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# GEOMETRIC ANALOGUE OF THE MUMFORD-TATE CONJECTURE FOR STABLY NON-DEGENERATE ABELIAN VARIETIES (A NOTE ON MUSTAFIN'S PAPER)

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**Introduction.** In his article [6], Mustafin gives, among other things, a proof of the so-called "geometric analogue of the Mumford-Tate Conjecture" for certain family of abelian varieties. However, there he imposes on the generic fiber some conditions about the dimension of the irreducible components of its first cohomology group as the representation space of its Hodge group. The purpose of this note is to remove these restrictions and to generalize some of his results. In fact, we show that the "stable non-degeneracy" (see (1.1) below) of the generic fiber is sufficient for the validity of the Conjecture, if its simple components are not of type IV. (As for the reason why we must exclude "type IV", see Remark (3.7).) In [3], we give a characterization of the stable non-degeneracy of an abelian variety and show that many interesting abelian varieties fall into this category. We also note that abelian varieties satisfying Mustafin's conditions are stably non-degenerate, but that the converse does not always hold. So this note gives some new examples of abelian varieties for which the Conjecture holds. Moreover, our argument shows that the most suitable category for which Mustafin's argument goes well is that of stably non-degenerate abelian varieties.

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NOTATION. For a projective variety $X$ over $\boldsymbol{C}$, we denote by $\mathscr{B}^*(X)$ the Hodge ring $\bigoplus_{i=0}^{\dim X}\mathscr{B}^i(X) = \bigoplus_{i=0}^{\dim X} H^{2i}(X, \boldsymbol{Q}) \cap H^{i,i}(X)$ and by $\mathscr{D}^*(X)$ the subalgebra of $\mathscr{B}^*(X)$ generated by the divisor classes. For a group $G$ (resp. Lie algebra $\mathfrak{g}$) and its representation space $V$, we denote by $[V]^G$ (resp. $[V]^{\mathfrak{g}}$) the subspace of $G$-(resp. $\mathfrak{g}$-) invariant elements of $V$. For an abelian variety $A$, we denote by $\mathrm{End}\,A$ the endomorphism ring of $A$ and put $\mathrm{End}^0 A = \mathrm{End}\,A \otimes \boldsymbol{Q}$. Finally, the "reduced dimension", $\mathrm{rdim}\,A$, of $A$ is defined as follows: When $A$ is simple,

$$\operatorname{rdim} A = \begin{cases} \dim A & \text{if } A \text{ is of type I}, \\ (1/2)\dim A & \text{if } A \text{ is of type II}, \\ \dim A & \text{if } A \text{ is of type III}, \\ (1/d)\dim A & \text{if } A \text{ is of type IV with } d^2 = [\mathrm{End}^0 A : \mathrm{Cent}(\mathrm{End}^0 A)] \end{cases}$$

(for the definition of "type", see [5]). When $A$ is isogenous to $\prod_{i=1}^{k} A_i^{m_i}$, where $A_i$ $(1 \leqq i \leqq k)$ are simple and mutually non-isogenous, we define $\operatorname{rdim} A = \sum_{i=1}^{k} \operatorname{rdim} A_i$.

**1. Stably non-degenerate abelian varieties.** In this section we review some results in [3].

DEFINITION 1.1. Let $A$ be an abelian variety over $\boldsymbol{C}$. $A$ is said to be stably non-degenerate if $\mathscr{B}^*(A^n) = \mathscr{D}^*(A^n)$ holds for any $n \geqq 1$.

In [3], we give the following characterization of stably non-degenerate abelian varieties:

THEOREM 1.2. *An abelian variety $A$ is stably non-degenerate if and only if* $\operatorname{rank}(\mathscr{L}_{ie}(\mathrm{Hg}(A)_C)) = \operatorname{rdim} A$, *where* $\mathrm{Hg}(A)$ *denotes the Hodge group of $A$.*

Among the examples of stably non-degenerate abelian varieties, the following are contained: generic abelian varieties, a power of an elliptic curve, the jacobian variety of the modular curve $X_0(N)$ for any level $N$, abelian varieties of prime dimension, etc. (See [3] for the reason why these are stably non-degenerate.)

**2. Preliminaries.** Let $S$ be a normal, irreducible algebraic variety over $\boldsymbol{C}$ and let $\eta$ denote its generic point. A finite, surjective morphism $S' \to S$ from another normal irreducible algebraic variety to $S$ is said to be a *finite covering* of $S$. In this section we establish some results related to the problem of extension of various objects on the generic fiber of an abelian scheme over $S$.

PROPOSITION 2.1. *Let $A$, $B$ be abelian schemes over $S$. Suppose that we are given a homomorphism $u_\eta\colon A_\eta \to B_\eta$ of their generic fibers. Then it can be prolonged to a unique homomorphism $u\colon A \to B$ of abelian schemes.*

PROOF. By [2, Théorème B], the induced homomorphism $T_l(u_\eta)$: $T_l(A_\eta) \to T_l(B_\eta)$ of their Tate modules can be prolonged to a homomorphism $u_l\colon T_l(A) \to T_l(B)$. Further, by the main theorem of [2], there exists a homomorphism $u\colon A \to B$ of abelian schemes such that $T_l(u) = u_l$. Then this $u$ coincides with $u_\eta$ on the generic fiber, since the $l^n$-division points $(A_\eta)_{l^n}$ for all $n \geqq 0$ constitute a dense subset of $A_\eta$. q.e.d.

PROPOSITION 2.2. *Notation being as above, suppose that $A_\eta$ is a product $\prod_{i=1}^k X_i$ of abelian subvarieties $X_i$ $(i = 1, \cdots, k)$ defined over $C(\eta)$. Then there exist abelian subschemes $B_i$ $(i = 1, \cdots, k)$ over $S$ such that $(B_i)_\eta = X_i$ for any $i$ and that $A = B_1 \times_S \cdots \times_S B_k$.*

PROOF. Let $p_{i,\eta}$ denote the $i$-th projection $A_\eta \to X_i$ $(i = 1, \cdots, k)$ which we consider as endomorphisms of $A_\eta$. Note that they have the properties $p_{i,\eta} \circ p_{i,\eta} = p_{i,\eta}$ and $\mathrm{Im}(p_{i,\eta}) = X_i$. Then, by Proposition (2.1), there exist endomorphisms $p_i: A \to A$ over $S$ which induces $p_{i,\eta}$ on the generic fiber for any $i$. By the uniqueness of prolongation we have the equalities $p_i \circ p_i = p_i$ for any $i$. Therefore, if we put $B_i = \mathrm{Im}(p_i)$, then we obtain the desired decomposition. q.e.d.

PROPOSITION 2.3. *Let $X$ be an abelian subvariety of $A_\eta$. Then there exists an abelian subscheme $B$ of $A$ such that $B_\eta = X$.*

PROOF. Let $B'$ denote the scheme-theoretic closure of $X$ in $A$. Then, by the same argument as in [4, Remark 20.9], there exists an open subset $U$ of $S$ such that $B'_U = B' \times_S U \to U$ has a (unique) structure of abelian scheme. Note that for any $n \geqq 0$, $(B'_U)_{l^n}$ extends to an étale covering over $S$, since it is contained in $(A_U)_{l^n}$ which does extend. So, by [2, Corollary 4.2], $B'_U$ extends to an abelian scheme $B'' \to S$. Then it follows from Proposition 2.1 that there exists a homomorphism $u: B'' \to A$ of abelian schemes. The image $\mathrm{Im}(u)$ of $u$ has the desired property. q.e.d.

PROPOSITION 2.4. *Let $A$, $B$ be abelian schemes over $S$ and let $u$: $A \to B$ be a homomorphism of abelian schemes. If the induced homomorphism $u_\eta: A_\eta \to B_\eta$ on the generic fibers is an isogeny, then $u$ is also an isogeny*, i.e., *a finite, flat, surjective homomorphism.*

PROOF. Since $u$ is smooth, proper and the dimension of $\ker(u_\eta)$ is equal to zero, $u$ has the desired properties. q.e.d.

Combining these propositions, we obtain the following:

PROPOSITION 2.5. *Let $A$ be an abelian scheme over $S$. Suppose that there exist abelian varieties $X_i$ over $C(\eta)$ $(i = 1, \cdots, k)$ and an isogeny $\varphi: \prod_{i=1}^k X_i \to A_\eta$ over $C(\eta)$. Then there exist abelian subschemes $B_i$ of $A$ $(i = 1, \cdots, k)$ and an isogeny $u: \prod_{i=1}^k B_i \to A$ such that $(B_i)_\eta$ is isogenous to $X_i$.*

PROOF. Put $Y_i = \mathrm{Im}(\varphi|_{X_i}) \subset A_\eta$. Then $\prod_{i=1}^k Y_i$ is also isogenous to $A_\eta$. It follows from Proposition 2.3 that every $Y_i$ extends to an abelian subscheme $B_i$ of $A$. Since the product $u$ of the inclusion map of $B_i$ into $A$ satisfies the assumption of Proposition 2.4, $u$ is an isogeny. q.e.d.

Now we define "simplicity" of an abelian scheme:

DEFINITION 2.6. An abelian scheme $A$ over $S$ is said to be simple if for any finite covering $S' \to S$, the induced abelian scheme $A' = A \times_S S' \to S'$ has no non-trivial abelian subschemes.

We have the following correspondence between the simplicity of an abelian scheme and the simplicitity (in the classical sense) of its generic fiber:

PROPOSITION 2.7. *Let $A$ be an abelian scheme over $S$. Then $A$ is simple if and only if $A_\eta$ is simple.*

PROOF. The if-part is obvious. Conversely, suppose that $A_\eta$ is not simple. Then there exists a finite extension $K'$ of $\boldsymbol{C}(\eta)$ such that $A_\eta \times_{\boldsymbol{C}(\eta)} K'$ has a non-trivial abelian subvariety $X$ defined over $K'$. Let $S'$ denote the normalization of $S$ in $K'$. Then $A \times_S S'$ has a non-trivial abelian subscheme by Proposition 2.3. q.e.d.

Now we introduce one more terminology:

DEFINITION 2.8. Let $A$ be an abelian scheme over $S$. If for some finite covering $S'$ of $S$ there exist simple abelian schemes $B_i$ over $S'$ $(i = 1, \cdots, k)$ and an isogeny $\varphi\colon \prod_{i=1}^k B_i \to A \times_S S'$, then we call $\varphi$ an isogeny decomposition of $A$.

Combining all the preceding results, we obtain the following:

PROPOSITION 2.9. *Let $A$ be an abelian scheme over $S$. Then an isogeny decomposition of $A$ in the above sense induces an isogeny decomposition of the generic fiber $A_\eta$ in the classical sense and vice versa.*

## 3. Statement of the main theorem.

Let $S$ be a normal, irreducible algebraic variety over $\boldsymbol{C}$ and let $\eta$ be the generic point of $S$.

DEFINITION 3.1. An abelian scheme $A$ over $S$ is said to be constant if there exists an abelian variety $A_o$ over $\boldsymbol{C}$ such that $A$ is isomorphic to $A_o \times_{\boldsymbol{C}} S$ over $S$.

DEFINITION 3.2. An abelian scheme $A$ over $S$ is said to be potentially trivial (resp. iso-trivial) if there exists a finite (resp. finite étale) covering $S' \to S$ so that $A \times_S S' \to S'$ becomes a constant abelian scheme.

PROPOSITION-DEFINITION 3.3. *Let $A$ be an abelian scheme over $S$. Let $G(A \to S; s)$ denote the connected component of the unity of the Zariski* *closure of* $\pi_1(S, s)$ $(s \in S)$ *in* $\mathrm{Aut}(H^1(A_s, \boldsymbol{Q}))$. *Then there exists a countable union* $T$ *of proper closed subsets of* $S$ *such that* $G(A \to S; s) \subset \mathrm{Hg}(A_s)$ *for* $s \in S - T$. *The points in* $S - T$ *are called general in the sense of Hodge* ([6, p. 256]).

REMARK 3.4. It follows from [9, Proof of 7.5] that $\mathrm{Hg}(A_s)$ is locally constant on $S - T$. On the other hand, $S - T$ is arcwise connected if $S$ is quasi-projective ([6, p. 276]). So, we have $\mathrm{Hg}(A_s) = \mathrm{Hg}(A_\eta)$ for $s \in S - T$ in this case.

Moreover, we have the following:

THEOREM 3.5 ([8, 7.3]). *Notation being as above,* $G(A \to S; s)$ *is a normal subgroup of* $\mathrm{Hg}(A_s)$.

Thus if $\mathrm{Hg}(A_s)$ is $\boldsymbol{Q}$-simple and the abelian scheme $A$ is not iso-trivial, then we have $G(A \to S; s) = \mathrm{Hg}(A_s)$. However, in order to ensure the $\boldsymbol{Q}$-simplicity of $\mathrm{Hg}(A_s)$, Mustafin imposed the following conditions: (1) $A_s$ is simple, (2) when $H^1(A_s, \boldsymbol{C})$ is decomposed into irreducible components as a representation space of $\mathrm{Hg}(A_s)_{\boldsymbol{C}}$, their common dimension is not divisible by four. Our purpose is to show that the stable non-degeneracy of $A_s$ (which is much weaker than the above two conditions) is "almost" sufficient. More precisely, we prove the following:

THEOREM 3.6 (Geometric analogue of the Mumford-Tate Conjecture). *Let* $S$ *be a normal, irreducible quasi-projective variety over* $\boldsymbol{C}$ *and let* $A$ *be an abelian scheme over* $S$. *Let* $s \in S$ *be a point which is general in the sense of Hodge. Then* $G(A \to S; s) = \mathrm{Hg}(A_s)$, *if the following two conditions are satisfied:*

(3.6.1) *the simple components which appear in the isogeny decomposition (see (2.8)) are not potentially trivial,*

(3.6.2) $A_s$ *is stably non-degenerate and has no simple components of type* IV.

REMARK 3.7. It is known that $G(A \to S; s)$ is always semi-simple ([1]) and that $\mathrm{Hg}(A_s)$ has a non-trivial central torus if $A_s$ has a component of type IV ([3, p. 503]). This is the reason why we must exclude those of type IV in the above theorem.

REMARK 3.8. If $A_s$ satisfies the condition (3.6.2), then so does $A_\eta$, since this condition is expressible in term of the Hodge group and $\mathrm{Hg}(A_s) = \mathrm{Hg}(A_\eta)$ by (3.4).

REMARK 3.9. The condition (3.6.1) is satisfied, for example, when $A \times_{\boldsymbol{C}(\eta)} K$ has no non-trivial $K/\boldsymbol{C}$-trace for any finite extension $K$ of $\boldsymbol{C}(\eta)$.

COROLLARY 3.10. *Under the same conditions as in* (3.6), $\mathscr{B}^p(A_s^n) = [H^{2p}(A_s^n, \boldsymbol{Q})]^{\pi_1(S,s)}$ *holds for all* $n$, $p$. *Further, we have* $\mathrm{End}_{S'}(A') = \mathrm{End}(R^1\pi_{A'}\boldsymbol{Z})$ *for any finite étale covering* $S' \to S$, *where* $A' = A \times_S S'$ *and* $\pi_{A'}: A' \to S'$ *is the natural projection. (Here* $\mathrm{End}(R^1\pi_{A'}\boldsymbol{Z})$ *denotes the algebra of endomorphisms of the local system* $\{H^1(A_{s'}, \boldsymbol{Z})\}_{s' \in S'}$ *over* $S'$.)

PROOF OF (3.10). First note that $G(A \times_S \cdots \times_S A \to S; s)$ coincides with $G(A \to S; s)$. Indeed, a proof similar to the one for the Hodge group goes well (see [3, (1.10)]: Replace $T$ by $\pi_1(S, s)$ and the representation space $H^1(A, \boldsymbol{Q})$ by $H^1(A_s, \boldsymbol{Q})$). So the first part of the corollary follows from the fact that the space of Hodge cycles is that of invariant elements in the cohomology space under the action of the Hodge group. Further, the second part follows from (3.6) and the result of [6] that the equality $G(A \to S; s) = \mathrm{Hg}(A_s)$ is equivalent to the last assertion of (3.10). q.e.d.

4. **Proof of the main theorem.** First we note the following:

PROPOSITION 4.1. *Let* $A$ *be an abelian scheme over* $S$ *and let* $U$ *be an open subset of* $S$ *which contains a given point* $s \in S$. *Then* $G(A \to S; s) = G(A|_U \to U; s)$ *holds in* $\mathrm{Aut}(H^1(A_s, \boldsymbol{Q}))$.

PROOF. Since $S - U$ is of real codimension $\geqq 2$, we see that the induced homorphism $\pi_1(U, s) \to \pi_1(S, s)$ is surjective. Further, the action of $\pi_1(U, s)$ on $H^1(A_s, \boldsymbol{Q})$ factors through the action of $\pi_1(S, s)$ since the abelian scheme $A|_U \to U$ extends to $A \to S$. Therefore the images of $\pi_1(U, s)$ and $\pi_1(S, s)$ in $\mathrm{Aut}(H^1(A_s, \boldsymbol{Q}))$ coincide. Hence $G(A|_U \to U; s) = G(A \to S; s)$ q.e.d.

PROPOSITION 4.2. *Let* $A$ *be an abelian scheme over* $S$ *and let* $f: S' \to S$ *be an étale covering which sends a given point* $s' \in S'$ *to* $s \in S$. *Then* $G(A \times_S S' \to S'; s') = G(A \to S; s)$ *in* $\mathrm{Aut}(H^1((A \times_S S')_{s'}, \boldsymbol{Q})) = \mathrm{Aut}(H^1(A_s, \boldsymbol{Q}))$.

PROOF. Let $G$ (resp. $G'$) denote the Zariski closure of $\pi_1(S, s)$ (resp. $\pi_1(S', s')$) in $\mathrm{Aut}(H^1((A \times_S S')_{s'}, \boldsymbol{Q}))$ (resp. $\mathrm{Aut}(H^1(A_s, \boldsymbol{Q}))$). *If* $\pi_1(S', s')$ is a normal subgroup of $\pi_1(S, s)$, then $\pi_1(S, s)$ normalizes $G'$, too. For, $xG'x^{-1}$ is also a $\boldsymbol{Q}$-subgroup of $\mathrm{Aut}(H^1((A \times_S S')_{s'}, \boldsymbol{Q})) = \mathrm{Aut}(H^1(A_s, \boldsymbol{Q}))$ for any $x \in \pi_1(S, s)$, and this implies the equality $G' = xG'x^{-1}$ by the minimality of $G'$. So $G$ coincides with the subgroup generated by $\{xG'\}_{x \in \pi_1(S,s)}$, which contains $G'$ as a subgroup of finite index. Thus, we obtain $G(A \times_S S' \to S'; s') = G(A \to S; s)$ in this case. The general case when $\pi_1(S', s')$ is not necessarily normal subgroup of $\pi_1(S, s)$ is reduced to the proceding one by considering $\bigcap_{x \in \pi_1(S,s)} x\pi_1(S', s')x^{-1}$ which is of finite index in $\pi_1(S, s)$.

q.e.d.

PROPOSITION 4.3. *Let $A$ be an abelian scheme over $S$. Then $G(A \to S; s) = \{1\}$ if and only if there exists a finite étale covering $S'$ of $S$ such that $A \times_S S' \to S'$ is constant.*

PROOF. This follows from the fact that $G(A \to S; s)$ is of finite index in the Zariski closure of $\pi_1(S, s)$ in $\mathrm{Aut}(H^1(A_s, \boldsymbol{Q}))$. q.e.d.

PROPOSITION 4.4. *Let $A$ be an abelian scheme over $S$ and let $f: S' \to S$ be a finite covering which sends a given point $s' \in S'$ to $s \in S$. Suppose that $f$ is unramified at $s'$. Then $G(A \times_S S' \to S'; s') = G(A \to S; s)$ in $\mathrm{Aut}(H^1((A \times_S S')_{s'}, \boldsymbol{Q})) = \mathrm{Aut}(H^1(A_s, \boldsymbol{Q}))$.*

PROOF. Let $U \subset S$ be the complement of the ramification locus of $f$ and put $U' = f^{-1}(U) \subset S'$. Then

$$\begin{aligned} G(A \times_S S' \to S'; s') &= G(A|_U \times_U U' \to U'; s') && \text{(by Prop. 4.1)} \\ &= G(A|_U \to U; s) && \text{(by Prop. 4.2)} \\ &= G(A \to S; s) && \text{(by Prop. 4.1)}\,. \end{aligned}$$ q.e.d.

PROPOSITION 4.5. *Let $u: A \to B$ be an isogeny of abelian schemes $A$, $B$ over $S$ and let $s$ be a point of $S$. Then $G(A \to S; s) \cong G(B \to S; s)$ in $\mathrm{Aut}(H^1(A_s, \boldsymbol{Q})) \cong \mathrm{Aut}(H^1(B_s, \boldsymbol{Q}))$.*

PROOF. This follows from the fact that $u$ induces a $\pi_1(S, s)$-equivariant isomorphism $H^1(A_s, \boldsymbol{Q}) \xrightarrow{\sim} H^1(B_s, \boldsymbol{Q})$. q.e.d.

Now let us start the proof of the main theorem. First we consider the case where $A_s$ is simple.

PROPOSITION 4.6. *Notation being as in* (3.6), *suppose that $A_s$ is a simple stably non-degenerate abelian variety which is not of type* IV. *Then* $\mathrm{Hg}(A_s)$ *is $\boldsymbol{Q}$-simple.*

PROOF. We recall the following lemma due to Mustafin:

LEMMA 4.7 ([6, §4, Lemma 3]). *Let $\rho: \mathfrak{g} \to \mathrm{End}\, V$ be a faithful $\boldsymbol{Q}$-irreducible representation of a semi-simple Lie algebra $\mathfrak{g}$ over $\boldsymbol{Q}$ and put $e = \dim_{\boldsymbol{Q}}(\mathrm{Cent}(\mathrm{End}_{\mathfrak{g}} V))$. Suppose that $(*)$ the number of simple components of $\mathfrak{g}_{\boldsymbol{C}}$ is equal to $e$. Then $\mathfrak{g}$ is $\boldsymbol{Q}$-simple.*

Let us put $\mathfrak{g} = \mathscr{L}ie(\mathrm{Hg}(A_s))$ and $V = H^1(A_s, \boldsymbol{Q})$. It is known that $\mathfrak{g}$ is semi-simple and that $\mathrm{End}_{\mathfrak{g}} V = \mathrm{End}^{\circ} A_s$ (see [7]). So we are reduced to showing that, if $A_s$ is a simple stably non-degenerate abelian variety which is not of type IV, then $\mathfrak{g}$ satisfies the condition $(*)$ in (4.7). But this is proved in [3, p. 499] when $A_s$ is of type I, and in [3, p. 502] when $A_s$ is of type II. Further we know that an abelian variety of type III cannot be stably non-degenerate ([3, p. 502, Case 3]). Thus Proposition (4.6) is proved. q.e.d.

Hence we obtain the $\boldsymbol{Q}$-simplicity of $\mathrm{Hg}(A_s)$ in this case. This implies by (3.5) that $G(A \to S; s) = \mathrm{Hg}(A_s)$. This completes the proof of (3.6) when $A_s$ is simple.

Next we consider the general case where $A_s$ is not necessarily simple. By (2.9), (3.8), (4.4) and (4.5), we may assume that there exist simple, mutually non-isogenous abelian schemes $B_i$ $(i = 1, \cdots, k)$ over $S$ such that $A = \prod_{i=1}^{k} B_i^{m_i}$. Then we have $\mathrm{Hg}(A_s) \cong \prod_{i=1}^{k} \mathrm{Hg}((B_i)_s)$ by [3, p. 507]. Note that $\mathrm{Hg}((B_i)_s)$ is $\boldsymbol{Q}$-simple for every $i$ by the preceding proof. Since $G(A \to S; s)$ is normal in $\mathrm{Hg}(A_s)$, we have $G(A \to S; s) = \prod_{j=1}^{l} \mathrm{Hg}((B_{h(j)})_s)$ for some increasing function $h: \{1, \cdots, l\} \to \{1, \cdots, k\}$ $(l \leqq k)$, However, if there exists a number, $m \notin \mathrm{Im}(h)$, then for some finite covering $S'$ the abelian scheme $A_m$ has to be constant, which contradicts the assumption of (3.6). So we obtain the equality $G(A \to S; s) = \prod_{i=1}^{k} \mathrm{Hg}((B_i)_s) = \mathrm{Hg}(A_s)$. This completes the proof of (3.6). q.e.d.

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