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THE PRODUCT OF OPERATORS WITH CLOSED RANGE AND AN EXTENSION OF THE REVERSE ORDER LAW

SAICHI IZUMINO

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1. Introduction. Let A be a (bounded linear) operator on a Hilbert space H. If A has closed range, then there is a unique operator $A^\dagger$ called the Moore-Penrose inverse or generalized inverse of A, which satisfies the following four identities [2, p. 321]:

$$AA^\dagger A = A\,, \qquad A^\dagger AA^\dagger = A^\dagger\,, \qquad (A^\dagger A)^* = A^\dagger A \qquad \text{and}$$
$$(AA^\dagger)^* = AA^\dagger\,.$$

We denote by (CR) the set of all operators on H with closed range (or equivalently, that of all operators with Moore-Penrose inverses). For two operators A and B in (CR), one problem is to find the condition under which the product AB is in (CR). Bouldin [3] [5] gave a geometric characterization of the condition in terms of the angle between two linear subspaces, and recently Nikaido [16] showed a topological characterization of it (for Banach space operators). Another problem is to represent the Moore-Penrose inverse $(AB)^\dagger$ in a reasonable form, that is, to generalize the reverse order law $(AB)^{-1} = B^{-1}A^{-1}$ for invertible operators. Many authors [1], [4], [6], [9], [10], [18]-[20], etc. (some of them in the setting of matrices) studied this problem. Barwick and Gilbert [1], Bouldin [4] [6], Galperin and Waksman [9], etc. proved some necessary and sufficient conditions which guarantee the "generalized" reverse order law $(AB)^\dagger = B^\dagger A^\dagger$ holds.

In this paper we shall treat the product of two operators with closed range. In Section 2 we shall show some norm inequalities for the product to have closed range, which enable us to refine the results in [3] and [16]. In Section 3, using our result in [12], we shall present an extension of the (generalized) reverse order law, and extend some main results in [1], [4], [6] and [9].

Throughout this paper all operators are bounded linear. A projection is a selfadjoint idempotent operator, and it is an orthogonal projection onto a closed linear subspace of H. For projections P and Q onto the closed linear subspaces M and N, we write, in lattice theoretic notations, $P^\perp$, $P \vee Q$ and $P \wedge Q$ the projections onto the orthocomplement $M^\perp$ of

M, the norm closure $(M+N)^-$ of $M+N$ and the intersection $M\cap N$, respectively. For an operator A we shall denote by $\ker A$ and $\operatorname{ran} A$ the kernel and the range, respectively. The lower bound $\gamma(A)$ of A $(A\neq 0)$ is defined by

$$\gamma(A)=\inf\{\|Ax\|: x\in(\ker A)^{\perp},\ \|x\|=1\}\ .$$

It is well-known [2, p. 311] that $A\in(CR)$ if and only if $\gamma(A)>0$, and in this case [2, p. 325]

$$\gamma(A)=\|A^{\dagger}\|^{-1}\ . \tag{1.1}$$

If $A\in(CR)$, then $A^*\in(CR)$ and $A^{*(\dagger)}=A^{\dagger(*)}$ [2, p. 320] ($A^{\alpha(\beta)}$ means $(A^{\alpha})^{\beta}$). Moreover, $A^{\dagger}A$ $(=A^*A^{*(\dagger)})$ and $AA^{\dagger}$ $(=A^{*(\dagger)}A^*)$ are the projections onto $(\ker A)^{\perp}$ $(=\operatorname{ran} A^*)$ and $\operatorname{ran} A$ $(=(\ker A^*)^{\perp})$, respectively. For further basic properties of Moore-Penrose inverses we refer to [2], [11] or [15].

We would like to express our thanks to the referee for his kind advice.

2. The closedness of range of the product operator. An operator $A\in(CR)$ is easily characterized as an operator satisfying $AXA=A$ for some operator X (cf. $AA^{\dagger}A=A$ for such an A), i.e., a relatively regular element of the operator algebra on H. Hence by [17, Result 3.1] (cf. [14, Theorem 1]) on relative regularity we, at once, have the following proposition, which shows that the problem on the closedness of $\operatorname{ran} AB$ is reduced to that of $\operatorname{ran} A^{\dagger}ABB^{\dagger}$, the range of the product of two *projections*.

PROPOSITION 2.1. *Let $A, B\in(CR)$. Then $AB\in(CR)$ if and only if $A^{\dagger}ABB^{\dagger}\in(CR)$.*

The following result shows a norm characterization of the closedness of $\operatorname{ran} PQ$ for two projections P and Q.

PROPOSITION 2.2. *Let P and Q be projections. If $PQ\neq 0$, then*

$$\gamma(PQ)^2+\|P^{\perp}Q(P\vee Q^{\perp})\|^2=1\ . \tag{2.1}$$

Hence (even if $PQ=0$) $PQ\in(CR)$ if and only if

$$\|P^{\perp}Q(P\vee Q^{\perp})\|<1\ . \tag{2.2}$$

PROOF. Since $\operatorname{ran} QP\subset\operatorname{ran} Q(Q^{\perp}\vee P)\subset(\operatorname{ran} QP)^-=(\ker PQ)^{\perp}$, we have

$$(\ker PQ)^{\perp}=\operatorname{ran} Q(Q^{\perp}\vee P)\ . \tag{2.3}$$

Let $x\in(\ker PQ)^{\perp}$ and $\|x\|=1$. Then, since $x=Q(Q^{\perp}\vee P)x=Qx$, we have $\|PQx\|^2 + \|P^\perp Q(P \vee Q^\perp)\|^2 \geqq \|PQx\|^2 + \|P^\perp Qx\|^2 = \|Qx\|^2 = 1$. By definition, the infimum of $\|PQx\|$ is $\gamma(PQ)$. Hence we have $\gamma(PQ)^2 + \|P^\perp Q(P \vee Q^\perp)\|^2 \geqq 1$. To show the converse inequality, note $\gamma(PQ) \leqq \|PQx\|$ $(x \in (\ker PQ)^\perp,\ \|x\| = 1)$. Hence $\gamma(PQ)^2 + \|P^\perp Qx\|^2 \leqq \|PQx\|^2 + \|P^\perp Qx\|^2 = 1$. Since the supremum of $\|P^\perp Qx\|$ is $\|P^\perp Q(P \vee Q^\perp)\|$, we obtain $\gamma(PQ)^2 + \|P^\perp Q(P \vee Q^\perp)\|^2 \leqq 1$. Now, the equivalence $PQ \in (CR) \Leftrightarrow$ (2.2) (between $PQ \in (CR)$ and (2.2)) is clear if $PQ \neq 0$. If $PQ = 0$, then $Q(P \vee Q^\perp) = 0$ (say, by (2.3)), so that (2.2) is clear. q.e.d.

By (1.1) we easily see $\gamma(A) = \gamma(A^*)$ for an operator $A \neq 0$, in perticular, $\gamma(PQ) = \gamma(QP)$ $(PQ \neq 0)$. Hence by (2.1) we have (even if $PQ = 0$)

$$\|P^\perp Q(P \vee Q^\perp)\| = \|Q^\perp P(Q \vee P^\perp)\| . \tag{2.4}$$

Between two closed linear subspaces M and N we define the angle $\alpha(M, N)$ $(0 \leqq \alpha(M, N) \leqq \pi/2)$ as the arccosine of

$$\sup\{|(x, y)| : \|x\| = \|y\| = 1,\ x \in M,\ y \in N\} ,$$

and $\alpha(M, N) = \pi/2$ when either M or N is $\{0\}$.

Suppose $A, B \in (CR)$, and write $P = A^\dagger A$, $Q = BB^\dagger$. Then $P^\perp(P \vee Q^\perp) = P^\perp \wedge (P^\perp \wedge Q)^\perp$ is the projection onto $L := \ker A \cap (\ker A \cap \operatorname{ran} B)^\perp$. If neither L nor $\operatorname{ran} B$ is $\{0\}$, then

$$\begin{aligned}\|P^\perp Q(P \vee Q^\perp)\| &= \|QP^\perp(P \vee Q^\perp)\| \\ &= \sup\{|(P^\perp(P \vee Q^\perp)x, Qy)| : \|x\| = \|y\| = 1\} \\ &= \sup\{|(x, y)| : \|x\| = \|y\| = 1, x \in L, y \in \operatorname{ran} B\} .\end{aligned}$$

Hence, by Propositions 2.1 and 2.2 we have the following result due to Bouldin [3] (cf. [5]).

COROLLARY 2.3 [3, Theorem]. *Let $A, B \in (CR)$. Then $AB \in (CR)$ if and only if $\alpha(\ker A \cap (\ker A \cap \operatorname{ran} B)^\perp, \operatorname{ran} B) > 0$.*

For another characterization of the closedness of $\operatorname{ran} PQ$, we have

PROPOSITION 2.4. *Let P and Q be projections. If $PQ \neq 0$, then*

$$\gamma(PQ) \geqq \gamma(P^\perp + Q) \geqq (1 - \|P^\perp Q(P \vee Q^\perp)\|)^2 . \tag{2.5}$$

Hence (even if $PQ = 0$) $PQ \in (CR)$ if and only if $P^\perp + Q \in (CR)$.

PROOF. Note first that $(\ker QP)^\perp = \operatorname{ran} P(P^\perp \vee Q)$, $(\ker (P^\perp + Q))^\perp = \operatorname{ran}(P^\perp \vee Q)$, and that both the subspaces are not $\{0\}$. Let $x \in (\ker QP)^\perp$ and $\|x\| = 1$. Then $x = P(P^\perp \vee Q)x = Px$ and

$$\|QPx\| = \|(P^\perp + Q)Px\| = \|(P^\perp + Q)x\| \geqq \gamma(P^\perp + Q) .$$

The last inequality follows from the fact $x = (P^\perp \vee Q)x \in (\ker (P^\perp + Q))^\perp$.

Hence we have $\gamma(QP) \geqq \gamma(P^\perp + Q)$. Since $\gamma(PQ) = \gamma(QP)$, we have the left hand side inequality of (2.5). Next, note $((P^\perp + Q)x, x) \geqq ((P^\perp Q^\perp P^\perp + Q)x, x)$ for any $x \in H$ and $P^\perp Q^\perp P^\perp + Q = 1 - (Q^\perp P + PQ^\perp) + PQ^\perp P$. Hence, if $x = (P^\perp \vee Q)x$ and $\|x\| = 1$ then

$$\begin{aligned}\|(P^\perp + Q)x\| \geqq ((P^\perp + Q)x, x) &\geqq 1 - 2\operatorname{Re}(Q^\perp Px, x) + (PQ^\perp Px, x)\\ &\geqq 1 - 2\|Q^\perp Px\| + \|Q^\perp Px\|^2 = (1 - \|Q^\perp Px\|)^2\\ &\geqq (1 - \|Q^\perp P(P^\perp \vee Q)\|)^2 .\end{aligned}$$

Hence $\gamma(P^\perp + Q) \geqq (1 - \|Q^\perp P(Q \vee P^\perp)\|)^2$. By (2.4) this implies the right hand side inequality of (2.5). Now the equivalence $PQ \in (CR) \Leftrightarrow P^\perp + Q \in (CR)$ is clear by (2.5) and (2.2) if $PQ \neq 0$. If $PQ = 0$, then $\operatorname{ran}(P^\perp + Q) = \operatorname{ran} P^\perp(1 + Q) = \operatorname{ran} P^\perp$, so that $P^\perp + Q \in (CR)$. q.e.d.

Before an application we remark that $A \in (CR)$ if and only if $AA^* \in (CR)$. This is seen by the facts $\operatorname{ran} AA^* \subset \operatorname{ran} A \subset (\operatorname{ran} AA^*)^-$, and $\operatorname{ran} A = \operatorname{ran} A\cdot(A^\dagger A)^* = \operatorname{ran} AA^*A^{\dagger(*)} \subset \operatorname{ran} AA^* \subset \operatorname{ran} A$ for $A \in (CR)$.

The equivalence (1) $\Leftrightarrow$ (3) of the following corollary was shown by Nikaido [16, Corollary 1].

COROLLARY 2.5. *Let* $A, B \in (CR)$. *Write* $P = A^\dagger A$ *and* $Q = BB^\dagger$. *Then the following conditions are equivalent.*

(1) $AB \in (CR)$.

(2) $P^\perp + Q \in (CR)$.

(3) $\ker A + \operatorname{ran} B$ *is closed.*

PROOF. (1) $\Leftrightarrow$ (2) By Propositions 2.1 and 2.4.

(2) $\Leftrightarrow$ (3) We employ a technique in [7, Theorem 2.2]. Let $T = \begin{Bmatrix} P^\perp & Q \\ 0 & 0 \end{Bmatrix}$ be a operator matrix on the product Hilbert space $H \oplus H$. Then $\operatorname{ran} T = (\operatorname{ran} P^\perp + \operatorname{ran} Q) \oplus \{0\}$ and $\operatorname{ran} TT^* = \operatorname{ran}(P^\perp + Q) \oplus \{0\}$. Hence by the above remark we have the desired equivalence. q.e.d.

COROLLARY 2.6. *Let P and Q be projections. Then* $\operatorname{ran} P + \operatorname{ran} Q$ *is closed if and only if* $\|PQ(P^\perp \vee Q^\perp)\| < 1$.

PROOF. By Corollary 2.5 and Proposition 2.4. q.e.d.

For a pair of two closed linear subspaces M and N, the gap $g(M, N)$ is defined (cf. [13, p. 219]) by

$$g(M, N) = \inf\{d(x, N)/d(x, M \cap N) : x \in M \setminus N\} ,$$

where $d(x, L)$ is the distance from x to L. We set $g(M, N) = 1$ when $M \subset N$. Let P and Q be the projections onto M and N, respectively. Then by a simple calculation we have $g(M, N) = \gamma(Q^\perp P)$ $(M \not\subset N)$, or by

(2.1) (even if $M \subset N$)

$$g(M, N) = (1 - \| PQ(P^{\perp} \vee Q^{\perp}) \|^2)^{1/2} .$$

Clearly, Corollary 2.6 says that $g(M, N) > 0$ if and only if $M + N$ is closed, which is a well-known result [13, IV, Theorem 4.3] (on a Banach space).

3. The reverse order law.

We state a result which we proved in [12].

LEMMA 3.1 [12, Lemmas 2.1 and 3.2]. *Let $A \in (CR)$, and let R be a projection commuting with $A^{\dagger}A$. Then $AR \in (CR)$, $C := 1 - ARA^{\dagger} + ARA^*$ is invertible and*

$$(AR)(AR)^{\dagger} = C^{-1}ARA^* . \tag{3.1}$$

Using the above lemma we have

LEMMA 3.2. *Let $A, B, AB \in (CR)$. Write $P = A^{\dagger}A$ and $Q = BB^{\dagger}$. Then $C := 1 - A(P^{\perp} \vee Q)A^{\dagger} + A(P^{\perp} \vee Q)A^*$ is invertible, and*

$$(AB)(AB)^{\dagger} = C^{-1}A(P^{\perp} \vee Q)A^* . \tag{3.2}$$

PROOF. Put $R = P^{\perp} \vee Q$. Since $\operatorname{ran} AB = \operatorname{ran} AQ \subset \operatorname{ran} AR \subset (\operatorname{ran} AR)^- = (\operatorname{ran} AQ)^- = \operatorname{ran} AB$, we have $\operatorname{ran} AB = \operatorname{ran} AR$, i.e., $(AB)(AB)^{\dagger} = (AR)(AR)^{\dagger}$. Since R commutes with $P = A^{\dagger}A$, we have, by Lemma 3.1, the required assertions. q.e.d.

COROLLARY 3.3. *Let P and Q be projections. If $PQ \in (CR)$, then*

$$(PQ)(PQ)^{\dagger} = P(P^{\perp} \vee Q) , \qquad (PQ)^{\dagger}(PQ) = Q(Q^{\perp} \vee P) . \tag{3.3}$$

We remark that the second identity of (3.3) can be also obtained from (2.3).

For the Moore-Penrose inverse of $(PQ)^{\dagger}$, we have the following result which is considered as an extention of [10, Theorem 3].

LEMMA 3.4. *Let P and Q be projections. If $PQ \in (CR)$, then $R := 1 - (P \vee Q^{\perp})Q + PQ$ is invertible and*

$$(PQ)^{\dagger} = R^{-1}P(P^{\perp} \vee Q) . \tag{3.4}$$

PROOF. Since $R = 1 - (P \vee Q^{\perp} - P)Q = 1 - (P \vee Q^{\perp})P^{\perp}Q$ and since $\|(P \vee Q^{\perp})P^{\perp}Q\| < 1$ by (2.1), we see that R is invertible. By (3.3) we see $(P \vee Q^{\perp})Q(PQ)^{\dagger} = (PQ)^{\dagger}(PQ)(PQ)^{\dagger} = (PQ)^{\dagger}$. Hence we have

$$R(PQ)^{\dagger} = (PQ)^{\dagger} - (P \vee Q^{\perp})Q(PQ)^{\dagger} + PQ(PQ)^{\dagger} = (PQ)(PQ)^{\dagger} = P(P^{\perp} \vee Q) .$$

This implies the desired identity. q.e.d.

Now we state the main theorem of this section.

THEOREM 3.5. *Let $A, B \in (CR)$. If $AB \in (CR)$, then*

$$\begin{aligned}(AB)^\dagger &= (AB)^\dagger(AB)\cdot B^\dagger\cdot(PQ)^\dagger\cdot A^\dagger\cdot(AB)(AB)^\dagger \\ &= f(B^*, Q^\perp \vee P)\cdot B^\dagger\cdot\{1-(P\vee Q^\perp)Q+PQ\}^{-1} \\ &\qquad \times (P^\perp \vee Q)\cdot A^\dagger\cdot f(A, P^\perp \vee Q)\ ,\end{aligned}$$

where $P = A^\dagger A$, $Q = BB^\dagger$ and $f(S, T) = (1 - STS^\dagger + STS^*)^{-1}STS^*$.

PROOF. Note $PQ \in (CR)$ by Proposition 2.1. The first identity is obtained from the fact:

$$\begin{aligned}(AB)B^\dagger(PQ)^\dagger A^\dagger(AB) &= A(A^\dagger ABB^\dagger)(A^\dagger ABB^\dagger)^\dagger(A^\dagger ABB^\dagger)B \\ &= A(A^\dagger ABB^\dagger)B = AB\ .\end{aligned}$$

The second identity is shown by (3.2), (3.4) and the identity $(AB)^\dagger(AB) = (B^*A^*)(B^*A^*)^\dagger$. q.e.d.

In each of the following two corollaries, $(AB)^\dagger$ is represented by a rational function in A, $A^\dagger$, B, $B^\dagger$ and their adjoints under a certain condition which is satisfied for invertible operators. Hence our theorem is, in a sense, a reasonable extention of the reverse order law.

COROLLARY 3.6. *Let $A, B, AB \in (CR)$. If $P := A^\dagger A$ and $Q := BB^\dagger$ commute, then*

$$(AB)^\dagger = f(B^*, P)B^\dagger A^\dagger f(A, Q) \quad (f \text{ is defined in Theorem } 3.5)\ .$$

PROOF. Since P and Q commute, we see that PQ is a projection. Hence $(PQ)^\dagger = PQ$ $(= QP)$, because $R^\dagger = R$ for a projection R. Since $(AB)(AB)^\dagger = (AQ)(AQ)^\dagger$, and since Q commutes with $A^\dagger A = P$, we have, by (3.1),$(AB)(AB)^\dagger = f(A, Q)$. Similarly we have $(AB)^\dagger(AB) = f(B^*, P)$. Hence by the first identity of Theorem 3.5 we have the desired representation of $(AB)^\dagger$. q.e.d.

We remark that the assumption $AB \in (CR)$ is not needed in Corollary 3.6. For, if P and Q commute then PQ is a projection and $PQ \in (CR)$, so that $AB \in (CR)$ (say, by Proposition 2.1).

COROLLARY 3.7. *Let $A, B, AB \in (CR)$. If $P^\perp \vee Q = P \vee Q^\perp = 1$, i.e., $\ker A$ and $\operatorname{ran} B$ are complementary, then*

$$(AB)^\dagger = B^\dagger(1 - Q + PQ)^{-1}A^\dagger\ .$$

PROOF. By assumption $f(A, P^\perp \vee Q) = f(A, 1) = (1 - AA^\dagger + AA^*)^{-1}AA^*$. Since $(1 - AA^\dagger + AA^*)AA^\dagger = AA^*$ (cf. $A^*AA^\dagger = A^*$), we have $f(A, 1) = AA^\dagger$. Similarly we have $f(B^*, Q^\perp \vee P) = B^\dagger B$. Hence by the second identity of Theorem 3.5 we have the required equation. q.e.d.

The following result was essentially shown in [16, Proposition 1] (for Banach space operators).

COROLLARY 3.8. *Let* $A, B \in (CR)$ *and let* $AB \neq 0$. *Then*

$$\gamma(AB) \geqq \gamma(A)\gamma(B)\gamma(PQ) . \tag{3.5}$$

PROOF. If $AB \in (CR)$, then by Theorem 3.5 $\|(AB)^\dagger\| \leqq \|B^\dagger\|\,\|(PQ)^\dagger\|\,\|A^\dagger\|$. Hence by (1.1) we obtain (3.5). If $AB \notin (CR)$, then $PQ \notin (CR)$. Hence (3.5) is clear. q.e.d.

The next two propositions extend (or refine) Bouldin [4, Theorem 3.1] [6, Theorem 3.3], Barwick and Gilbert [1, Theorems 1 and 2], Shinozaki and Sibuya [18, Propositions 3.2 and 4.3].

First we state a useful lemma for our discussion.

LEMMA 3.9 [8, Theorem 2]. *Let* T *be an idempotent operator with* $\|T\| \leqq 1$. *Then* T *is a projection.*

PROPOSITION 3.10. *Let* $A, B, AB \in (CR)$. *Then the following conditions are equivalent.*

(1) $A^\dagger A$ *commutes with* BB^*.

(2) $(AB)^\dagger(AB) = B^\dagger A^\dagger AB$.

(3) $C := 1 - A^{*(\dagger)}BB^\dagger A^* + ABB^\dagger A^*$ *is invertible, and*

$$(AB)^\dagger = B^\dagger A^* C^{-1} .$$

PROOF. (1) $\Rightarrow$ (2) Since $A^* = A^\dagger AA^*$ (and $B^* = B^\dagger BB^*$), we have

$$\begin{aligned}(AB)^\dagger(AB) &= (AB)^*(AB)^{\dagger(*)} = B^*A^*(AB)^{\dagger(*)} = B^\dagger BB^*\cdot A^\dagger AA^*\cdot(AB)^{\dagger(*)}\\ &= B^\dagger A^\dagger ABB^*A^*(AB)^{\dagger(*)} = B^\dagger A^\dagger(AB)(AB)^*(AB)^{\dagger(*)}\\ &= B^\dagger A^\dagger(AB)(AB)^\dagger(AB) = B^\dagger A^\dagger AB .\end{aligned}$$

(2) $\Rightarrow$ (3) We first show that $P := A^\dagger A$ and $Q := BB^\dagger$ commute. Since $AB = (AB)(AB)^\dagger(AB) = AB\cdot B^\dagger A^\dagger AB$, we have $PQ = A^\dagger ABB = A^\dagger\cdot ABB^\dagger A^\dagger AB\cdot B^\dagger = (PQ)^2$. Besides, clearly $\|PQ\| \leqq 1$. Hence by Lemma 3.9 PQ is a projection, so that P and Q commute. Now by Corollary 3.6 we see $(AB)^\dagger = f(B^*, P)B^\dagger A^\dagger f(A, Q)$. Since $f(B^*, P) = (AB)^\dagger(AB) = B^\dagger A^\dagger AB$, and since $f(A, Q) = f(A, Q)^* = AQA^*C^{-1}$, we have $(AB)^\dagger = B^\dagger A^\dagger ABB^\dagger A^\dagger AQA^*C^{-1} = B^\dagger A^*C^{-1}$.

(3) $\Rightarrow$ (1) Let $(AB)^\dagger = B^\dagger A^*C^{-1}$. Then $(AB)^\dagger C = B^\dagger A^*$ or

$$(AB)^\dagger - (AB)^\dagger A^{\dagger(*)}QA^* + (AB)^\dagger AQA^* = B^\dagger A^* . \tag{3.6}$$

Since $(AB)(AB)^\dagger AQA^* = ABB^\dagger A^*$, multiplying (3.6) by AB from the left, we have $(AB)(AB)^\dagger - (AB)(AB)^\dagger A^{\dagger(*)}QA^* + ABB^\dagger A^* = ABB^\dagger A^*$. Hence

$$(AB)(AB)^\dagger = (AB)(AB)^\dagger A^{\dagger(*)}QA^* \;. \tag{3.7}$$

If we multiply (3.7) by $(AB)^\dagger$ from the left, then we obtain $(AB)^\dagger = (AB)^\dagger A^{\dagger(*)}QA^*$. Hence by (3.6) we see

$$(AB)^\dagger AQA^* = B^\dagger A^* \;. \tag{3.8}$$

Now, if we assume that P and Q commute, then by (3.8)

$$\begin{aligned} PBB^* &= PQBB^* = QPBB^* = BB^\dagger A^\dagger ABB^* = BB^\dagger A^* A^{\dagger(*)}BB^* \\ &= B\cdot(AB)^\dagger AQA^*\cdot A^{\dagger(*)}BB^* = B(AB)^\dagger AQA^\dagger ABB^* \\ &= B(AB)^\dagger(AB)B^* \;. \end{aligned}$$

This shows that PBB^* is selfadjoint. Hence P and BB^* commute, which is the assertion (1). To see that P and Q commute, take the adjoints in (3.7). Then we have $(AB)(AB)^\dagger = AQA^\dagger(AB)(AB)^\dagger$. Multiplying by AB from the right, we have $AB = AQA^\dagger AB$. By this identity we easily see $PQ = (PQ)^2$, so that P and Q commute (cf. Proof of $(2)\Rightarrow(3)$). q.e.d.

Similarly to Proposition 3.10 we have:

PROPOSITION 3.10′. *Let $A, B, AB \in (CR)$. Then the following conditions are equivalent.*

(1) $BB^\dagger$ *commutes with* A^*A.

(2) $(AB)(AB)^\dagger = ABB^\dagger A^\dagger$.

(3) $D := 1 - B^*A^\dagger AB^{*(\dagger)} + B^*A^\dagger AB$ *is invertible, and*

$$(AB)^\dagger = D^{-1}B^*A^\dagger \;.$$

PROOF. Replace, in Proposition 3.10, A and B by B^* and A^* respectively, and take the adjoints. q.e.d.

COROLLARY 3.11 [6, Theorem 3.3]. *Let $A, B, AB \in (CR)$. Then the following conditions are equivalent.*

(1) $A^\dagger A$ *commutes with* BB^* *and* $BB^\dagger$ *commutes with* A^*A.

(2) $(AB)^\dagger(AB) = B^\dagger A^\dagger AB$ *and* $(AB)(AB)^\dagger = ABB^\dagger A^\dagger$.

(3) $(AB)^\dagger = B^\dagger A^\dagger$.

PROOF. The equivalence $(1)\Leftrightarrow(2)$ is clear by Propositions 3.10 and 3.10′. If (2) is assumed, then $A^\dagger A$ and $BB^\dagger$ commute (cf. Proof of Proposition 3.10 $(2)\Rightarrow(3)$). Hence $(AB)^\dagger = (AB)^\dagger(AB)(AB)^\dagger = B^\dagger A^\dagger AB(AB)^\dagger = B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger$, which is the assertion (3). The implication $(3)\Rightarrow(2)$ is clear. q.e.d.

The following proposition is a Hilbert space version of a result due to Galperin and Waksman ([9, Theorem 2]).

PROPOSITION 3.12. *Let $A, B, AB \in (CR)$. Then the following conditions* *are equivalent.*

(1) $\operatorname{ran} B^\dagger A^* = \operatorname{ran} B^* A^*$ *and* $\operatorname{ran} A^{*(\dagger)} B = \operatorname{ran} AB$.

(2) $(AB)^\dagger = B^\dagger (A^\dagger A B B^\dagger)^\dagger A^\dagger$.

PROOF. Note first that B^*A^*, $B^\dagger A^*$, $A^{*(\dagger)}B \in (CR)$, say, by Proposition 2.1. Write $P = A^\dagger A$ and $Q = BB^\dagger$, and let $X = B^\dagger (PQ)^\dagger A^\dagger$. Then clearly $XABX = X$, so that $X \in (CR)$. Next we want to show

$$\operatorname{ran} X = \operatorname{ran} B^\dagger A^* \quad \text{and} \quad \operatorname{ran} X^* = \operatorname{ran} A^{*(\dagger)} B \,. \tag{3.9}$$

Since $(PQ)^\dagger P = (PQ)^\dagger$ by (3.4), and since $\operatorname{ran} B^\dagger (Q^\perp \vee P) = \operatorname{ran} B^\dagger P$ (cf. Proof of Lemma 3.2), we have

$$\begin{aligned} \operatorname{ran} X &= \operatorname{ran} B^\dagger (PQ)^\dagger A^\dagger = \operatorname{ran} B^\dagger (PQ)^\dagger P = \operatorname{ran} B^\dagger (PQ)^\dagger = \operatorname{ran} B^\dagger (PQ)^\dagger (PQ) \\ &= \operatorname{ran} B^\dagger (Q^\perp \vee P) = \operatorname{ran} B^\dagger P = \operatorname{ran} B^\dagger A^* \,. \end{aligned}$$

Similarly we have the other identity of (3.9). Now, if we assume (1), then by (3.9) we obtain

$$\operatorname{ran} X = \operatorname{ran} B^* A^* \quad \text{and} \quad \operatorname{ran} X^* = \operatorname{ran} AB \,, \tag{3.10}$$

or equivalently

$$XX^\dagger = (AB)^\dagger (AB) \quad \text{and} \quad X^\dagger X = (AB)(AB)^\dagger \,. \tag{3.11}$$

Hence $X = XX^\dagger X X^\dagger X = (AB)^\dagger (AB) \cdot X \cdot (AB)(AB)^\dagger = (AB)^\dagger (AB)(AB)^\dagger = (AB)^\dagger$, which is the assertion (2). Conversely, if we assume (2), i.e., $X = (AB)^\dagger$, then clearly (3.11) and hence (3.10) are valid. Hence by (3.9) we have the assertion (1). q.e.d.

We remark that the condition $P^\perp \vee Q = P \vee Q^\perp = 1$ ($P = A^\dagger A$, $Q = BB^\dagger$) taken in Corollary 3.7 implies the assertion (2) (hence also (1)) of the above proposition.

The following result adds to Corollary 3.11 another condition in order that $(AB)^\dagger = B^\dagger A^\dagger$ holds.

COROLLARY 3.13 (cf. [9, Theorem 3]). *Let* $A, B, AB \in (CR)$. *Then* $(AB)^\dagger = B^\dagger A^\dagger$ *if and only if*

(3.12) $A^\dagger A$ *and* $BB^\dagger$ *commute, and* (1) (*or equivalently* (2)) *of Proposition* 3.12 *holds.*

PROOF. If $(AB)^\dagger = B^\dagger A^\dagger$ then $(AB)^\dagger AB = B^\dagger A^\dagger AB$, so that $A^\dagger A$ and $BB^\dagger$ commute. Since $(A^\dagger A B B^\dagger)^\dagger = A^\dagger A B B^\dagger = B B^\dagger A^\dagger A$, we have $B^\dagger (A^\dagger A B B^\dagger)^\dagger A^\dagger = B^\dagger A^\dagger = (AB)^\dagger$, the assertion (2) of Proposition 3.12. Conversely, if (3.12) is assumed then $(AB)^\dagger = B^\dagger (A^\dagger A B B^\dagger)^\dagger A^\dagger = B^\dagger B B^\dagger A^\dagger A A^\dagger = B^\dagger A^\dagger$, as desired. q.e.d.

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FACULTY OF EDUCATION
TOYAMA UNIVERSITY
TOYAMA, 930
JAPAN