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ON A RESULT OF K. MASUDA CONCERNING REACTION-DIFFUSION EQUATIONS

ALAIN HARAUX AND AMAR YOUKANA

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Abstract. We give a simplified proof of a recent result due to K. Masuda concerning the global existence and asymptotic behavior of non-negative solutions to some reaction-diffusion systems. This new method also provides an analogous result under weaker growth assumptions on the nonlinear terms.

Introduction. Let Ω be an open, bounded domain of class C^1 in $\boldsymbol{R}^n$, with boundary $\Gamma = \partial\Omega$. Let d_1, d_2 be two positive constants with $d_1 \neq d_2$ and $\alpha_1(x)$, $\alpha_2(x)$ two nonnegative functions of class $C^1(\Gamma)$ with $\alpha_1 \leqq 1$, $\alpha_2 \leqq 1$. Let $\varphi \in C^1(\boldsymbol{R}^+)$ be a nonnegative function. We consider the reaction-diffusion system

$$(1) \qquad \begin{cases} \partial u/\partial t - d_1\Delta u + u\varphi(v) = 0 & \text{on} \quad \boldsymbol{R}^+ \times \Omega \\ \partial v/\partial t - d_2\Delta v - u\varphi(v) = 0 & \text{on} \quad \boldsymbol{R}^+ \times \Omega \end{cases}$$

with the homogeneous boundary conditions

$$(2) \qquad \begin{cases} \alpha_1 \partial u/\partial n + (1-\alpha_1)u = 0 & \text{on} \quad \boldsymbol{R}^+ \times \Gamma \\ \alpha_2 \partial v/\partial n + (1-\alpha_2)v = 0 & \text{on} \quad \boldsymbol{R}^+ \times \Gamma \end{cases}$$

A basic question, initially raised by Martin in [5], is the existence of global solutions in $C(\bar{\Omega})$ for the initial-value problem associated to the system (1)-(2). This question has been studied successively by Alikakos [1] who gave a positive answer when $\varphi(v) \leqq C(1 + |v|^{n+2/n})$, and by Masuda [6] who solved the question when $\varphi(v) \leqq C(1 + |v|^{\beta})$ with $\beta > 0$ arbitrarily large without any restriction on n.

In this paper, we show that the method of K. Masuda can be generalized to handle any non-linearity $\varphi(v)$ such that

$$(3) \qquad \lim_{v \to +\infty} [(1/v)\mathrm{Log}(1 + \varphi(v))] = 0\,.$$

Also the proof given here is slightly simpler than Masuda's argument and is therefore still interesting when $\varphi(v) = cv^{\beta}$.

1. Notation and preliminary observations. Throughout the text we shall denote by $\|\ \ \|_p$ the norm in $L^p(\Omega)$, $\|\ \ \|_\infty$ the norm in $C(\bar{\Omega})$ or $L^\infty(\Omega)$

and $\| \ \|_{1,\infty}$ the norm in $C^1(\bar{\Omega})$. The study of *local* existence and uniqueness of solutions to the initial-value problem for (1)–(2) in the framework of $C(\bar{\Omega})$ or $L^p(\Omega)$ is classical. As a consequence of the theory of analytical semi-groups the solutions are *classical* on $]0, T^*[$, where T^* denotes the eventual blowing-up time in $L^\infty(\Omega)$. It also follows from standard methods that if the initial data $u(0, x) = u_0(x)$ and $v(0, x) = v_0(x)$ are nonnegative, then u and v are nonnegative on $]0, T^*[\times\Omega$. Finally we note that as a consequence of the methods of [2], to prove that the solutions of (1)–(2) are *global* it is sufficient to derive a uniform estimate of $\|u\varphi(v)\|_p$ on $]0, T^*[$ for some $p > n/2$. Since $\|u(t)\|_\infty$ is obviously bounded by $\|u_0\|_p$, it is therefore good enough to obtain a bound on $\|\varphi(v)\|_n$ on $]0, T^*[$. Moreover if the bound does not depend on t, it will follow from the results of [2] that $\|u(t)\|_{1,\infty}$ and $\|v(t)\|_{1,\infty}$ are bounded for $t > 0$.

2. Statement and proof of the main result. The main result of the paper can be stated as follows.

THEOREM 1. *For any solution of* (1)–(2) *there exist two positive numbers* ε *and* δ, *depending only on* $\|u(0)\|_\infty$, *such that*

$$\int_\Omega \{1 + \delta[u(t, x) + u^2(t, x)]\} e^{\varepsilon v(t,x)} dx \quad \text{is non-increasing on }]0, T^*[\ . \tag{4}$$

PROOF. For any $f \in C^2(\boldsymbol{R}^+)$ such that $f \geqq 0$ and $f' \geqq 0$ on $\boldsymbol{R}^+$ and any solution (u, v) of (1)–(2) it is straightforward to check the following inequalities.

$$(d/dt)\left[\int_\Omega f(v)dx\right] \leqq -d_2\int_\Omega f''(v)|\nabla v|^2 dx + \int_\Omega uf'(v)\varphi(v)dx\ . \tag{5}$$

$$\begin{aligned}(d/dt)\left[\int_\Omega (u+u^2)f(v)dx\right] \leqq & -2d_1\int_\Omega f(v)|\nabla u|^2 dx - d_2\int_\Omega f''(v)|\nabla v|^2(u+u^2)dx \\ & -(d_1+d_2)\int_\Omega f'(v)(1+2u)\nabla u\cdot\nabla v dx + \int_\Omega u[(u+u^2)f'(v) - (1+2u)f(v)]\varphi(v)dx\ .\end{aligned} \tag{6}$$

[Note that $\partial u/\partial n \leqq 0$ and $\partial v/\partial n \leqq 0$ on $]0, T^*[\times\Gamma$ and (5) is an equality if $\alpha_2 \equiv 1$, while (6) is an equality if $\alpha_1 \equiv \alpha_2 \equiv 1$].

We now choose in (5) $f(v) = e^{\varepsilon v}$ where $\varepsilon > 0$ will be determined later on. By the Cauchy-Schwarz inequality we note that

$$\begin{aligned}(d_1 + d_2)\int_\Omega \varepsilon e^{\varepsilon v}(1 + 2u)|\nabla u||\nabla v|dx - 2d_1\int_\Omega e^{\varepsilon v}|\nabla u|^2 dx \\ \leqq [\varepsilon^2(d_1 + d_2)^2/(8d_1)]\int_\Omega e^{\varepsilon v}(1 + 2u)^2|\nabla v|^2 dx\end{aligned} \tag{7}$$

and since here $f'' \geqq 0$, from (7) and (6) we deduce immediately the following inequality:

$$(8)\qquad (d/dt)\int_\Omega (u+u^2)e^{\varepsilon v}dx \leqq \int_\Omega u[\varepsilon(u+u^2)-(1+2u)]e^{\varepsilon v}\varphi(v)dx + [\varepsilon^2(d_1+d_2)^2/(8d_1)]\int_\Omega e^{\varepsilon v}(1+2u)^2|\nabla v|^2dx\,.$$

On the other hand from (5) we deduce

$$(9)\qquad (d/dt)\int_\Omega e^{\varepsilon v}dx \leqq \int_\Omega \varepsilon u e^{\varepsilon v}\varphi(v)dx - \varepsilon^2 d_2\int_\Omega e^{\varepsilon v}|\nabla v|^2dx\,.$$

Let $\delta = 8d_1d_2(d_1+d_2)^{-2}(1+2\|u_0\|_\infty)^{-2}$. It follows immediately from (8) and (9), since $(1+2u)^2 \leqq (1+2\|u_0\|_\infty)^2$ that we have

$$(10)\qquad (d/dt)\int_\Omega [1+\delta(u+u^2)]e^{\varepsilon v}dx \leqq \int_\Omega \{\varepsilon+\delta[\varepsilon(u+u^2)-(1+2u)]\}ue^{\varepsilon v}\varphi(v)dx\,.$$

The integrand on the right-hand side of (10) is $\leqq 0$ a.e. on $]0, T^*[\times\Omega$ as soon as we have $\varepsilon+\delta\varepsilon(\|u_0\|_\infty+\|u_0\|_\infty^2)\leqq\delta$. Hence for any $\varepsilon>0$ such that

$$(11)\qquad \varepsilon \leqq \delta[1+\delta(\|u_0\|_\infty+\|u_0\|_\infty^2)]^{-1}$$

the inequality (10) yields

$$(12)\qquad (d/dt)\int_\Omega [1+\delta(u(t,x)+u^2(t,x)]e^{\varepsilon v}dx \leqq 0 \quad \text{on} \quad]0, T^*[\,.$$

This clearly implies (4) and Theorem 1 is completely proved.

COROLLARY 2. *If φ satisfies condition* (3), *all the solutions of* (1)-(2) *with nonnegative initial data u_0, v_0 in $L^\infty(\Omega)$ are global and uniformly bounded on* $]0, +\infty[\times\Omega$.

PROOF. (3) in particular implies that for any $(u_0, v_0)\in L^\infty(\Omega)\times L^\infty(\Omega)$, there exists a constant K such that

$$(13)\qquad 1+\varphi(v)\leqq Ke^{(\varepsilon/n)v}\,, \quad \text{for all} \quad v\geqq 0$$

where $\varepsilon=\varepsilon(\|u_0\|_\infty)$ is chosen as in the proof of Theorem 1. Then from (4) it follows in particular that $\varphi(v)\in L^\infty(]0, T^*[, L^n(\Omega))$ and therefore $u\varphi(v)$ is bounded in $L^n(\Omega)$ for $t\in[0, T^*[$. By the preliminary remarks we conclude that the solution is global and bounded on $]0, +\infty[\times\Omega$.

REMARK 3. In [8], the problem of global existence of nonnegative solutions to (1)-(2) is studied when $\varphi(v)=e^{\alpha v}$. It turns out that global existence can be established when $n=1$ or 2 under a smallness assumption on $\|u_0\|_\infty$.

3. Behavior as $t \to +\infty$. To conclude the paper we give a simple proof of another result of [6] in the spirit of [2].

PROPOSITION 4. *Let (u, v) be any non-negative solution of* (1)-(2) *such that u and v belong to $C_B(]0, +\infty[\times\bar{\Omega})$. Then as $t \to +\infty$ we have*

$$\|u(t) - u^*\|_\infty \to 0 \tag{14}$$

$$\|v(t) - v^*\|_\infty \to 0\ , \tag{15}$$

where u^* and v^* are two real nonnegative constants such that $u^*\varphi(v^*) = 0$.

PROOF. We have the obvious inequalities

$$\begin{aligned}(d/dt)\int_\Omega u dx &= d_1\int_\Omega \Delta u dx - \int_\Omega u\varphi(v)dx \\ &= d_1\int_\Gamma [\partial u/\partial n]d\sigma - \int_\Omega u\varphi(v)dx \leqq 0\end{aligned} \tag{16}$$

$$(d/dt)\int_\Omega (u + v)dx = d_1\int_\Gamma [\partial u/\partial n]d\sigma + d_2\int_\Gamma [\partial v/dn]d\sigma \leqq 0 \tag{17}$$

and the equality

$$\begin{aligned}(d/dt)\int_\Omega u^2 dx &= d_1\int_\Omega \Delta u\cdot u dx - \int_\Omega u^2\varphi(v)dx \\ &= -d_1\int_\Omega |\nabla u|^2 dx + d_1\int_\Gamma u[\partial u/dn]d\sigma - \int_\Omega u^2\varphi(v)dx\ .\end{aligned} \tag{18}$$

In particular the following functions of t:

$$\int_\Gamma |\partial u/dn|\,d\sigma,\ \int_\Gamma |\partial v/dn|\,d\sigma,\ \int_\Omega u\varphi(v)dx,\ \int_\Omega |\nabla u|^2 dx \quad \text{and} \quad \int_\Omega |\nabla v|^2 dx$$

are in $L^1(\boldsymbol{R}^+)$. Therefore $[\operatorname{meas}(\Omega)]^{-1}\int_\Omega u(t, x)dx$ and $[\operatorname{meas}(\Omega)]^{-1}\int_\Omega v(t, x)dx$ tend to a limit respectively denoted by u^* and v^*.

Then by using the above inequalities [together with an additional formula for the t-derivative of the spatial integral of v^2] in conjunction with Poincaré's inequality, the conclusion follows easily.

REMARK 5. The only case where (u^*, v^*) can be different from $(0, 0)$ is the case of Neumann boundary conditions for *both* u and v, i.e., when $\alpha_1 \equiv \alpha_2 \equiv 1$. In this case all nonnegative constants (u^*, v^*) such that $u^*\varphi(v^*) = 0$ are stationary solutions of (1) and therefore the result of Proposition 4 is then optimal.

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ANALYSE NUMÉRIQUE, TOUR 55–65, 5ÈME ÉTAGE
UNIVERSITÉ PIERRE ET MARIE CURIE, 4, PLACE JUSSIEU
75252 PARIS CEDEX 05
FRANCE