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ON LINEAR TOPOLOGICAL PROPERTIES OF SOME C^*-ALGEBRAS

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The purpose of this note is to give Banach space-like characterizations of the following special algebraic properties with respect to a C^*-algebra A and a von Neumann algebra M:

I. each irreducible representation of A is finite dimensional;

II(resp. II′). each irreducible representation of M (resp. A) is finite dimensional with bounded degree;

III(resp. III′). M (resp. A^{**}) is a finite atomic von Neumann algebra, where A^{**}, the bidual of A, is identified with the enveloping von Neumann algebra of A, i.e. the von Neumann algebra generated by the image of the representation of A which is the direct sum of all cyclic representations induced by states of A.

We shall single out these properties by means of the (DP) property and properties $(*)$, $(**)$ defined below. For a Banach space E, let us consider the following conditions:

(DP_1) for any Banach space F, each weakly compact linear map of E into F transforms each weakly relatively compact subset of E to a relatively compact subset of F;

(DP_2) each $\sigma(E, E^*)$-convergent sequence in E is $\tau(E^{**}, E^*)$-convergent;

(DP_3) each $\sigma(E^*, E^{**})$-convergent sequence in E^* is $\tau(E^*, E)$-convergent;

$(*)$ each $\sigma(E, E^*)$-convergent sequence in E is norm-convergent;

$(**)$ for any locally convex topological vector space F, each weakly compact linear map of E into F is compact.

The implications among these conditions are as follows:

$(\mathrm{DP}_1) \Leftrightarrow (\mathrm{DP}_2) \Leftrightarrow (\mathrm{DP}_3)$ ([3] or [2; Chapter 9]), $(*)$ or $(**) \Rightarrow (\mathrm{DP}_1)$, and $(**) \Leftrightarrow$ "$(*)$ is satisfied with E replaced by E^*." (See Lemma 6 below.) We say that a Banach space E has the (DP) (=Dunford-Pettis) property [resp. property $(*)$, $(**)$] if E satisfies any one of the equivalent conditions (DP_1)-(DP_3) [resp. $(*)$, $(**)$]. The definition and three equivalent forms of the (DP) property are due to A. Grothendieck [3].

The (DP) property can be considered to afford a sort of axiomatization of characteristics of weakly compact linear maps acting on $C(K)$, the Banach space of continuous functions on a compact Hausdorff space K; in fact, a continuous linear map of $C(K)$ into a Banach space F is weakly compact if and only if it transforms each weakly relatively compact subset of $C(K)$ to a relatively compact subset of F([3] or [2; Chapter 9]). This fact shows that the commutative C^*-algebra $C(K)$ has the (DP) property. Another example with the (DP) property is the predual L^1 of the commutative von Neumann algebra L^∞, and the sequence space l^1 (resp. c_0) is the typical example with property $(*)$[resp. $(**)$]. Therefore it will be meaningful to consider which C^*-algebras or preduals of von Neumann algebras have the (DP) property, property $(*)$ or $(**)$. But the example in S. Sakai [7] shows that generally these Banach spaces do not have the (DP) property.

Throughout this note, topological isomorphisms between Banach spaces always mean linear homeomorphisms between them, and for a Banach space E, E_1 denotes the unit ball of E.

Our theorems are stated as follows:

THEOREM 1. *For a C^*-algebra A, the following are equivalent*:

(1) *A^* has the (DP) property*;

(2) [*resp.* (2′)] *A^{**} is finite* (*resp. finite type I*);

(3) *each irreducible representation of A is finite dimensional.*

THEOREM 2. *For a von Neumann algebra M, the following are equivalent*:

(1) *M is topologically isomorphic to a $C_0(\Omega)$, the Banach space of continuous functions on a locally compact Hausdorff space Ω*;

(2) *M^* has the (DP) property*;

(3) *M satisfies the property II above, or equivalently M is of the form $\sum_{i=1}^{n} \oplus Z_i \otimes B(H_i)$, where the Z_i are commutative von Neumann algebras and the $B(H_i)$ are full matrix algebras on the i-dimensional Hilbert spaces H_i*;

(4) *M_* is topologically isomorphic to an L^1.*

THEOREM 3. *For a von Neumann algebra M, the following are equivalent*:

(1) *M is finite atomic*;

(2) *M_* has property $(*)$*;

(3) *the strong and weak-operator topologies coincide on M_1.*

In Theorem 3, the implication (1)$\Rightarrow$(2) is due to K. Saitô [6; Theorems 1 and 2]. These results mean that properties I, II′ and III′

are preserved by topological isomorphisms between C^*-algebras and that property II (resp. III) is preserved by a topological isomorphism (resp. σ-weakly bicontinuous linear isomorphism) between von Neumann algebras.

Proofs of Theorems and their Corollaries. To prove Theorems 1 and 2, we prepare 5 Lemmata.

LEMMA 1. *The predual M_* of a finite type I von Neumann algebra M has the (DP) property.*

PROOF. We first note that, for any von Neumann algebra N, N_* has the (DP) property if and only if, for each sequence (x_n) in N, $x_n \to 0$ $\sigma(N, N^*)$ implies $x_n^* x_n \to 0$ $\sigma(N, N_*)$. To see this, take N_* as E in the condition (DP_3) and remark that $x_n \to 0$ $\sigma(N, N^*)$ implies uniform boundedness of (x_n) and that $x_n \to 0$ $\tau(N, N_*) \Leftrightarrow x_n^* x_n,\ x_n x_n^* \to 0$ $\sigma(N, N_*)$ [1; Theorem II. 7].

By hypothesis, M is of the form $\sum_{i=1}^{\infty} {}^{\oplus} Z_i \otimes B(H_i)$, where the Z_i are commutative von Neumann algebras and the $B(H_i)$ are the $i \times i$ full matrix algebras on the i-dimensional Hilbert spaces H_i. Let z_j be the central projection in M such that $Mz_j = \sum_{i=1}^{j} {}^{\oplus} Z_i \otimes B(H_i)$. Then $z_j \uparrow 1$ strongly and the predual $(Mz_j)_*$ of Mz_j has the (DP) property since it is topologically isomorphic to the Banach space of type L^1

$$(Z_1)_* \oplus [\underbrace{(Z_2)_* \oplus \cdots \oplus (Z_2)_*}_{2^2}] \oplus \cdots \oplus [\underbrace{(Z_j)_* \oplus \cdots \oplus (Z_j)_*}_{j^2}] \ (l^1\text{-sum})\,.$$

Therefore, by the first paragraph, for a sequence (x_n) in M,

$$\begin{aligned} x_n \to 0 \ \sigma(M, M^*) &\Rightarrow x_n z_j \to 0 \ \sigma(Mz_j, (Mz_j)^*) \\ &\Rightarrow x_n^* x_n z_j \to 0 \ \sigma(Mz_j, (Mz_j)_*) \\ &\quad \text{for all } j. \end{aligned}$$

Given $\varepsilon > 0$ and a positive φ in M_*, there exists a j_0 with $\varphi(1 - z_{j_0}) \leqq \varepsilon / \sup_n \|x_n\|^2$; hence

$$\overline{\lim_{n\to\infty}}\, \varphi(x_n^* x_n) \leqq \lim_{n\to\infty} \varphi(x_n^* x_n z_{j_0}) + \varepsilon = \varepsilon\,.$$

Thus we have $x_n^* x_n \to 0$ $\sigma(M, M_*)$ as desired. q.e.d.

REMARK. As shown in the proof above, the $(Mz_j)_*$ are topologically isomorphic to L^1, but so is not M_* (see Theorem 2).

The following Lemmata 2 and 3 are a modification of the argument in [7] and an immediate consequence of the proof of Lemma 1, respectively.

LEMMA 2. *If $C(H)$, the algebra of compact operators in a Hilbert space H, has the (DP) property, then H is finite dimensional.*

LEMMA 3. *Let M be a von Neumann algebra and N a σ-weakly closed *-subalgebra of M. Then if M_* has the (DP) property, so does N_* too.*

LEMMA 4. *If the predual M_* of a von Neumann algebra M has the (DP) property, then M is finite.*

PROOF. If M is not finite, there exists a σ-weakly closed *-subalgebra N of M which is *-isomorphic to $B(H)$ for some infinite dimensional Hilbert space H. By Lemma 3, $T(H) \cong N_*$ has the (DP) property. Then it is readily verified from the definition of the (DP) property and the fact that $C(H)^* = T(H)$ that $C(H)$ has the (DP) property, contradictory to Lemma 2. q.e.d.

LEMMA 5. *Let A be a C^*-algebra and A^{**} the enveloping von Neumann algebra of A. Then the following are equivalent:*

(i) *[resp. (i′)] A^{**} is finite (resp. finite type I);*

(ii) *each irreducible representation of A is finite dimensional.*

PROOF. (i) $\Rightarrow$ (ii) and (i′) $\Rightarrow$ (i) are obvious. (ii) $\Rightarrow$ (i′): Let z be the central projection in A^{**} corresponding to the finite type I part of A^{**}. Assuming $z \neq 1$(1, the unit of A^{**}), we deduce a contradiction. Let $I = Az \cap A$. Then I is a proper closed two-sided ideal in A and A/I is a nonzero C^*-algebra whose irreducible representations are all finite dimensional. Hence by [4; Theorem 6.1] there exists a closed two-sided ideal J in A such that $J \supsetneqq I$ and J/I satisfies a polynomial identity (or equivalently all irreducible representations of J/I are finite dimensional with bounded degree). There exist central projections p, q in A^{**} with $\bar{I}^{\sigma-w} = A^{**}p$, $\bar{J}^{\sigma-w} = A^{**}q$ and $p \lneqq q$. We have $p \leqq z$ since $\bar{I}^{\sigma-w} = (Az \cap A)^{-\sigma-w} \subset A^{**}z$. On the other hand, since J/I satisfies a polynomial identity, so does $(J/I)^{**} \cong A^{**}(q-p)$. Hence $q - p \leqq z$. Thus $q = p \vee (q-p) \leqq z$, $J = Aq \cap A \subset Az \cap A = I$, a contradiction. This completes the proof. q.e.d.

PROOF OF THEOREM 1. (1) $\Rightarrow$ (2) by Lemma 4, (2) $\Leftrightarrow$ (2′) $\Leftrightarrow$ (3) by Lemma 5, and (2′) $\Rightarrow$ (1) by Lemma 1. q.e.d.

PROOF OF THEOREM 2. (1) $\Rightarrow$ (2): If M is topologically isomorphic to a $C_0(\Omega)$, so is M^* to $C_0(\Omega)^* = \mathfrak{M}^1(\Omega)$ (the Banach space of bounded Radon measures on Ω). Since $\mathfrak{M}^1(\Omega)$ is a Banach space of type L^1, M^* has the (DP) property [3].

(2) $\Rightarrow$ (3): By Theorem 1, (2) implies that each irreducible representation is finite dimensional, so that M is a CCR algebra. Thus we obtain (3) [4; Theorem 9.1].

(3) $\Rightarrow$ (4): Immediate from the proof of Lemma 1.

(4) $\Rightarrow$ (1): Suppose (4). Then M is topologically isomorphic to $(L^1)^* = L^\infty \cong C(K)$, where K is the spectrum of the commutative C^*-algebra L^∞.
q.e.d.

COROLLARY TO THEOREM 2. *For a C^*-algebra A, the following are equivalent:*

(1) *each irreducible representation of A is finite dimensional with bounded degree;*

(2) *A^{***} has the (DP) property;*

(3) *A^* is topologically isomorphic to an L^1.*

Theorem 3 will be proved after the statement of some easy Lemmata.

LEMMA 6. *Let E be a Banach space. Then:*

(i) *The following are equivalent:*

(i1) *E has property $(*)$;*

(i2) *the topologies $\tau(E^*, E)$ and $\sigma(E^*, E)$ coincide on E_1^*.*

(ii) *The following are equivalent:*

(ii1) *E has property $(**)$;*

(ii2) *E_1^{**} is $\tau(E^{**}, E^*)$-compact;*

(ii3) *E^* has property $(*)$.*

PROOF. (i) is readily proved, so we omit the proof. (ii1) $\Rightarrow$ (ii2): The canonical injection j: $(E, \text{norm}) \to (E^{**}, \tau(E^{**}, E^*))$ is weakly compact because $(E^{**}, \tau(E^{**}, E^*))^* = E^*$ and $j(E_1)^{-\tau(E^{**},E^*)} = E_1^{**}$ is $\sigma(E^{**}, E^*)$-compact. Hence j is compact, i.e. E_1^{**} is $\tau(E^{**}, E^*)$-compact. (ii2) $\Rightarrow$ (ii1): Let $T: E \to F$ be any weakly compact linear map of E into a locally convex space F. Then ${}^{tt}T: E^{**} \to {}^{tt}T(E^{**}) \subset F$ is $\tau(E^{**}, E^*)$-$\tau(F, F^*)$-continuous. Hence if E_1^{**} is $\tau(E^{**}, E^*)$-compact, $T(E_1) \subset {}^{tt}T(E_1^{**})$ is relatively compact in F, i.e. T is compact. (ii2) $\Leftrightarrow$ (ii3) follows from (i). q.e.d.

LEMMA 7. *Let N be a von Neumann algebra which is not atomic. Then:*

(i) *There exists a weakly relatively compact subset in N which is not relatively compact.*

(ii) *There exist a sequence (u_n) of self-adjoint unitary operators in N and a projection p in N with $p \neq 1$ such that $u_n \to p$ $\sigma(N, N_*)$.*

PROOF. (i) We may assume that N is non-atomic and there exists a faithful normal state φ on N. Then there exist orthogonal families

$\{p_0, p_1\}$, $\{p_{00}, p_{01}, p_{10}, p_{11}\}$, $\cdots$ etc. of projections in N such that

$$p_0 + p_1 = 1\,, \quad \varphi(p_0) = \varphi(p_1) = 1/2\,;$$
$$p_{00} + p_{01} = p_0\,, \quad p_{10} + p_{11} = p_1\,,$$
$$\varphi(p_{00}) = \varphi(p_{01}) = \varphi(p_{10}) = \varphi(p_{11}) = 1/4, \cdots \text{etc.}$$

Put $a_1 = p_0 - p_1$, $a_2 = p_{00} - p_{01} + p_{10} - p_{11}$, $\cdots$ etc. and $\varphi_n = \varphi(a_n \cdot) \in N_*$. Then $\varphi(a_n a_m) = \delta_{nm}$, $\|a_n\| = 1$ $(n, m = 1, 2, \cdots)$ and

$$\|\varphi_n - \varphi_m\| \geqq |\varphi_n(a_m) - \varphi_m(a_m)| = |\varphi(a_n a_m) - \varphi(a_m^2)| = 1$$

for $n \neq m$, so $K = \{\varphi_n\} \subset N_*$ is not relatively compact. But K is weakly relatively compact because, for each x in N,

$$\varphi((x - \sum_{n=1}^{m} \varphi_n(x)a_n)^*(x - \sum_{n=1}^{m} \varphi_n(x)a_n)) = \varphi(x^*x) - \sum_{n=1}^{m} |\varphi_n(x)|^2 \quad \text{for all } m$$
$$\Rightarrow \sum_{n=1}^{\infty} |\varphi_n(x)|^2 < \infty \Rightarrow \varphi_n(x) \to 0 \quad \text{as} \quad n \to \infty.$$

(ii) follows from the above construction of the a_n. q.e.d.

PROOF OF THEOREM 3. As stated before, (1) $\Rightarrow$ (2) follows from [6; Theorems 1 and 2]. We have (2) $\Leftrightarrow$ (3) by Lemma 6 and [1; Theorem II. 7]. (2) $\Rightarrow$ (1): Suppose (2). Since property ($*$) implies the (DP) property, M_* has the (DP) property, so that M is finite (Lemma 4). Lemma 7(i), combined with Lemma 6(i), shows that M is atomic. q.e.d.

COROLLARY TO THEOREM 3. *Let A be a C^*-algebra. Then A^{**} is finite atomic if and only if A has property* ($**$).

PROOF. Immediate from Theorem 3 and Lemma 6(ii). q.e.d.

REMARKS. It follows from Theorem 3 and Lemma 7(ii) that for a von Neumann algebra M, the property III above is also equivalent to compactness in the weak-operator topology of M_u(resp. M_p, ext (M_1)), where M_u(resp. M_p, ext (M_1)) denotes the unitary group of M(resp. the set of projections in M, the set of extreme points of M_1). The equivalence of the property III and compactness in the weak-operator topology of M_u is known in [5; Corollary to Proposition 5].

In Lemma 6, (ii3) $\Rightarrow$ (ii1) is already obtained in [8; Lemma 9].

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