Tôhoku Math. Journ.
30 (1978), 455-461.

ALGEBRAIC VARIETIES BIHOLOMORPHIC TO $C^* \times C^*$

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(Received April 11, 1977, revised June 20, 1977)

All algebraic varieties etc. considered in this paper are over the field C of complex numbers.

1. Introduction. On the complex manifold $C^* \times C^*$, one can put the structure of an algebraic variety which is not rational: if $T = C \bmod (Z + Z\tau)$ is any complex 1-torus, then the map $z \mapsto (\exp z, \exp(z/\tau))$ is an embedding of the additive group C in $C^* \times C^*$ such that the quotient group is isomorphic to T, and by GAGA the algebraic structure of T induces one on $C^* \times C^*$ ([5], p. 108, Remark). Let us call such an algebraic structure on $C^* \times C^*$ a *Serre structure*.

Clearly a Serre structure is not rational. This paper is motivated by the conjecture that the converse is also true:

CONJECTURE 1.1.* Every non-rational structure of an algebraic variety on $C^* \times C^*$ is a Serre structure.

Our result in this direction (Theorem 4.1 below) is: if a nonsingular algebraic surface is biholomorphic to $C^* \times C^*$ and is the complement of an *irreducible* curve in a complete nonsingular surface, then it is a Serre variety.

The Serre varieties also have the interesting property that the only regular functions on them are the constants (see e.g. [2], pp. 232-234, for a discussion). We note in particular that they answer a question of Goodman ([1], p. 162) in the negative: being Stein manifolds, they are of pure codimension one in any nontrivial open embedding in a variety, but are clearly not proper over any affine variety. In §3 below, we give easy examples of *affine* nonsingular surfaces which are holomorphically isomorphic but not algebraically, after showing in §2 that, for reduced algebraic curves, holomorphic isomorphism implies algebraic isomorphism.

I thank S. Ramanan and M. V. Nori for helpful discussions. I am

* This conjecture is true; some recent results of T. Ueda of Kyoto University reduce the generel case of (1.1) to the special case dealt with here. See (4.10).

also grateful to Professor T. Oda for pointing out a mistake in an earlier version of the paper, and for other useful comments.

2. Holomorphic maps between algebraic curves.

PROPOSITION 2.1. *Any proper holomorphic map between reduced algebraic curves is algebraic.*

COROLLARY 2.2. *Holomorphically isomorphic reduced algebraic curves are algebraically isomorphic.*

2.3 PROOF OF 2.1. Let $f\colon X \to Y$ be a proper holomorphic map of reduced algebraic curves. Suppose first that X and Y are nonsingular, and let $\bar{X}$, $\bar{Y}$ be their nonsingular compactifications. It is enough to show that f extends to a holomorphic map $\bar{f}\colon \bar{X} \to \bar{Y}$, for then $\bar{f}$ and hence f is algebraic. Now let $W_1, W_2, \cdots,$ be disjoint disc-like neighbourhoods of the finitely many points of $\bar{Y} - Y$, and $W = \bigcup W_i$. Then the properness of f implies that $K = f^{-1}(Y - W) \subset X$ is compact. Now let U be a connected neighbourhood of any $p \in \bar{X} - X$ with $U \cap K = \varnothing$. Then $U - p$ is also connected, and $f(U - p) \subset W$. Hence $f(U - p) \subset W_i$ for some i. Hence f is holomorphic at p by Riemann's theorem.

The general case can be reduced to the above case by using normalizations and the fact that a holomorphic map between reduced algebraic varieties is algebraic iff its graph is algebraic ([4], Proposition 8).

REMARK 2.4 (M. V. Nori). Corollary 2.2, hence Proposition 2.1, is false for nonreduced noncomplete curves. For example, let X be the complement of one point in a generic complete nonsingular curve of genus $\geqq 3$ and L the total space of a nontrivial algebraic line bundle on X. Let Y be the first infinitesimal neighbourhood in L of the zero-section of L. Then, since X has no nontrivial automorphisms, it is easy to see that Y is not algebraically isomorphic to $X \times \operatorname{Spec} \boldsymbol{C}[T]/(T^2)$. However, since all line boundles on an open Riemann surface are holomorphically trivial, Y is holomorphically isomorphic to $X \times \operatorname{Spec} \boldsymbol{C}[T]/(T^2)$.

3. Surfaces. Let X be a nonsingular affine curve. By the genus of X, we shall mean the genus of its nonsingular compactification. X is of genus zero iff all algebraic line bundles on it are algebraically trivial.

PROPOSITION 3.1. *Let X be a nonsingular affine curve, and L the total space of an algebraic line bundle $\mathscr{L}$ over X. Then $\mathscr{L}$ is algebraically trivial if and only if L is isomorphic as a variety to $X \times \boldsymbol{C}$.*

PROOF. We may assume that the genus of X is at least one. Let

$f: L \to X \times \boldsymbol{C}$ be an isomorphism of varieties, and g the composite of f with the natural projection $X \times \boldsymbol{C} \to X$. If $p: L \to X$ is the natural bundle projection, then g is constant on the fibres of p by Lüroth's theorem, hence induces a map $h: X \to X$ such that $g = h \cdot p$. Clearly h is an isomorphism. By changing f suitably, we may thus assume that h is the identity. This means that $\mathscr{L}$ becomes trivial, when its structure group is extended to $\mathrm{Aut}(\boldsymbol{C})$. But then $\mathscr{L}$ itself must be trivial, since $\mathrm{Aut}(\boldsymbol{C})$ is a semi-direct product of $\boldsymbol{C}^*$ and the normal subgroup $\boldsymbol{C}$. q.e.d.

REMARK 3.2. The above proposition is valid for complete curves X as well. Indeed, the above proof applies if the genus of X is at least one. Here is a proof (due to M. V. Nori) applicable to all complete X: any isomorphism of L with $X \times \boldsymbol{C}$ must carry the zero-section of L to $X \times$ (a point), hence $\mathscr{L}$, which is isomorphic to the normal bundle of the zero-section of L, must be trivial.

A similar argument proves the following proposition which we shall need later.

PROPOSITION 3.3. *A line bundle $\mathscr{L}$ on a compact Riemann surface X is topologically trivial if (and only if) the total space of $\mathscr{L}$ is homeomorphic to $X \times \boldsymbol{C}$.*

PROOF. If the total space L of $\mathscr{L}$ is homeomorphic to $X \times \boldsymbol{C}$, then the intersection form on $H_2(L, \boldsymbol{Z})$ vanishes identically, so that the self-intersection of the zero-section of L must be zero. q.e.d.

PROPOSITION 3.4. *Let X be a nonsingular affine curve of genus at least one, and L the total space of a nontrivial algebraic line bundle $\mathscr{L}$ over X. Then L is a nonsingular affine surface which is holomorphically but not algebraically isomorphic to $X \times \boldsymbol{C}$.*

PROOF. As is well known, L is affine (since $\mathscr{L}$ is a quotient of a trivial bundle). Also, $\mathscr{L}$ is holomorphically trivial, since X is an open Riemann surface; hence L is biholomorphic with $X \times \boldsymbol{C}$. Finally X is not algebraically isomorphic with $X \times \boldsymbol{C}$, on account of 3.1. q.e.d.

4. The main result.

THEOREM 4.1. *Let X be a complete nonsingular algebraic surface, and $C \subset X$ an irreducible curve such that $V = X - C$ is biholomorphic with $\boldsymbol{C}^* \times \boldsymbol{C}^*$. Then:*

(i) $q(X) = \dim H^1(X, \mathscr{O}) = 1$;

(ii) *C is nonsingular and of genus one;*

(iii) *the Albanese map $\alpha: X \to T$ maps C isomorphically onto T, and makes X a P^1-bundle over T;*

(iv) $\alpha: V = X - C \to T$ *is the non-trivial principal C-bundle over* T.

PROOF. Consider the exact cohomology sequence

$$(*) \qquad \cdots \to H_c^i(V) \to H^i(X) \to H^i(C) \to H_c^{i+1}(V) \to \cdots$$

(with coefficients $\boldsymbol{C}$ say), H_c^* being cohomology with compact support. We know that $b_3(X)$ has to be even, while $b_2(C) = 1$ ($b_i(\cdot)$ denotes the i-th Betti number) and $\dim H_c^3(V) = 2$. Hence we see from $(*)$ that $H_c^3(V) \xrightarrow{\sim} H^3(X)$; in particular, $b_3(X) = b_1(X) = 2$. Hence $q(X) = 1$, and (i) is proved.

4.2. PROOF OF (ii). Since $b_1(X) = 2$, $H_c^2(V) = \boldsymbol{C}$, and $H_c^1(V) = 0$, $(*)$ implies that $2 \leqq b_1(C) \leqq 3$. Let $\tilde{C}$ be the normalization of C. Then $b_1(\tilde{C}) = 0$ or 2, since it must be even and $\leqq 3$.

We claim that $b_1(\tilde{C}) = 2$. First note that, if $\alpha: X \to T$ is the Albanese map, then T is a 1-torus ($q(X) = 1$), and α is surjective. Now if $b_1(\tilde{C}) = 0$, α is constant (Lüroth), so that $X - \alpha^{-1}(\alpha(C))$ is a nonempty open subset of V. But this is impossible, since by assumption V contains no compact analytic sets.

Thus $b_1(\tilde{C}) = 2$, and $\alpha(C) = T$ as shown above. We now assert that $b_1(C) = 2$. Indeed suppose $b_1(C) = 3$. Then the exact sequence $(*)$ implies that $H^2(X) \to H^2(C)$ is an isomorphism, i.e., $b_2(X) = 1$. But we know that $b_2(X) \geqq 2$, since C and a fibre F of α define independent elements of $H_2(X, \boldsymbol{Z})$ ($(F \cdot F) = 0$ and $(F \cdot C) \neq 0$, where $(\cdot)$ denotes the intersection number).

Hence $b_1(C) = 2 = b_1(\tilde{C})$. It follows easily that the normalization map $p: \tilde{C} \to C$ is a homeomorphism. But then C must be nonsingular, since the composite map $\alpha \cdot p: \tilde{C} \to T$, being a nonconstant map of smooth curves of genus one, is smooth. This proves (ii).

4.3. THE FIBRES OF α ARE CONNECTED. This is a known property of the Albanese map when its image is a curve ([6], Ch. IV, §2, Theorem 4). In our case, the argument simplifies slightly. Namely, let $X \to \tilde{T} \to T$ be the Stein factorisation of α. Then $\tilde{T}$ is a *normal* curve, hence smooth. Now we know that the map $C \to \tilde{T}$ is nonconstant, hence $\tilde{T}$ is also of genus one. By the universal property of the Albanese, $\tilde{T} \to T$ is therefore an isomorphism. q.e.d.

4.4. α IS SMOOTH. Let E be the set of points at which α is not smooth. Then E has no one-dimensional components: such a component would have to meet C (since $X - C$ contains no compact analytic sets), whereas $\alpha | C$ is smooth.

Thus E is a finite set. Hence every fibre of α is reduced, i.e., is of the form $\sum D_i$, with the D_i distinct reduced irreducible curves in X.

Now let F be a generic (nonsingular) fibre of α and $F_0 = \sum D_i$ a (possibly) singular fibre. Then:

4.5. $\chi(F_0) \geqq \chi(F)$. (χ denotes the topological Euler-Poincaré characteristic).

PROOF. Since F_0 is connected, we have ([6], Ch., IV, Lemma 4)

$$\chi(F_0) \geqq -(F_0 \cdot (F_0 + K)) ,$$

where K denotes the canonical divisor of X. But $(F_0 \cdot F_0) = 0$, and $(F_0 \cdot K) = (F \cdot K)$, hence

$$\chi(F_0) \geqq -(F \cdot K) = \chi(F) ,$$

where the last equality again follows from the lemma of [6] quoted above, since F is nonsingular and $(F \cdot F) = 0$.

4.6. It follows from (*) of 4.1 that $b_2(X) = 2$, so that $\chi(X) = 0$. Now, if the F_i are all the possible (finitely many) singular fibres of α, we have ([6], Ch. IV, Theorem 6)

$$\chi(X) = \chi(F) \cdot \chi(T) + \sum_i (\chi(F_i) - \chi(F)) .$$

Since $\chi(T) = \chi(X) = 0$, it follows by 4.5 that $\chi(F_i) = \chi(F)$ for all i. Hence $\chi(F_i) = -(F_i \cdot K) = -(F_i \cdot (F_i + K))$ for all i, so that the F_i must be nonsingular irreducible curves ([6], Ch. IV, Lemma 4). Thus we have proved that α is smooth.

4.7. PROOF OF (iii). Since α is smooth, $X \to T$ is differentiably locally trivial with fibre F a compact connected oriented surface. Also $\alpha | C$ is smooth; let n be the degree of $\alpha: C \to T$. Then it is easy to see that $\alpha: V = X - C \to T$ is also locally differentiably trivial, with fibre $F' = F - n$ points. But now, since $\pi_2(T) = 0$, and $\pi_1(V) = \boldsymbol{Z}^2 = \pi_1(T)$, it follows from the homotopy sequence of the fibration $V \to T$ that F' is simply connected. But then F must also be simply connected, i.e., $X \to T$ is a P^1-bundle, and n must be 1, i.e., $C \to T$ is an isomorphism. q.e.d.

Thus, to conclude the proof of theorem 4.1, it is enough to prove

4.8. Let $\alpha: X \to T$ be a P^1-bundle over the 1-torus T, and suppose $C \subset X$ gives a section of α such that $V = X - C$ is biholomorphic to $\boldsymbol{C}^* \times \boldsymbol{C}^*$. Then V is the nontrivial principal $\boldsymbol{C}$-bundle over T.

PROOF. $V \to T$ is *a priori* an Aut($\boldsymbol{C}$)-bundle. There is a vector bundle W of rank two over T, and an exact sequence.

$$(**) \qquad 0 \to 1 \to W \to L \to 0$$

such that $X = P(W)$, 1 is the trivial line bundle on T, C is the section of $P(W) = X$ defined by $1 \hookrightarrow W$, and L is a line bundle on T which can be identified as follows. Regard V as an element of $H^1(T, \mathrm{Aut}(\boldsymbol{C}))$ and let p_* be the map $H^1(T, \mathrm{Aut}(\boldsymbol{C})) \to H^1(T, \mathcal{O}^*)$ induced by the exact sequence $0 \to \boldsymbol{C} \to \mathrm{Aut}(\boldsymbol{C}) \to \boldsymbol{C}^* \to 1$. Then $L = p_*(V)$.

We now claim that $\deg(L) = 0$. Indeed, using a topological splitting of $(**)$, it is easy to see that the total space of L is *homeomorphic* to V, which by assumption is homeomorphic to $T \times \boldsymbol{C}$. Hence $\deg(L) = 0$ by 3.3.

Now the obstruction to the splitting of $(**)$ lies in $H^1(T, L^*) \approx H^0(T, L)$ (since T is of genus one). Thus if $L \neq 1$, $(**)$ would split, i.e. α would have another section disjoint from C, contradicting the assumption that V contains no compact analytic sets. Hence $L = p_*(V) = 1$. But this means precisely (cf. the definition of p_*) that V comes from $H^1(X, \mathcal{O})$, i.e., is a principal $\boldsymbol{C}$-bundle. Clearly this bundle cannot be trivial since V contains no compact analytic sets. This proves 4.8.

REMARK 4.9. We have only used the fact that V is homeomorphic to $\boldsymbol{C}^* \times \boldsymbol{C}^*$ and contains no compact analytic sets. Also X need only be assumed Kähler. Thus Theorem 4.1 is valid under these weaker assumptions.

4.10. The referee has kindly informed me that T. Ueda of Kyoto University has proved the following theorem:

Let S be a possibly non-Kähler compactification of $\boldsymbol{C}^* \times \boldsymbol{C}^*$ such that $C = S - \boldsymbol{C}^* \times \boldsymbol{C}^*$ is *minimal*, i.e., contains no exceptional curves of the first kind. Then there are only three possibilities:

(i) S is rational, and the irreducible components of C are all rational; the graph of C contains a loop;

(ii) S is a primary Hopf surface and C is irreducible nonsingular elliptic;

(iii) S is a ruled surface, and C is irreducible nonsingular elliptic.

(It is well-known that each of these cases actually occurs.)

Clearly, Ueda's theorem together with our main theorem implies that Conjecture 1.1 is true. We note finally that any (algebraic) isomorphism of Serre surfaces clearly induces one of the associated elliptic curves, so that the isomorphism classes of non-rational algebraic structures on $\boldsymbol{C}^* \times \boldsymbol{C}^*$ are in bijection with isomorphism classes of nonsingular elliptic curves.

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