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SOME SYSTEM OF DIFFERENTIAL EQUATIONS ON RIEMANNIAN MANIFOLDS AND ITS APPLICATIONS TO CONTACT STRUCTURES

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1. Introduction. Let (M, g) be a connected $m(\geq 2)$-dimensional Riemannian manifold of class C^∞. The existence of a function satisfying some differential equations on (M, g) sometimes determines differential geometric properties of (M, g). For example, a complete (M, g) admits a non-trivial (global) solution f of the system of differential equations

$$\nabla_j\nabla_i f + Kfg_{ji} = 0 \tag{1.1}$$

for some positive constant K, if and only if (M, g) is isometric to a Euclidean sphere $S^m(K)$ of radius $1/\sqrt{K}$ (S. Ishihara and Y. Tashiro [7], M. Obata [11], [12]).

Furthermore, M. Obata [12] announced the following.

OBATA'S THEOREM (*). *Let (M, g) be a complete and simply connected Riemannian manifold of dimension m. In order for M to admit a non-trivial solution f for the system of differential equations*

$$\nabla_j\nabla_i\nabla_h f + K(2\nabla_j f g_{ih} + \nabla_i f g_{hj} + \nabla_h f g_{ji}) = 0 \tag{1.2}$$

for some positive constant K, it is necessary and sufficient that (M, g) be isometric to a Euclidean sphere $S^m(K)$.

This system of differential equations is of order 3, and this result is itself of great importance. However, the complete proof has not yet been published unfortunately.

On the other hand, a unit Killing vector field ξ on (M, g) is called a normal contact structure or simply a Sasakian structure, if it satisfies

$$R(X, \xi)Y = g(\xi, Y)X - g(X, Y)\xi \tag{1.3}$$

for any vector fields (or tangent vectors) X and Y, where R denotes the Riemannian curvature tensor field. In other words, ξ is a Sasakian structure if and only if it belongs to the 1-nullity distribution of (M, g). In this case, the dimension m of M is odd.

Let ξ and η be two Sasakian structures on (M, g). If $g(\xi, \eta) = f$ is not constant, then f satisfies (1.2) for $K = 1$ (S. Tachibana and W. N.

Yu [17]). Applying Theorem (*), S. Tachibana and W. N. Yu [17] showed the following.

THEOREM (**). *Let (M, g) be a complete and simply connected Riemannian manifold. If (M, g) admits two Sasakian structures ξ and η with non-constant $g(\xi, \eta)$, then (M, g) is isometric to $S^m(1)$.*

In this paper, without assuming Theorem (*) we give a proof of Theorem (**) for the dimension $m \leq 11$. For this purpose, first we prove Theorem A as a weaker version of Theorem (*), and then we apply a Theorem by D. Ferus [4] on 1-nullity distributions.

THEOREM A. *Let (M, g) be a complete Riemannian manifold. If (M, g) admits a non-trivial function f satisfying* (1.2) *and*

$$(1.4) \qquad (1/K)g^{hj}\nabla_h f \nabla_j \nabla_i f = -4f\nabla_i f \, ,$$

then (M, g) is of constant curvature K.

This is a special case of Theorem (*) as will be explained in the Section 4. However, our proof is differential-geometric and we can give a nice frame at a point concretely, which plays an important rôle in the outline of Obata's proof of Theorem (*).

2. **Preliminaries.** Let (M, g) be a Riemannian manifold with a normal contact structure or a Sasakian structure ξ (cf. S. Sasaki [15], S. Tanno [18], [19]). Since ξ is a Killing vector field, (1.3) is written as

$$(2.1) \qquad \nabla_X(\nabla\xi)\cdot Y = g(\xi, Y)X - g(X, Y)\xi \, ,$$

where X and Y are vector fields or tangent vectors. If we put $\nabla\xi = -\phi$ (i.e., $\nabla_X\xi = -\phi X$), then the (1, 1)-tensor field ϕ satisfies the following.

$$(2.2) \qquad \phi\xi = 0 \, , \quad \phi\phi X = -X + g(\xi, X)\xi \, ,$$

$$(2.3) \qquad g(\phi X, \phi Y) = g(X, Y) - g(\xi, X)g(\xi, Y) \, ,$$

$$(2.4) \qquad \nabla_j\phi_i^h = g_{ji}\xi^h - \delta_j^h\xi_i \, .$$

The 1-form which is dual to ξ with respect to g is called a contact form on M.

We pick up some known results for our later arguments.

[i] If φ is an isometry of (M, g) and ξ is a Sasakian structure on (M, g), then $\varphi\xi$ is also a Sasakian structure on (M, g), where φ denotes also its differential (S. Tanno [19]).

[ii] If ξ and η are Sasakian structures on (M, g), then $g(\xi, \eta) = f$ satisfies (1.2) for $K = 1$ (S. Tachibana and W. N. Yu [17]).

[iii] If ξ and η are two Sasakian structures on (M, g) and if $g(\xi, \eta) = f$

is constant ($\neq 1, \neq -1$), then putting $\xi_{(1)} = \xi$,

$$\xi_{(2)} = (\eta - f\xi)/|\eta - f\xi| ,$$

and $\xi_{(3)} = [\xi_{(1)}, \xi_{(2)}]/2$, we have a Sasakian 3-structure $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ (cf. Y. Y. Kuo [9]). A Riemannian manifold admitting a Sasakian 3-structure is an Einstein space (T. Kashiwada [8]).

The Riemannian curvature tensor field R is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z . \tag{2.5}$$

We define a curvature-like tensor field T by

$$T(X, Y)Z = R(X, Y)Z - K(g(Y, Z)X - g(X, Z)Y) . \tag{2.6}$$

Let x be a point of M and let M_x be the tangent space to M at x. Then the K-nullity space N_x at x is defined by

$$N_x = \{X \in M_x ;\, T(X, Y)Z = 0 \quad \text{for all} \quad Y, Z \in M_x\} . \tag{2.7}$$

$\dim N_x = \mu(x)$ is called the index of K-nullity at x (S. S. Chern and N. H. Kuiper [2], T. Otsuki [13]). The minimum μ of $\mu(x)$ on M is called the index of K-nullity of (M, g).

[iv] $\mu(x)$ is upper semi-continuous, and hence the set G of M where $\mu(x) = \mu$ is open.

[v] If $\mu(x)$ is constant on an open submanifold M^0 of M, then the K-nullity distribution: $x \to N_x$ is integrable on M^0, and leaves (maximal integral submanifolds) are totally geodesic. Therefore leaves are of constant curvature (K. Abe [1], A. Gray [6], R. Maltz [10], etc.).

[vi] If (M, g) is complete and if $\mu \geq 1$, then leaves on G are complete (K. Abe [1], Y. N. Clifton and R. Maltz [3], D. Ferus [5], etc.).

[vii] Let N be a μ-dimensional involutive (integrable) distribution. Assume that:

(a) each leaf of N is complete and totally geodesic,

(b) sectional curvature $K(X, Y) = \text{constant} > 0$ for $X \in N, Y \in N^\perp$, where $N^\perp$ denotes the distribution orthocomplementary to N (in case $\mu < m$),

(c) $\mu > \nu_m$, where ν_m is for example;

$$\begin{array}{llllllllll} m: & 2\,3\,4 & 5\,6\,7 & 8\,9\,10 & 11\,12 & 13\,14\,15\,16 & 17\,18\,19 \cdots 24\,25\,26\,27 \cdots \\ \nu_m: & 0\,1\,0 & 1\,2\,3 & 0\,1\,\;2 & 3\;\;4 & 5\;\;6\;\;7\;\;0 & 1\;\;2\;\;3 \cdots\;\;8\;\;1\;\;2\;\;3 \cdots . \end{array}$$

Then we have $\mu = m$ (D. Ferus [4]).

Next we summerise the fundamental facts on (1.2).

[viii] If a function f on (M, g) satisfies (1.2), then $(\nabla^i f) = (g^{ij} \nabla_j f)$ is an infinitesimal projective transformation. This follows from (1.2) and the Ricci identity for $\nabla_j \nabla_i \nabla_h f - \nabla_i \nabla_j \nabla_h f$ (M. Obata [12]).

[ix] Let $\{x(s)\}$ be a complete geodesic in (M, g) with arclength parameter s. Then the restriction of f satisfying (1.2) with $K=1$ to $\{x(s)\}$ is given by

$$f(s) = (f''(0)/2)\sin^2 s + (f'(0)/2)\sin 2s + f(0)\,.$$

This is verified by the fact that (1.2) on the geodesic is $f''' + 4f' = 0$, where the dash denotes d/ds. In particular, $f(s)$ is bounded and periodic $(f(\pi + s) = f(s))$. Furthermore, if $f(s_0)$ is the maximum, then $f(s_0 + \pi/2)$ is the minimum.

[x] If f satisfies (1.2), then $(\nabla^i f)$ belongs to the K-nullity distribution. This follows from the Ricci identity.

3. Proof of Theorem A. By replacing g by Kg, we can assume that $K=1$ in proving Theorem A. Let f be a non-trivial function on (M, g) satisfying

$$\nabla_j\nabla_i\nabla_h f + 2\nabla_j f g_{ih} + \nabla_i f g_{hj} + \nabla_h f g_{ji} = 0\,, \tag{3.1}$$

$$\nabla_j\nabla_i f\nabla^j f = -4f\nabla_i f\,, \tag{3.2}$$

where $\nabla^j f = g^{jh}\nabla_h f$, and ∇ denotes the Riemannian connection of (M, g). We define a vector field F by $F = \operatorname{grad} f = (\nabla^i f)$. By (3.2) we have $\nabla_F F = -4fF$, that is, each trajectory of F is a geodesic. Let $\varphi_t = \exp tF$ be the (local) 1-parameter group of (local) transformations generated by F. We take and fix a trajectory C of F and take a point p of C. We parameterise C so that

$$C = \{x(t)\} = \{\exp tF\cdot p\} = \{\varphi_t p\}\,.$$

Let $f = f(t)$ be the restriction of f to C. Then

$$f^*(t) = \frac{df}{dt} = F^i\frac{\partial f}{\partial x^i} = \nabla^i f\nabla_i f\,.$$

(3.2) is written as

$$\nabla_i(\nabla_j f\nabla^j f) = -4\nabla_i f^2\,.$$

Transvecting the last equation by $\nabla^i f$, we get

$$f^{**} = -4(f^2)^*\,,$$

where $**$ denotes d^2/dt^2. Solving this we have

$$f^* = -4f^2 + c \tag{3.3}$$

for some constant c. If $c = 0$, then

$$f(t) = (4t + \beta)^{-1} \tag{3.4}$$

for some constant β. If $c < 0$, then

$$f(t) = -b \tan 4b(t+\beta)\,, \quad b = (-c/4)^{1/2}\,. \tag{3.5}$$

And if $c > 0$, then

$$f(t) = a(e^{8at+\beta} - 1)/(e^{8at+\beta} + 1)\,, \quad 4a^2 = c\,, \tag{3.6}$$

or

$$f(t) = a(e^{8at+\beta} + 1)/(e^{8at+\beta} - 1)\,, \quad 4a^2 = c\,. \tag{3.7}$$

Since C is a geodesic and (M, g) is complete, we see that the parameter t can be extended as far as $f^* \neq 0$. Therefore (3.4), (3.5) and (3.7) can not be bounded. Hence, by [ix] of §2 we get (3.6). This shows that $f(t)$ takes the value 0 once. So we assume that $f(p) = 0$.

LEMMA 3.1. *A non-trivial trajectory C of F is parameterised so that $C = \{\exp tF\cdot p,\ -\infty < t < \infty\}$, $f(p) = f(0) = 0$, and*

$$f^*(t) = -4f^2 + 4a^2\,, \tag{3.8}$$

$$f(t) = a(e^{8at} - 1)/(e^{8at} + 1) \tag{3.9}$$

for some positive constant a. In particular, $-a < f(t) < a$ and $f(t) \to a$ (as $t \to \infty$), $f(t) \to -a$ (as $t \to -\infty$).

PROOF. This follows from (3.6) and $f(p) = 0$. q.e.d.

LEMMA 3.2. *The tensor field $(\nabla_j F^i) = (\nabla_j \nabla^i f)$ is symmetric with respect to g and has real eigenvalues at each point of C. There are at most three different eigenvalues at each point:*

$$-4f\,, \quad -2f + 2a\,, \quad -2f - 2a\,,$$

where the multiplicity of $-4f$ is one.

PROOF. F is an eigenvector at each point by (3.2) for the eigenvalue $-4f$. Let Z be a (local) field of eigenvectors along C which is orthogonal to F. We put

$$Z^j \nabla_j \nabla^i f = \lambda Z^i\,. \tag{3.10}$$

Operating ∇_h to (3.2) we obtain

$$\nabla_h \nabla_j \nabla_i f \nabla^j f + \nabla_j \nabla_i f \nabla_h \nabla^j f = -4\nabla_h f \nabla_i f - 4f \nabla_h \nabla_i f\,.$$

By (3.1) we have

$$4f \nabla_h \nabla_i f = -\nabla_j \nabla_i f \nabla_h \nabla^j f - \nabla_h f \nabla_i f + (\nabla_r f \nabla^r f) g_{ih}\,.$$

Transvecting the last equation with $Z^h Z^i$ and using (3.10), we get

$$\lambda^2 + 4f\lambda - (\nabla_r f \nabla^r f) = 0\,. \tag{3.11}$$

Hence, λ can take at most two values. Since $(\nabla_r f \nabla^r f) = f^*$, applying (3.8) we can solve λ from (3.11) and get $\lambda = -2f \pm 2a$. q.e.d.

LEMMA 3.3. *By L_F we denote the Lie derivation by F. Then*

$$L_F(\nabla_j \nabla^i f) = -2(\nabla_r f \nabla^r f)\delta^i_j - 2\nabla^i f \nabla_j f \ .$$

PROOF. By definition of L_F we have

$$L_F(\nabla_j \nabla^i f) = F^h \nabla_h \nabla_j \nabla^i f - \nabla_h F^i \nabla_j \nabla^h f + \nabla_j F^h \nabla_h \nabla^i f \ .$$

Applying (3.1) we get the required relation. q.e.d.

LEMMA 3.4. *Let D_1, D_2, D_3 be the fields of subspaces along C defined by the eigenspaces for $-4f$, $-2f + 2a$, $-2f - 2a$, respectively. Then D_1, D_2, D_3 are orthogonal and invariant by $\varphi_t = \exp tF$.*

PROOF. It is clear that D_1, D_2, D_3 are orthogonal. Since $L_F F = [F, F] = 0$, we have $\varphi_t D_1 = D_1$. Next we show that $D_2 + D_3$ is invariant by φ_t. Let Z_p be a tangent vector at p such that $Z_p \in (D_2 + D_3)(p)$. Define a vector field Z along C by

$$Z_{\varphi_t p} = \varphi_t Z_p \ ,$$

where φ_t denotes also its differential. Define a function $G(t)$ of t by $G(t) = g_{\varphi_t p}(Z, F)$. Since $L_F Z = 0$, $L_F F = 0$ and $L_F g = (2\nabla_i \nabla_j f)$, we get

$$\begin{aligned} G^*(t) &= \frac{dG(t)}{dt} = L_F G(t) \\ &= 2\nabla_i \nabla_j f Z^i \nabla^j f = -8fG(t) \end{aligned}$$

by (3.2). Since actually $G(0) = 0$, we have $G(t) = 0$ and $Z \in D_2 + D_3$. Next we put

$$Z = Z_2 + Z_3 \tag{3.12}$$

where Z_2 and Z_3 are vector fields along C such that $Z_2 \in D_2$ and $Z_3 \in D_3$. It is easily verified that

$$Z_2 = (\nabla_Z F + (2f + 2a)Z)/4a \ ,$$

from which the differentiability of Z_2 on t follows. Operating L_F to the both sides of the last equation, applying Lemma 3.3, and using (3.8) and $L_F Z = 0$, we obtain

$$L_F Z_2 = 0 \ .$$

By (3.12) Z_3 is also differentiable and hence we get $L_F Z_3 = 0$. Hence, D_2 and D_3 are invariant by φ_t. q.e.d.

Let $\{(E_i)_p, i = 1, \cdots, m\}$ be an orthonormal frame at p such that

$$(E_1)_p \in D_1(p)\,,$$
$$(E_u)_p \in D_2(p)\,, \quad u = 2, \cdots, r\,,$$
$$(E_v)_p \in D_3(p)\,, \quad v = r+1, \cdots, m\,.$$

We define a field of frames $\{E_i\}$ along C by $(E_i)_{\varphi_t p} = \varphi_t(E_i)_p$. Then $\{E_u\}$ is a field of basis of D_2 and $\{E_v\}$ is a field of basis of D_3.

LEMMA 3.5. *With respect to the length of E_i, we have*

$$G_2(t) = g_{\varphi_t p}(E_u, E_u) = 2e^{4at}/(e^{4at} + e^{-4at})\,, \quad E_u \in D_2\,,$$
$$G_3(t) = g_{\varphi_t p}(E_v, E_v) = 2e^{-4at}/(e^{4at} + e^{-4at})\,, \quad E_v \in D_3\,.$$

In particular, $G_2(t) \to 2$ (as $t \to \infty$) and $G_2(t) \to 0$ (as $t \to -\infty$), while $G_3(t) \to 0$ (as $t \to \infty$) and $G_3(t) \to 2$ (as $t \to -\infty$).

PROOF. Let $E_u \in D_2$. Then

$$\begin{aligned} G_2^*(t) = L_F G_2(t) &= 2\nabla_j \nabla_i f E_u^j E_u^i \\ &= 2(2a - 2f)G_2(t)\,. \end{aligned}$$

Consequently

$$\log G_2(t) = \int 4(a - f)dt + L$$

for some constant L. By (3.9), we get

$$\int 4(a - f)dt = 4at - \log(e^{4at} + e^{-4at})\,,$$

and hence,

$$G_2(t) = e^L e^{4at}/(e^{4at} + e^{-4at})\,.$$

Since $G_2(0) = 1$, we get $e^L = 2$. $G_3(t)$ is similarly obtained. q.e.d.

Similarly we have $g_{\varphi_t p}(E_i, E_j) = 0$ for $i \neq j$. Thus, $\{E_i\}$ is a field of orthogonal frames.

PROOF OF THEOREM A. Now we can apply Obata's method. Since the field of orthogonal frames $\{E_i\}$ is invariant by φ_t, its dual $\{w^i\}$ is also invariant by φ_t. We put

$$*E_i = E_i/|E_i|\,, \quad *w^i = |E_i|w^i\,.$$

Since the projective curvature tensor field P is invariant by φ_t (cf. [viii] of §2), $P(E_i, E_j, E_k, w^l)$ is constant on C. By Lemma 3.1 and $|E_1|(0) = 1$, we obtain

$$|E_1|^2 = f^*/4a^2 = 4/(e^{8at} + e^{-8at} + 2)\,. \tag{3.13}$$

Since the length of C is $\pi/2$ ([ix] of §2), $P(*E_i, *E_j, *E_k, *w^l)$ is bounded on C. On the other hand,

$$P(*E_i, *E_j, *E_k, *w^l) = \frac{|E_l|}{|E_i||E_j||E_k|}P(E_i, E_j, E_k, w^l)$$

holds on C. If $P(E_i, E_j, E_k, w^l) \neq 0$ for some (i, j, k, l), then by Lemma 3.5 and (3.13) the right hand side can not be bounded when $t \to \infty$ or $t \to -\infty$. This is a contradiction. Thus, $P = 0$ and (M, g) is of constant curvature K_1. Since $(\nabla^i f)$ belongs to the 1-nullity distribution, $K_1 = 1$.

q.e.d.

4. **Remarks.** (1) As M. Obata explained in [12], the standard model of f satisfying (1.2) on $S^m(1)$ is

$$f = \sum a_\alpha (y^\alpha)^2 + b \,, \tag{4.1}$$

where a_α, b are constant ($a_0 \geq a_1 \geqq \cdots \geq a_m$), and ($y^\alpha$; $\alpha = 0, 1, \cdots, m$) is the standard coordinate system of an $(m+1)$-dimensional Euclidean space E^{m+1}. In this case, at a critical point ($y^0 = 1$), $(\nabla_i \nabla^h f)$ has eigenvalues $2(a_i - a_0)$.

PROPOSITION 4.1. *A function f of the form* (4.1) *satisfies* (1.4) *for $K = 1$, if and only if*

$$\begin{aligned} f = a(y^0)^2 + \cdots + a(y^r)^2 \\ + (a+h)(y^{r+1})^2 + \cdots + (a+h)(y^m)^2 - a - h/2 \,, \end{aligned} \tag{4.2}$$

where a and h are constant and $1 \leq r \leq m-1$.

PROOF. Let (U, x^i) be a local coordinate neighborhood such that $x^i = y^i (i = 1, \cdots, m)$, $y^0 = [1 - \Delta]^{1/2} (\Delta = \sum (x^i)^2)$, $y^0 > 0$. Then we have

$$\begin{aligned} g_{ij} &= \delta_{ij} + x^i x^j/(1 - \Delta) \,, \\ g^{jk} &= \delta^{jk} - x^j x^k \,, \\ \Gamma^i_{jk} &= x^i \delta_{jk} + x^i x^j x^k/(1 - \Delta) \,. \end{aligned}$$

Since $f = \sum (a_i - a_0)(x^i)^2 + a_0 + b$, calculating (1.4) we get $(a_i - a_0)^2 x^i = -2(a_0 + b)(a_i - a_0)x^i$, and hence (4.2). The converse is now clear.

q.e.d.

Hence, our Theorem A is modeled on functions (4.2).

For a function f of the form (4.1), it is also verified that if each trajectory of grad f is a geodesic, then it is of the form (4.2) up to an additive constant.

(2) Next we explain about Sasakian structures, for simplicity, on a 3-dimensional sphere $S^3(1)$. We identify $S^3(1)$ with the space of unit quaternions Q_0. Let $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ be units such that $\boldsymbol{i}^2 = -1$, $\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}$,

etc. For a point $q \in Q_0$, we define two curves

$$a(t) = \cos tq + \sin tqi ,$$
$$b(t) = \cos tq + \sin tiq .$$

We define ξ_1 by $(\xi_1)_q = (da(t)/dt)_{t=0}$ and η_1 by $(\eta_1)_q = (db(t)/dt)_{t=0}$. Then ξ_1 and η_1 are Sasakian structures. Similarly for j and k, we have ξ_2, η_2, ξ_3, η_3. (ξ_1, ξ_2, ξ_3) defines a Sasakian 3-structure. By our definition, the actions of $\exp t\xi_1$ and $\exp s\eta_1$ are commutative. Hence $[\xi_1, \eta_1] = 0$. Similarly $[\xi_\alpha, \eta_\beta] = 0$, $\alpha, \beta = 1, \cdots, 3$.

$\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$ form a basis of the Lie algebra of Killing vector fields on $S^3(1) = Q_0$.

$\xi_1, \eta_1, \eta_2, \eta_3$ form a basis of the Lie algebra of infinitesimal automorphisms of the Sasakian structure ξ_1 on $S^3(1)$.

$g(\xi_1, \eta_1)$ is not constant and satisfies (1.2) and (1.4) (cf. Lemma 5.3).

(3) If (M, g) is complete and real analytic, a paper by A. S. Sodovnikov [16] is remarkable.

5. Applications to Sasakian structures.

THEOREM 5.1. *Let (M, g) be a 3-dimensional Riemannian manifold. If ξ and η are Sasakian structures on (M, g) with $\xi \neq \eta$ and $\xi \neq -\eta$, then (M, g) is of constant curvature* 1.

PROOF. Except for a set of measure zero, we have a 2-plane field defined by ξ and η. Therefore, 1-nullity index is equal to 2 or greater than 2, and hence (M, g) is of constant curvature 1. q.e.d.

LEMMA 5.2. *Let ξ and η be two Sasakian structures on (M, g). If we put $\nabla\xi = -\phi$ and $\nabla\eta = -\psi$, then $f = g(\xi, \eta)$ satisfies*

$$\nabla^j f \nabla_j \nabla^i f = -3f\nabla^i f - \psi^i_j \phi^j_r \psi^r_s \xi^s - \phi^i_j \psi^j_r \phi^r_s \eta^s . \tag{5.1}$$

PROOF. First we have

$$\nabla_i f = \nabla_i(\xi^r \eta_r) = \psi_{ir}\xi^r + \phi_{ir}\eta^r , \tag{5.2}$$

where $\psi_{ir} = g_{is}\psi^s_r$, etc. Then

$$\begin{aligned}\nabla_j \nabla_i f &= -\nabla_j \psi^r_i \xi_r - \psi_{ir}\phi^r_j - \phi_{ir}\psi^r_j - \eta^r \nabla_j \phi_{ri} \\ &= -2fg_{ij} + \eta_i \xi_j + \xi_i \eta_j - \psi_{ir}\phi^r_j - \phi_{ir}\psi^r_j .\end{aligned}$$

By these two relations we get

$$\nabla^j f \nabla_j \nabla_i f = -3f(\psi_{ir}\xi^r + \phi_{ir}\eta^r) - \psi_{ir}\phi^r_j \psi^j_s \xi^s - \phi_{ir}\psi^r_j \phi^j_s \eta^s ,$$

from which we have (5.1).

LEMMA 5.3. *In Lemma 5.2, if $[\xi, \eta] = 0$, then*

$$(5.3) \qquad \nabla^j f \nabla_j \nabla^i f = -4f\nabla^i f .$$

PROOF. $[\xi, \eta] = \nabla_\xi \eta - \nabla_\eta \xi = 0$ implies $\phi^i_r \eta^r = \psi^i_s \xi^s$. Then, by (5.1) and (5.2), we obtain (5.3). q.e.d.

THEOREM 5.4. *Let (M, g) be a complete Riemannian manifold. If (M, g) admits two Sasakian structures ξ and η with non-constant $g(\xi, \eta)$ and satisfying $[\xi, \eta] = 0$, then (M, g) is of constant curvature* 1.

PROOF. If $[\xi, \eta] = 0$, by Lemma 5.3 we have (1.4) for $K = 1$. By [ii] of §2, $f = g(\xi, \eta)$ satisfies (1.2) for $K = 1$. Therefore, Theorem A shows that (M, g) is of constant curvature 1.

THEOREM 5.5. *Let (M, g) be a complete Riemannian manifold of dimension $2n + 1 = m$; $5 \leq m \leq 11$ (or more generally, $m = 5, 7, 9, 11, 17, 19, 25, 27, 33, \cdots$). If (M, g) admits two Sasakian structures ξ and η with non-constant $g(\xi, \eta)$, then (M, g) is of constant curvature* 1.

PROOF. If $[\xi, \eta] = 0$, Theorem 5.5 follows from Theorem 5.4. Now we assume that $[\xi, \eta] \neq 0$. First we show that four vector fields $\xi, \eta, [\xi, \eta], F = \operatorname{grad} f$ are linearly independent almost everywhere. Since ξ is a Killing vector field, we get

$$\begin{aligned} L_\xi f &= L_\xi(g(\xi, \eta)) = g(\xi, [\xi, \eta]) \\ &= g(\xi, \nabla_\xi \eta - \nabla_\eta \xi) \\ &= g(\xi, -\psi\xi + \phi\eta) = 0 . \end{aligned}$$

Similarly we have $L_\eta f = 0$. Denoting by d the exterior differentiation we get $L_\xi df = dL_\xi f = 0$ and $L_\eta df = 0$. These relations show

$$\begin{aligned} [\xi, F] &= L_\xi F = 0 , \\ [\eta, F] &= L_\eta F = 0 . \end{aligned}$$

At the same time, we obtain

$$0 = L_\xi f = df(\xi) = g(\xi, F) .$$

Similarly, $g(\eta, F) = 0$. Then we get

$$\begin{aligned} 0 &= L_\xi(g(\eta, F)) = g(L_\xi \eta, F) = g([\xi, \eta], F) , \\ 0 &= L_\xi(g(\eta, \eta)) = 2g([\xi, \eta], \eta) , \end{aligned}$$

and $g([\xi, \eta], \xi) = 0$. F vanishes only on the set of measure zero. Since different two Killing vector fields can not be identical on any open set, ξ, η, F, $[\xi, \eta]$ are linearly independent almost everywhere by $\xi \perp F$, $\eta \perp F$, $[\xi, \eta] \perp \xi$, $[\xi, \eta] \perp \eta$, $[\xi, \eta] \perp F$. Assume that ξ, η, F, $[\xi, \eta]$ are linearly independent at each point of an open set U. ξ, η, F belong to the 1-nullity distribution D on U. Since D is integrable on U, we have $[\xi, \eta] \in D$. Therefore the index of 1-nullity $\mu(x) \geq 4$ on U, and consequently $\mu \geq 4$ on M. Then we can apply [vii] of §2, and we get Theorem 5.5. q.e.d.

COROLLARY 5.6. *Let (M, g) be a complete Riemannian manifold of dimension m; $3 \leq m \leq 11$. If it admits non-proportional two Sasakian structures, then it is an Einstein space.*

PROOF. This follows from Theorem 5.5 and [iii] of §2. q.e.d.

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ADDED IN PROOF: Theorem (*) in the introduction has been completely proved by the present author (: Differential equations of order 3 on Riemannian manifolds [to appear]), and consequently Theorem (**) is also verified.