\infty}^{+\infty} \mathscr{D}_{b^n}$ and, using Corollary 2.6, we deduce:

$$b_\alpha = b^\alpha\,.$$

q.e.d.

In particular, we obtain the Stone representation theorem:

$$u_t = b^{it}\,, \qquad t \in \boldsymbol{R}\,.$$

We remark that, applying Corollary 3.5 with $\varepsilon = 1$ in this case, we obtain [5], Lemma 4.3.

Now we consider our second particular case.

Let H be a Hilbert space and $\{u_t\}$, $\{v_t\}$ strongly continuous one-parameter groups of unitaries on H. It is known that in $B(H)^*$ there exists a unique norm-closed linear subspace $B(H)_*$ such that the duality between $B(H)$ and $B(H)^*$ induces the relation $B(H) = (B(H)_*)^*$ (see for example [12], Corollary 1.13.3). The linear forms

$$B(H) \ni x \mapsto (x\xi \mid \eta)\,, \qquad \xi, \eta \in H\,,$$

form a total set in $B(H)_*$ in the norm-topology. Define for every $t \in \boldsymbol{R}$ an isometry U_t in $B_{B(H)_*}(B(H))$ by

$$U_t x = v_t x u_t\,.$$

It is easy to see that $\{U_t\}$ is a $B(H)_*$-continuous one-parameter group.

THEOREM 6.2. *Let H be a Hilbert space, $\{u_t\}_{t \in \boldsymbol{R}}$, $\{v_t\}_{t \in \boldsymbol{R}}$ strongly continuous one-parameter groups of unitary operators on H, b, d their analytic generators, $\{U_t\}_{t \in \boldsymbol{R}}$ the above defined $B(H)_*$-continuous one-parameter group of isometries in $B_{B(H)_*}(B(H))$ and B_α its analytical extension in $\alpha \in \boldsymbol{C}$. Then for every $\alpha \in \boldsymbol{C}$*

$$\mathscr{D}_{B_\alpha} = \{x \in B(H);\ \overline{b^\alpha \mid \mathscr{D}_{d^\alpha x b^\alpha}} = b^\alpha \text{ and } d^\alpha x b^\alpha \text{ is bounded}\}$$

and if $x \in \mathscr{D}_B$ then

$$\mathscr{D}_{d^\alpha x b^\alpha} = \mathscr{D}_{b^\alpha}\,,$$
$$B_\alpha x = \overline{d^\alpha x b^\alpha}\,.$$

PROOF. We consider, for example, that $\operatorname{Re} \alpha \geqq 0$. In the case $\operatorname{Re} \alpha \leqq 0$ the proof is similar.

Suppose that $x \in B(H)$ is such that $\overline{b^\alpha \mid \mathscr{D}_{d^\alpha x b^\alpha}} = b^\alpha$ and

$$\|d^\alpha x b^\alpha \xi\| \leqq c\|\xi\|\,, \qquad \xi \in \mathscr{D}_{d^\alpha x b^\alpha}\,.$$

Then

$$|(x b^\alpha \xi \mid d^{\bar\alpha} \eta)| \leqq c\|\xi\|\,\|\eta\|\,, \qquad \xi \in \mathscr{D}_{d^\alpha x b^\alpha},\ \eta \in \mathscr{D}_{d^{\bar\alpha}}\,.$$

Since $\overline{b^\alpha \mid \mathscr{D}_{d^\alpha x b^\alpha}} = b^\alpha$, we deduce that the above inequality holds for every $\xi \in \mathscr{D}_{b^\alpha}$, $\eta \in \mathscr{D}_{d^{\bar\alpha}}$. Using the Phragmen-Lindelöf theorem, there results:

$$|(x b^\gamma \xi \mid d^{\bar\gamma} \eta)| \leqq \max\{\|x\|, c\}\|\xi\|\,\|\eta\|\,,$$
$$\xi \in \mathscr{D}_{b^\alpha}\,, \quad \eta \in \mathscr{D}_{d^{\bar\alpha}}, \quad 0 \leqq \operatorname{Re} \gamma \leqq \operatorname{Re} \alpha\,.$$

Hence for $0 \leqq \operatorname{Re} \gamma \leqq \operatorname{Re} \alpha$ there exists $F(\gamma) \in B(H)$ such that

$$\|F(\gamma)\| \leqq \max\{\|x\|, c\}\,,$$
$$(F(\gamma)\xi \mid \eta) = (x b^\gamma \xi \mid d^{\bar\gamma} \eta),\ \xi \in \mathscr{D}_{b^\alpha},\ \eta \in \mathscr{D}_{d^{\bar\alpha}}\,.$$

It is easy to see that $\gamma \mapsto F(\gamma)$ is an $B(H)_*$-regular extension of $it \mapsto v_t x u_t$

on $\{\gamma \in \boldsymbol{C};\ 0 \leqq \operatorname{Re}\gamma \leqq \operatorname{Re}\alpha\}$. Consequently $x \in \mathscr{D}_{B_\alpha}$.

Conversely, let $x \in \mathscr{D}_{B_\alpha}$. For $\xi \in \mathscr{D}_{b^\alpha}$, $\eta \in \mathscr{D}_{d^{\bar\alpha}}$

$$\gamma \mapsto (F_x(\gamma)\xi \mid \eta) ,$$
$$\gamma \mapsto (xb^\gamma\xi \mid d^{\bar\gamma}\eta)$$

are regular extensions of $it \mapsto (v_t x u_t \xi \mid \eta)$ on $\{\gamma \in \boldsymbol{C};\ 0 \leqq \operatorname{Re}\gamma \leqq \operatorname{Re}\alpha\}$, hence they coincide. In particular,

$$((B_\alpha x)\xi \mid \eta) = (xb^\alpha\xi \mid d^{\bar\alpha}\eta) .$$

For every $\xi \in \mathscr{D}_{b^\alpha}$

$$|(xb^\alpha\xi \mid d^{\bar\alpha}\eta)| \leqq \|(B_\alpha x)\xi\| \ \|\eta\| , \qquad \eta \in \mathscr{D}_{d^{\bar\alpha}} ,$$

so $\eta \mapsto (xb^\alpha\xi \mid d^{\bar\alpha}\eta)$ defines a bounded antilinear functional on H. Consequently

$$xb^\alpha\xi \in \mathscr{D}_{(d^{\bar\alpha})^*} = \mathscr{D}_{d^\alpha} ,$$

that is

$$\xi \in \mathscr{D}_{d^\alpha x b^\alpha} .$$

Moreover,

$$((B_\alpha x)\xi \mid \eta) = (d^\alpha x b^\alpha \xi \mid \eta) , \qquad \eta \in \mathscr{D}_{d^{\bar\alpha}} ,$$
$$(B_\alpha x)\xi = d^\alpha x b^\alpha \xi .$$

In conclusion, $\mathscr{D}_{d^\alpha x b^\alpha} = \mathscr{D}_{b^\alpha}$, $d^\alpha x b^\alpha$ is bounded and $B_\alpha x = \overline{d^\alpha x b^\alpha}$. q.e.d.

Applying Corollary 3.5 with $\varepsilon = 1$ to $\{U_t\}$ and B, it follows [5], Lemma 4.2.

We remark also that Theorem 4.4 is applied in this particular case in [15].

We are grateful to A. van Daele who communicated us an example of $\{u_t\}$ and $\{v_t\}$ for which $\sigma(B) = \boldsymbol{C}$. Hence he contributed to correct an error in our initial version. However, it seems that, using techniques from [14], it can prove that if b or d is bounded then $\sigma(B) \subset \boldsymbol{R}_+$.

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