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UNIFORM STABILITY FOR ONE-DIMENSIONAL DELAY-DIFFERENTIAL EQUATIONS WITH DOMINANT DELAYED TERM

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1. Introduction. For $q \geqq 0$ and $\alpha, \beta \in \boldsymbol{R}$, the one-dimensional differential-difference equation

$$x'(t) = -\alpha x(t) - \beta x(t-q) \tag{1.1}$$

is a simple example of a delay-differential equation and has been studied from early times in the development of the stability theory of delay-differential equations. For (1.1), the theory of characteristic equations is valid and it is known that the zero solution of (1.1) is uniformly stable if and only if α and β satisfy one of the following conditions:

(R_1) $\quad \alpha \geqq |\beta|$,

(R_2) $\quad \alpha = \beta \sin \eta, \quad 0 \leqq \beta q \leqq \left(\eta + \dfrac{\pi}{2}\right)\Big/\cos \eta, \quad -\dfrac{\pi}{2} < \eta < \dfrac{\pi}{2}$,

(R_3) $\quad -\alpha = \beta, \quad 0 \leqq \beta q < 1$,

that is, (α, β) is contained in the region (stability region) illustrated in Figure 1 with its boundary except for the point $(-1/q, 1/q)$. Moreover, the zero solution of (1.1) is uniformly asymptotically stable if and only if (α, β) is contained in the interior of $R_1 \cup R_2$ (cf. [3], [7]). It is a feature that R_2 and R_3 become smaller as q increases, while R_1 is independent of q.

On the other hand, the theory of characteristic equations is not applicable to the delay-differential equation such as

$$x'(t) = -a(t)x(t) - b(t)x(t - r(t)), \tag{1.2}$$

where $a, b: [0, \infty) \to \boldsymbol{R}$ and $r: [0, \infty) \to [0, q]$ are continuous functions. Liapunov's method seems to be the only way to investigate the behavior of solutions of (1.2). For (1.2), it is reasonable to expect a similar stability region for (α, β) under the conditions

$$0 \leqq \alpha \leqq a(t), \qquad |b(t)| \leqq \beta \tag{1.3}$$

or

$$\alpha \leqq a(t) \leqq 0, \qquad 0 \leqq b(t) \leqq \beta. \tag{1.4}$$

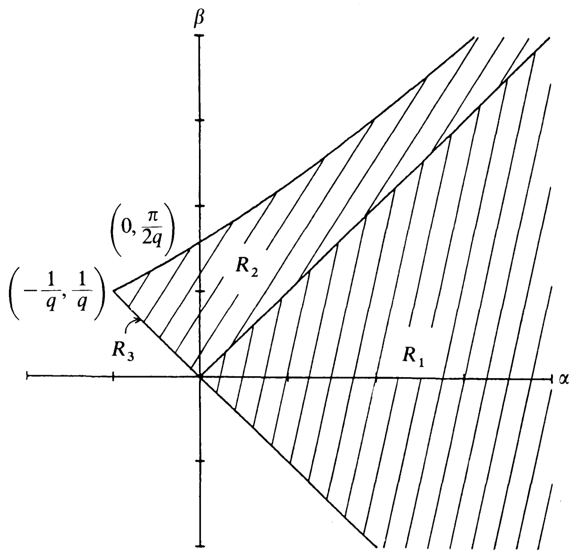

FIGURE 1

There are many works for the stability region which is independent of q like the region R_1 (cf. [6], [8], [9], [12], [13]). But there are few works for the stability region corresponding to R_2 ([2], [16]). In the case $a(t)\equiv 0$ in (1.2), i.e., for the equation

$$x'(t) = -b(t)x(t-r(t)), \tag{1.5}$$

the stability of the zero solution has been much studied (cf. [4], [5], [10], [11], [14]). It is interesting that under the condition $0 \leqq b(t) \leqq \beta$, the zero solution of (1.5) is uniformly stable if $\beta q \leqq 3/2$, but there are equations with unbounded solutions if $\beta q > 3/2$. The stability region $0 \leqq \beta q \leqq 3/2$ for (1.5) does not coincide with the stability region $0 \leqq \beta q \leqq \pi/2$ for the differential-difference equation

$$x'(t) = -\beta x(t-q).$$

For a general delay-differential equation

$$x'(t) = F(t, x_t), \tag{1.6}$$

where F: $[0, \infty) \times C^q \to \boldsymbol{R}$, Yorke [17] has shown the uniform stability of the zero solution under the conditions

$$-\beta M(\phi) \leqq F(t, \phi) \leqq \beta M(-\phi) \quad \text{and} \quad 0 \leqq \beta q \leqq 3/2, \tag{1.7}$$

where $M(\phi) = \max\{0, \sup_{s\in[-q,0]} \phi(s)\}$. The author [15] proved the uniform stability for (1.6) under conditions more general than (1.7).

For the special case

$$x'(t) = -a(t)(x(t) - x(t-q))$$

of (1.2), Atkinson-Haddock [1] has shown the uniform stability under the conditions including

$$|a(t)| \leqq \alpha \quad \text{and} \quad \alpha q < 1 ,$$

which corresponds to (R_3).

Recently, the author and Sugie [16] have established a stability region for an equation more general than (1.2) under the conditions (1.4) and $|a(t)| \leqq b(t)$.

In this paper, we study the stability region which depends on q for an equation more general than (1.2). Theorem 3.1 in Section 3 gives a stability region for (1.2) under (1.3), which corresponds to (R_2) with $0 \leqq \eta < \pi/2$. Theorem 4.1 in Section 4 gives a stability region for (1.2) under (1.4), which corresponds to (R_2) with $-\pi/2 < \eta < 0$ and includes the stability region given in [16]. The stability region obtained for (1.1) by the

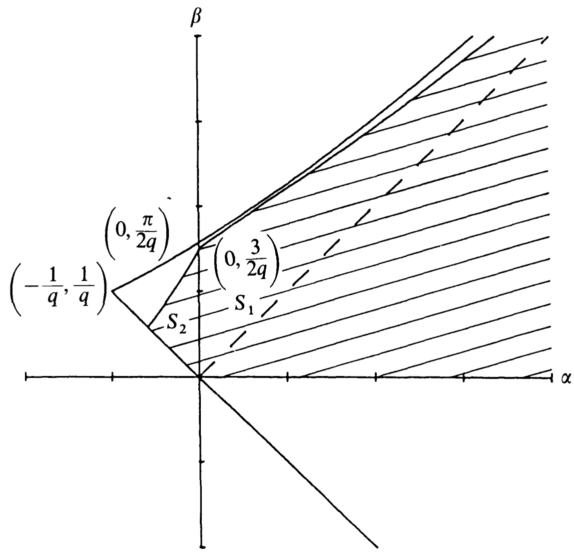

FIGURE 2

results in this paper is illustrated in Figure 2. We also give some results on the asymptotic stability of the zero solution of (1.2). If (α, β) is contained in the interior of the regions S_1 and S_2 in Figure 2, it will be shown that the zero solution of (1.2) is asymptotically stable. (Figures 1 and 2 were drawn accurately by a computer and an X–Y plotter.) Thanks are due to the referee for valuable comments.

2. Definitions and assumptions. For $q \geqq 0$, let C^q be the space of continuous functions on $[-q, 0]$, and define the norm

$$\|\phi\| = \sup_{s\in[-q,0]} |\phi(s)|$$

for $\phi \in C^q$. For $H > 0$, let

$$S(H) = \{x \in \boldsymbol{R} : |x| < H\} \quad \text{and} \quad C^q(H) = \{\phi \in C^q : \|\phi\| < H\}.$$

If $\psi(\cdot)$ is a continuous function defined on $[-q, T]$ for some $T > 0$, then for $0 \leqq t \leqq T$, we denote by $\psi_t \in C^q$ the function defined by $\psi_t(s) = \psi(t+s)$ for $s \in [-q, 0]$.

Consider the delay-differential equation

$$\text{(DDE)} \qquad x'(t) = F(t, x_t),$$

where F: $[0, \infty) \times C^q(H) \to \boldsymbol{R}$ is continuous and $x'(t)$ denotes the right-hand derivative of $x(t)$. For an initial function $\phi \in C^q(H)$ at $t_0 \geqq 0$, we denote by $x(\cdot; t_0, \phi)$ the solution of (DDE) such that $x_{t_0} = \phi$. We assume that $F(t, 0) \equiv 0$ so that $x(t) \equiv 0$ is a solution of (DDE), which is called the zero solution.

DEFINITION 2.1. The zero solution of (DDE) is said to be *stable* if for any $\varepsilon > 0$ and $t_0 \geqq 0$ there exists $\delta(t_0, \varepsilon) > 0$ such that if $\phi \in C^q(\delta)$, then

$$|x(t; t_0, \phi)| < \varepsilon \qquad \text{for all} \quad t \geqq t_0.$$

The zero solution of (DDE) is *uniformly stable* if the above δ is independent of t_0.

DEFINITION 2.2. The zero solution of (DDE) is said to be *asymptotically stable* if it is stable and if for any $t_0 \geqq 0$ there exists $\delta_0(t_0) \geqq 0$ such that if $\phi \in C^q(\delta_0)$, then

$$x(t; t_0, \phi) \to 0 \qquad \text{as} \quad t \to \infty.$$

DEFINITION 2.3. The zero solution of (DDE) is said to be *uniformly asymptotically stable* if it is uniformly stable and if there exists $\delta_0 > 0$ such that for each $\varepsilon > 0$, there exists $T(\varepsilon) > 0$ such that for any $t_0 \geqq 0$ and $\phi \in C^q(\delta_0)$,

$$|x(t; t_0, \phi)| < \varepsilon \qquad \text{for all} \quad t \geqq t_0 + T(\varepsilon).$$

The following condition on $F(t, \phi)$ was given by Yorke [17]: There exists $\beta \geqq 0$ such that

$$(\mathrm{A}_1) \qquad -\beta M(\phi) \leqq F(t, \phi) \leqq \beta M(-\phi) \qquad \text{for all} \quad t \geqq 0 \quad \text{and} \quad \phi \in C^q(H),$$

where $M(\phi) = \max\{0, \sup_{s\in[-q,0]} \phi(s)\}$.

The author and Sugie [16] modified the above condition and proposed the following condition to show a stability region of the delay-differential equation which will be studied in Section 4: There exist $\beta \geqq 0$ and a non-negative continuous function $b(t)$ on $[0, \infty)$ such that

$$(\mathrm{A}_2) \qquad \begin{aligned} &b(t) \leqq \beta \quad \text{and} \\ &-b(t) \sup_{s\in[-q,0]} \phi(s) \leqq F(t, \phi) \leqq b(t) \sup_{s\in[-q,0]} (-\phi(s)) \end{aligned}$$

for all $t \geqq 0$ and $\phi \in C^q(H)$.

We note that (A_2) implies (A_1), and if $F(t, \phi)$ satisfies (A_1) then $F(t, 0) \equiv 0$ and

$$|F(t, \phi)| \leqq \beta \|\phi\| \qquad \text{for all} \quad t \geqq 0 \quad \text{and} \quad \phi \in C^q(H).$$

EXAMPLE 2.1. Let f: $[0, \infty) \times S(H) \to \boldsymbol{R}$ and r: $[0, \infty) \times C^q(H) \to [0, q]$ be continuous functions such that $0 \leqq xf(t, x) \leqq \beta x^2$ for all $(t, x) \in [0, \infty) \times S(H)$. Then $F(t, \phi) = -f(t, \phi(-r(t, \phi)))$ satisfies (A_1).

For other examples of $F(t, \phi)$ satisfying (A_1), we refer to [17].

EXAMPLE 2.2. Let b: $[0, \infty) \to [0, \infty)$ be a continuous function such that $0 \leqq b(t) \leqq \beta$ for some $\beta \geqq 0$, and let $r(t, \phi)$ be a continuous function as in Example 2.1. Then $F(t, \phi) = -b(t)\phi(-r(t, \phi))$ satisfies (A_2).

EXAMPLE 2.3. Let $c(t, s)$ be a continuous non-negative function defined for all $-q \leqq s \leqq t$ such that

$$\int_{-q}^{0} c(t, t+s)ds \leqq \beta q .$$

Then

$$F(t, \phi) = -\frac{1}{q} \int_{-q}^{0} c(t, t+s)\phi(s)ds$$

satisfies (A_2).

Finally we give a lemma which is useful in later sections.

LEMMA 2.1. *Let $x(t)$ be a continuously differentiable function on $[T_1, T_2]$ such that $x(t_1) = 0$ for some $t_1 \in [T_1, T_2]$ and*

$$\frac{d}{dt}|x(t)| \leqq \alpha |x(t)| + c \qquad \text{for all} \quad t \in [T_1, T_2],$$

where $\alpha \neq 0$ and $c \geqq 0$. Then

$$|x(t)| \leqq \frac{c}{\alpha}(e^{\alpha|t-t_1|} - 1) \qquad \text{for all} \quad t \in [T_1, T_2]. \tag{2.1}$$

PROOF. Let $t \in [t_1, T_2]$. Then

$$|x(t)| = |x(t)| - |x(t_1)| \leqq \alpha \int_{t_1}^{t} |x(s)| ds + c(t - t_1),$$

and hence

$$\frac{d}{dt}\left\{\alpha e^{-\alpha(t-t_1)}\int_{t_1}^{t}|x(s)|\,ds\right\}\leqq \alpha c(t-t_1)e^{-\alpha(t-t_1)},$$

for all $t\in[t_1, T_2]$. Integrating both sides from t_1 to t, we have

$$\alpha e^{-\alpha(t-t_1)}\int_{t_1}^{t}|x(s)|\,ds\leqq -c(t-t_1)e^{-\alpha(t-t_1)}+\frac{c}{\alpha}(1-e^{-\alpha(t-t_1)})$$

and therefore

$$\alpha\int_{t_1}^{t}|x(s)|\,ds\leqq -c(t-t_1)+\frac{c}{\alpha}(e^{\alpha(t-t_1)}-1),$$

which proves (2.1). The proof in the case $t\in[T_1, t_1]$ is similar.

3. Stability region in Quadrant I. In this section, we consider the delay-differential equation

$$x'(t)=G(t, x(t))+F(t, x_t), \tag{3.1}$$

where G: $[0, \infty)\times S(H)\to \boldsymbol{R}$ and F: $[0, \infty)\times C^q(H)\to \boldsymbol{R}$ are continuous. We assume that there exists $\alpha>0$ such that

$$xG(t, x)\leqq -\alpha x^2 \qquad \text{for all} \quad (t, x)\in[0, \infty)\times S(H), \tag{3.2}$$

and that $F(t, \phi)$ satisfies (A_1) for some $\beta>0$.

Let $x(t)$ be a solution of (3.1) on $[t_1-q, t_1]$ such that $x(t)>0$ for all $t\in(t_1-q, t_1)$. Then $M(-x_{t_1})=0$ and hence

$$x(t_1)x'(t_1)=x(t_1)(G(t_1, x(t_1))+F(t, x_{t_1}))\leqq -\alpha x^2(t_1)\leqq 0.$$

Similarly if $x(t)$ is a solution of (3.1) on $[t_1-q, t_1]$ such that $x(t)<0$ for all $t\in(t_1-q, t_1)$, then $x(t_1)x'(t_1)\leqq 0$. Thus we have the following:

LEMMA 3.1. *For some $t_1\geqq 0$, let $x(t)$ be a solution of* (3.1) *on $[t_1-q, t_1]$ such that $|x(t)|>0$ for all $t\in(t_1-q, t_1)$. Then*

$$x(t_1)x'(t_1)\leqq 0.$$

The above lemma shows that if $x(t)=x(t; t_0, \phi)$ is a positive (or negative) solution of (3.1) defined on $[t_0, \infty)$ for $t_0\geqq 0$ and $\phi\in C^q(H)$, then $|x(t)|$ is a non-increasing function on $[t_0+q, \infty)$. Therefore, in order to show the uniform stability of the zero solution of (3.1), it suffices to investigate the behavior of solutions of (3.1) which cross the t-axis.

From now on we assume that $\alpha\geqq\beta$ or

$$\theta=\frac{\beta^2}{\alpha^2}\left\{1-\frac{\beta-\alpha}{\beta}\left(1+\frac{2\alpha\beta}{(\beta-\alpha)^2}e^{-\alpha q}\right)^{1/2}\right\}\leqq 1\,,$$

and that (A_1) and (3.2) hold.

In the case $\beta<\alpha/(1-e^{-\alpha q})$, we let $\tilde{\beta}=\alpha/(1-e^{-\alpha q})$. Then (A_1) is also satisfied for $\tilde{\beta}$, and the corresponding θ satisfies $\theta=1/(1+(1-\alpha^2/\tilde{\beta}^2)^{1/2})<1$. Thus we may assume without loss of generality that

$$\frac{\beta}{\alpha}(1-e^{-\alpha q})\geqq 1\,. \tag{3.3}$$

Let $x(t)$ be a solution of (3.1) on $[t_1-2q,\, T]$ for some $T>t_1\geqq 0$ such that

$$x(t_1)=0 \quad\text{and}\quad x(t)>0 \qquad \text{for all}\quad t\in(t_1,\, T]\,,$$

and let $r=\sup_{s\in[t_1-2q,\,t_1]}|x(s)|$. We shall show that

$$|x(t)|\leqq\theta r \qquad \text{for all}\quad t\in[t_1,\, T]\,. \tag{3.4}$$

It suffices to show that for each $\varepsilon>0$,

$$x(t)<\theta(r+\varepsilon) \qquad \text{for all}\quad t\in[t_1,\, T]\,.$$

Suppose that there exist $\varepsilon>0$ and $t_3\in(t_1,\, T]$ such that

$$x(t_3)=\theta(r+\varepsilon) \quad\text{and}\quad x(t)<\theta(r+\varepsilon) \qquad \text{for all}\quad t\in[t_1,\, t_3)\,. \tag{3.5}$$

In practice, we have $t_3\leqq t_1+q$ by Lemma 3.1. By (A_1) and (3.2),

$$\frac{d}{dt}|x(t)|\leqq -\alpha|x(t)|+\beta r \qquad \text{for all}\quad t\in[t_1-q,\, t_1]\,, \tag{3.6}$$

and

$$\frac{d}{dt}|x(t)|\leqq -\alpha x(t)+\beta(r+\varepsilon) \qquad \text{for all}\quad t\in[t_1,\, t_3]\,. \tag{3.7}$$

It follows from Lemma 2.1 and (3.7) that

$$x(t)\leqq\frac{\beta}{\alpha}(r+\varepsilon)(1-e^{-\alpha(t-t_1)}) \qquad \text{for all}\quad t\in[t_1,\, t_3]\,. \tag{3.8}$$

We also obtain by Lemma 2.1 and (3.6) that

$$|x(t)|\leqq\frac{\beta}{\alpha}r(1-e^{-\alpha(t_1-t)}) \qquad \text{for all}\quad t\in[t_1-q,\, t_1]\,.$$

Hence, it follows from (A_1) and (3.2) again that for $t\in[t_1,\, t_3]$,

$$x'(t) \leqq -\alpha x(t) + \beta M(-x_t) \leqq -\alpha x(t) + \beta \sup_{s \in [t-q, t_1]}(-x(s))$$

$$\leqq -\alpha x(t) + \beta \sup_{s \in [t-q, t_1]} |x(s)| \leqq -\alpha x(t) + \frac{\beta^2}{\alpha} r \sup_{s \in [t-q, t_1]}(1 - e^{-\alpha(t_1 - s)})$$

$$\leqq -\alpha x(t) + \frac{\beta^2}{\alpha} r(1 - e^{-\alpha(t_1 + q - t)}) . \tag{3.9}$$

By (3.3), we can choose $t_2 \geqq t_1$ in such a way that

$$\frac{\beta}{\alpha}(1 - e^{-\alpha(t_1 + q - t_2)}) = 1 . \tag{3.10}$$

Define a continuous function $y_\varepsilon(t)$ on $[t_1, \infty)$ by

$$y_\varepsilon(t) = \frac{\beta}{\alpha}(r + \varepsilon)(1 - e^{-\alpha(t - t_1)})$$

for $t \in [t_1, t_2]$ and

$$y_\varepsilon(t) = \frac{\beta^2}{\alpha^2} r \left\{ 1 - \left(\frac{\beta - \alpha}{2\beta} + \frac{\alpha}{\beta - \alpha} e^{-\alpha q} \right) e^{-\alpha(t - t_2)} - \frac{\beta - \alpha}{2\beta} e^{\alpha(t - t_2)} \right\}$$

$$+ \frac{\beta}{\alpha} \varepsilon \left(1 - \frac{\beta}{\beta - \alpha} e^{-\alpha q} \right) e^{-\alpha(t - t_2)}$$

for $t \geqq t_2$. Note that $y_\varepsilon(t)$ is the solution of

$$y' = -\alpha y + \frac{\beta^2}{\alpha} r(1 - e^{-\alpha(t_1 + q - t)})$$

on $[t_2, \infty)$ with

$$y_\varepsilon(t_2) = \frac{\beta}{\alpha}(r + \varepsilon)\left(1 - \frac{\beta}{\beta - \alpha} e^{-\alpha q} \right).$$

Moreover let

$$z_\varepsilon(t) = \frac{\beta^2}{\alpha^2} r \left\{ 1 - \left(\frac{\beta - \alpha}{2\beta} + \frac{\alpha}{\beta - \alpha} e^{-\alpha q} \right) e^{-\alpha(t - t_2)} - \frac{\beta - \alpha}{2\beta} e^{\alpha(t - t_2)} \right\} + \frac{\beta}{\alpha} \varepsilon \left(1 - \frac{\beta}{\beta - \alpha} e^{-\alpha q} \right)$$

for $t \geqq t_2$. Then $y_\varepsilon(t) \leqq z_\varepsilon(t)$ for all $t \geqq t_2$ and

$$\max_{t \in [t_2, T]} z_\varepsilon(t) = z_\varepsilon\left(t_2 + \frac{1}{2\alpha} \log\left(1 + \frac{2\alpha\beta}{(\beta - \alpha)^2} e^{-\alpha q} \right) \right)$$

$$= \frac{\beta^2}{\alpha^2} r \left\{ 1 - \frac{\beta - \alpha}{\beta} \left(1 + \frac{2\alpha\beta}{(\beta - \alpha)^2} e^{-\alpha q} \right)^{1/2} \right\} + \frac{\beta}{\alpha} \varepsilon \left(1 - \frac{\beta}{\beta - \alpha} e^{-\alpha q} \right)$$

$$= \theta r + \frac{\beta}{\alpha} \varepsilon \left(1 - \frac{\beta}{\beta - \alpha} e^{-\alpha q} \right),$$

since

$$z_\varepsilon'(t)=\frac{\beta^2}{\alpha}r\left\{\left(\frac{\beta-\alpha}{2\beta}+\frac{\alpha}{\beta-\alpha}e^{-\alpha q}\right)e^{-\alpha(t-t_2)}-\frac{\beta-\alpha}{2\beta}e^{\alpha(t-t_2)}\right\}.$$

In particular,

$$y_\varepsilon(t_2)=z_\varepsilon(t_2)<\theta r+\frac{\beta}{\alpha}\varepsilon\left(1-\frac{\beta}{\beta-\alpha}e^{-\alpha q}\right),$$

since $z_\varepsilon'(t_2)>0$. Hence

$$\frac{\beta}{\alpha}\left(1-\frac{\beta}{\beta-\alpha}e^{-\alpha q}\right)<\theta.$$

Observe that $y_\varepsilon(t)$ is increasing on $[t_1, t_2]$. We have

$$y_\varepsilon(t)<\theta(r+\varepsilon)\qquad \text{for all}\quad t\geqq t_1. \tag{3.11}$$

If $t_3\leqq t_2$, then by (3.8) and (3.11)

$$x(t)\leqq y_\varepsilon(t)<\theta(r+\varepsilon)\qquad \text{for all}\quad t\in[t_1, t_3],$$

which yields a contradiction at $t=t_3$. Hence $t_2<t_3$ and $x(t_2)\leqq y_\varepsilon(t_2)$. It follows from (3.9) and (3.11) that

$$x(t)\leqq y_\varepsilon(t)<\theta(r+\varepsilon)\qquad \text{for all}\quad t\in[t_2, t_3],$$

which contradicts (3.5) at $t=t_3$. Thus (3.4) is proved.

In a similar way, under $\theta\leqq 1$, we can show (3.4) for any solution $x(t)$ of (3.1) on $[t_1-2q, T]$ such that

$$x(t_1)=0 \quad \text{and} \quad x(t)<0 \qquad \text{for all}\quad t\in(t_1, T].$$

(3.4) means that if a solution of $x(t)$ for (3.1) crosses the t-axis at some t_1, then

$$|x(t)|\leqq\theta\sup\nolimits_{s\in[t_1-2q,t_1]}|x(s)| \quad \text{as long as} \quad |x(t)|\neq 0.$$

Therefore, together with Lemma 3.1, we obtain the following:

LEMMA 3.2. *Assume that $\theta\leqq 1$, and let $x(t)$ be a solution of (3.1) on $[t_1-2q, T]$ for some $T>t_1\geqq 0$ such that $x(t_1)=0$. Then*

$$|x(t)|\leqq\theta\sup\nolimits_{s\in[t_1-2q,t_1]}|x(s)| \qquad \text{for all}\quad t\in(t_1, T].$$

We now state a theorem on the uniform stability of the zero solution of (3.1).

THEOREM 3.1. *Suppose that there exist $\alpha>0$ and $\beta>0$ such that (A_1) and (3.2) hold and that $\alpha\geqq\beta$ or*

$$(3.12) \qquad \frac{\beta^2}{\alpha^2}\left\{1-\frac{\beta-\alpha}{\beta}\left(1+\frac{2\alpha\beta}{(\beta-\alpha)^2}e^{-\alpha q}\right)^{1/2}\right\}\leqq 1\,.$$

Then the zero solution of (3.1) *is uniformly stable.*

PROOF. For $t_0\geqq 0$ and $\phi\in C^q(He^{-2\beta q})$, let $x(t)=x(t;\,t_0,\,\phi)$ be a solution of (3.1). By (A_1) and (3.2),

$$x(t)x'(t)\leqq\beta\|x_t\|^2$$

and hence $\|x_t\|^2\leqq\|\phi\|^2+2\beta\int_{t_0}^{t}\|x_s\|^2 ds$. Then Gronwall's inequality implies that $\|x_t\|\leqq\|\phi\|e^{\beta(t-t_0)}$. Therefore $V(t)=\sup_{s\in[t-3q,\,t]}x^2(s)$ is well defined for $t\geqq t_0+2q$. We shall show that $V(t)$ is non-increasing for all $t\geqq t_0+2q$. Suppose not. Then there exist a solution $x(t)=x(t;\,t_0,\,\phi)$ and $t_2>t_0+2q$ such that

$$(3.13) \qquad V'(t_2)>0\,.$$

It is easy to see that $V(t_2)=x^2(t_2)$ and $V'(t_2)=2x(t_2)x'(t_2)>0$. By Lemma 3.1, there exists $t_1\in(t_2-q,\,t_2)$ such that $x(t_1)=0$. Then it follows from Lemma 3.2 that

$$|x(t)|\leqq\theta\sup_{s\in[t_1-2q,\,t_1]}|x(s)| \qquad \text{for all} \quad t\in(t_1,\,t_2]\,,$$

and hence $V'(t_2)\leqq 0$, which contradicts (3.13). The proof is now complete.

REMARK 3.1. The condition (3.12) is also written as

$$(3.14) \qquad \beta q\leqq-\frac{1}{\omega}\log\left(\frac{1}{2}(1-\omega)(2-\omega-\omega^2)\right),$$

where $\omega=\alpha/\beta$. The region S_1 of points $(\alpha,\,\beta)$ which satisfy the conditions of Theorem 3.1 is illustrated in Figure 2.

Letting $\alpha\to 0$ in (3.12), we have $\beta q-1/2\leqq 1$. This coincides with Yorke's condition below.

THEOREM [17, Theorem 1.1]. *Suppose that there exists $\beta\geqq 0$ such that* (A_1) *holds and $\beta q\leqq 3/2$. Then the zero solution of*

$$(3.15) \qquad x'(t)=F(t,\,x_t)$$

is uniformly stable.

Yorke [17] also proved the uniform asymptotic stability for (3.15) under the conditions (A_1), $0<\beta q<3/2$ and

(3.16) For all sequences $t_n\to\infty$ and $\phi_n\in C^q(H)$ converging to a constant nonzero function in $C^q(H)$, $F(t_n,\,\phi_n)$ does not converge to 0.

If (A_1) and (3.2) hold, then the right hand side of (3.1) satisfies (3.16), and so the uniform asymptotic stability for (3.1) will be proved in a way similar to [17], but under (A_1) and (3.2), the proof is slightly easier.

THEOREM 3.2. *Suppose that there exist* $\alpha>0$ *and* $\beta>0$ *such that* (A_1) *and* (3.2) *hold and that* $\alpha\geqq\beta$ *or*

$$\frac{\beta^2}{\alpha^2}\left\{1-\frac{\beta-\alpha}{\beta}\left(1+\frac{2\alpha\beta}{(\beta-\alpha)^2}e^{-\alpha q}\right)^{1/2}\right\}<1\,.$$

Then the zero solution of (3.1) *is uniformly asymptotically stable.*

PROOF. Suppose that the zero solution is not uniformly asymptotically stable. Then there exist $\varepsilon>0$, sequences $\{\tau_n'\}$, $\{\phi_n\}$, $\{T_n'\}$ and a sequence $\{x(\cdot;\tau_n',\phi_n)\}$ of solutions on $[\tau_n',\infty)$ such that $\tau_n'\geqq 0$, $\phi_n\in C^q(He^{-2\beta q})$, $T_n'\to\infty$ as $n\to\infty$ and $|x(\tau_n'+T_n';\tau_n',\phi_n)|>\varepsilon$. Let $x^n(t)=x(t;\tau_n',\phi_n)$. Then by the proof of Theorem 3.1 $v_n(t)=\sup_{s\in[t-3q,t]}|x^n(s)|$ is non-increasing for all $t\geqq\tau_n'+2q$ and

$$\varepsilon<|x^n(\tau_n'+T_n')|\leqq v_n(\tau_n'+T_n')\leqq v_n(\tau_n'+2q)<H\,.$$

Hence there exist sequences $\{\tau_n\}$ and $\{T_n\}$ such that

$$\tau_n'+2q\leqq\tau_n<\tau_n+T_n\leqq\tau_n'+T_n'\,,\qquad T_n\to\infty\qquad\text{as}\quad n\to\infty$$

and

$$v_n(\tau_n+T_n)>\theta v_n(\tau_n)\,. \tag{3.17}$$

We show that

$$x^n(t)\neq 0\qquad\text{for all}\quad t\in[\tau_n,\tau_n+T_n]\,. \tag{3.18}$$

Suppose that there exists $t_n\in[\tau_n,\tau_n+T_n]$ such that $x^n(t_n)=0$. Note that by the monotonicity of $v_n(t)$, we have

$$v_n(\tau_n+T_n)\leqq v_n(t_n)\leqq\sup_{s\geqq t_n}|x^n(s)|\,.$$

It follows from Lemma 3.1 that

$$\sup_{s\geqq t_n}|x^n(s)|\leqq\theta\sup_{s\in[t_n-2q,t_n]}|x^n(s)|\leqq\theta\sup_{s\geqq t_n-3q}|x^n(s)|\leqq\theta v_n(\tau_n)\,,$$

which contradicts (3.17). Therefore (3.18) is proved, and then by Lemma 3.1, $|x^n(t)|$ is non-increasing on $[\tau_n+q,\tau_n+T_n]$. Hence

$$|x^n(t)|>\varepsilon\qquad\text{for all}\quad t\in[\tau_n+q,\tau_n+T_n]\,.$$

We may assume that $x^n(t)>\varepsilon$ for all $t\in[\tau_n+q,\tau_n+T_n]$. Then by (A_1) and (3.2),

$$\frac{d}{dt}x^n(t)\leqq-\alpha x^n(t)+\beta M(-x_t^n)<-\alpha\varepsilon \tag{3.19}$$

for all $t\in[\tau_n+2q,\tau_n+T_n]$, which implies that

$$x^n(\tau_n+T_n)\leqq x^n(\tau_n+q)-\alpha\varepsilon(T_n-\tau_n-2q)\to-\infty \quad \text{as} \quad n\to\infty\,,$$

since $T_n\to\infty$. This is a contradiction and completes the proof.

REMARK 3.2. For (3.15), suppose that

$$(\mathrm{A}_2) \text{ holds for some } \quad 0<\beta q<3/2 \quad \text{and} \quad \liminf_{t\to\infty} b(t)>0\,.$$

Then $F(t, \phi)$ satisfies (3.16), and hence by [17] the zero solution of (3.15) is uniformly asymptotically stable. But we can also give a proof quite similar to the proof of Theorem 3.2 as follows: Suppose that $x^n(t)>\varepsilon$ for all $t\in[\tau_n+q, \tau_n+T_n]$. Then by (A_2)

$$\frac{d}{dt}x^n(t)\leqq b(t)\sup_{s\in[t-q,t]}(-x(s))<-\varepsilon b(t)$$

for all $t\in[\tau_n+q, \tau_n+T_n]$, which yields a contradiction.

EXAMPLE 3.1. Let $a, b: [0, \infty)\to[0, \infty)$ and $r: [0, \infty)\to[0, q]$ be continuous functions such that

$$a(t)\geqq\alpha \quad \text{and} \quad 0\leqq b(t)\leqq\beta\,,$$

for some $\alpha>0$ and $\beta>0$, and let $f, g: S(H)\to\boldsymbol{R}$ be continuous functions such that

$$|f(x)|\leqq|x| \quad \text{for} \quad x\in S(H) \quad \text{and} \quad xf(x)>0 \quad \text{for} \quad x\neq0\,,$$

and that

$$|g(x)|\geqq|x| \quad \text{for} \quad x\in S(H) \quad \text{and} \quad xg(x)>0 \quad \text{for} \quad x\neq0\,.$$

Then the delay-differential equation

$$x'(t)=-a(t)g(x(t))-b(t)f(x(t-r(t))) \tag{3.20}$$

satisfies (A_1) and (3.2). Hence, if

$$\alpha\geqq\beta \quad \text{or} \quad \frac{\beta^2}{\alpha^2}\left\{1-\frac{\beta-\alpha}{\beta}\left(1+\frac{2\alpha\beta}{(\beta-\alpha)^2}e^{-\alpha q}\right)^{1/2}\right\}\leqq1\,,$$

Theorems 3.1 and 3.2 imply that

(i) the zero solution of (3.20) is uniformly stable, and

(ii) if $\theta<1$, then the zero solution of (3.20) is uniformly asymptotically stable.

4. Stability region in Quadrant II. In this section, we consider the delay-differential equation

$$x'(t)=G(t, x(t))+F(t, x_t)\,, \tag{4.1}$$

where $G: [0, \infty)\times S(H)\to\boldsymbol{R}$ and $F: [0, \infty)\times C^q(H)\to\boldsymbol{R}$ are continuous. We assume that there exist $\alpha<0$, $\beta>0$ and a non-positive function $a(t)$ on $[0, \infty)$ such that

(4.2) $$a(t) \geqq \alpha \quad \text{and} \quad |G(t, x)| \leqq -a(t)|x|$$

for all $t \geqq 0$ and $x \in S(H)$, and that $F(t, \phi)$ satisfies (A_2). To avoid confusion, we let $\gamma = -\alpha$ and $c(t) = -a(t)$, so that we have

(4.2′) $$c(t) \leqq \gamma \quad \text{and} \quad |G(t, x)| \leqq c(t)|x| .$$

We further assume that

(4.3) $$-a(t) = c(t) \leqq b(t) \qquad \text{for all} \quad t \geqq 0$$

and

(4.4) $$-\alpha = \gamma \leqq \beta \quad \text{and} \quad \gamma q < 1 .$$

For $\alpha < 0$ and $\beta > 0$, the equation (1.1) is an example of (4.1) satisfying (4.2)–(4.3), and if either $-\alpha = \gamma \geqq \beta$ or $\gamma q > 1$, then the zero solution of (1.1) is unstable. Hence, it is quite reasonable to assume (4.4). For a continuous function ψ: $[0, \infty) \to \boldsymbol{R}$, we let $\psi_+(t) = \max\{0, \psi(t)\}$ and $\psi_-(t) = \max\{0, -\psi(t)\}$.

LEMMA 4.1. *For some $t_1 \geqq 0$ and $T > t_1 + q$, let $x(t)$ be a solution of* (4.1) *on* $[t_1, T]$ *such that $x(t) > 0$ for all $t \in (t_1, T]$. Then*

(4.5) $$\int_{t_1+q}^{t} x'_+(s)ds \leqq \frac{\gamma q}{1-\gamma q} \int_{t_1}^{t_1+q} x'_+(s)ds$$

for all $t \in [t_1 + q, T]$.

PROOF. For $t \in [t_1 + q, T]$, choose $q_1 \in [0, q]$ so that $x(t - q_1) = \inf_{s \in [-q, 0]} x(t + s)$. Then by (A_2), (4.2) and (4.3),

$$\begin{aligned} x'(t) &= G(t, x(t)) + F(t, x_t) \leqq c(t)x(t) + b(t) \sup_{s \in [-q, 0]}(-x(t+s)) \\ &\leqq c(t)x(t) - b(t) \inf_{s \in [-q, 0]}(x(t+s)) \leqq c(t)(x(t) - x(t - q_1)) \\ &\leqq \gamma \int_{t-q_1}^{t} x'(s)ds \leqq \gamma \int_{t-q}^{t} x'_+(s)ds . \end{aligned}$$

Hence, integrating $x'_+(t)$ from $t_1 + q$ to t, we have

$$\begin{aligned} \int_{t_1+q}^{t} x'_+(s)ds &\leqq \int_{t_1+q}^{t} \gamma \int_{s-q}^{s} x'_+(u)duds \leqq \gamma \int_{t_1}^{t} x'_+(u) \int_{u}^{u+q} dsdu \\ &\leqq \gamma q \int_{t_1}^{t_1+q} x'_+(u)du + \gamma q \int_{t_1+q}^{t} x'_+(u)du , \end{aligned}$$

which proves (4.5).

In a similar way, we have the following:

LEMMA 4.2. *For some $t_1 \geqq 0$ and $T > t_1 + q$, let $x(t)$ be a solution of* (4.1) *on* $[t_1, T]$ *such that $x(t) < 0$ for all $t \in (t_1, T]$. Then*

$$\int_{t_1+q}^{t} x'_-(s)ds \leqq \frac{\gamma q}{1-\gamma q} \int_{t_1}^{t_1+q} x'_-(s)ds \tag{4.6}$$

for all $t \in [t_1 + q, T]$.

LEMMA 4.3. *For $t_1 \geqq 0$ and $T > t_1$, let $x(t)$ be a solution of* (4.1) *on* $[t_1 - 2q, T]$ *such that $|x(t)| > 0$ for all $t \in [t_1 - q, T]$. Then*

$$|x(t)| \leqq \left(1 + \frac{\gamma q^2}{1-\gamma q}(\gamma+\beta)\right) \sup_{s \in [t_1-2q, t_1]} |x(s)|$$

for all $t \in [t_1, T]$.

PROOF. It is clear from (4.2) and (A_2) that

$$|x'(t)| \leqq (\gamma+\beta)\|x_t\| \qquad \text{for all} \quad t \in [t_1 - q, T].$$

Hence if $x(t) > 0$ for all $t \in (t_1 - q, T]$, then by Lemma 4.1,

$$x(t) \leqq x(t_1) + \int_{t_1}^{t} x'_+(s)ds \leqq x(t_1) + \frac{\gamma q}{1-\gamma q} \int_{t_1-q}^{t_1} (\gamma+\beta)\|x_s\| ds$$

$$\leqq x(t_1) + \frac{\gamma q^2}{1-\gamma q}(\gamma+\beta) \sup_{s \in [t_1-q, t_1]} \|x_s\| \leqq \left(1 + \frac{\gamma q^2}{1-\gamma q}(\gamma+\beta)\right) \sup_{s \in [t_1-2q, t_1]} |x(s)|.$$

The proof in the other case $x(t)$ negative on $(t_1 - q, T]$ is similar.

LEMMA 4.4. *Let $x(t)$ be a solution of* (4.1) *on* $[t_1 - 2q, T]$ *for some $t_1 \geqq 0$ and $T \geqq t_1 + q$ such that $x(t_1) = 0$ and $x(t) > 0$ for all $t \in (t_1, T]$, and let $r = \sup_{s \in [t_1-2q, t_1]} |x(s)|$. Suppose that there exists $\eta \geqq 0$ such that*

$$\frac{\beta(\gamma+\beta)}{\gamma^2}(e^{\gamma q} - 1 - \gamma q) \leqq 1 - \eta - \gamma q \tag{4.7}$$

or

$$\frac{\beta}{\gamma}(e^{\gamma q} - 1) > 1 \quad \text{and} \quad \frac{\beta}{\gamma}\left(e^{\gamma q} - \frac{\gamma+\beta}{\gamma} \log \frac{\gamma+\beta}{\beta}\right) \leqq 1 - \eta - \gamma q. \tag{4.8}$$

Then

$$x(t) \leqq \int_{t_1}^{t} x'_+(s)ds \leqq (1 - \eta - \gamma q)r \qquad \textit{for all} \quad t \in [t_1, t_1 + q].$$

PROOF. We will carry out the proof in a way similar to that for Lemma 3.2. Unfortunately we cannot assume $\beta(e^{\gamma q} - 1)/\gamma \geqq 1$, and for this reason the conditions of the lemma are complicated. Suppose that there exists $t_4 \in (t_1, t_1 + q]$ such that

$x(t_4) > (1-\eta-\gamma q)r$. Then we can choose $t_3 < t_4$ so that $x(t) < (1-\eta-\gamma q)r$ for all $t \in (t_1, t_3)$ and

$$x(t_3) = (1-\eta-\gamma q)r\,. \tag{4.10}$$

By (A_2) and (4.2),

$$x'(t) \leqq \gamma\,|x(t)| + \beta r \qquad \text{for all} \quad t \in [t_1 - 2q,\, t_3]\,,$$

and hence $x(t_1) = 0$ yields that $x(t) \leqq (e^{\gamma(t-t_1)} - 1)\beta r/\gamma$ for all $t \in [t_1, t_3]$. Therefore we have

$$x'_+(t) \leqq \beta r e^{\gamma(t-t_1)} \qquad \text{for all} \quad t \in [t_1, t_3]\,. \tag{4.11}$$

Further, it follows from Lemma 2.1 that

$$|x(t)| \leqq \frac{\beta}{\gamma} r(e^{\gamma(t_1-t)} - 1) \qquad \text{for all} \quad t \in [t_1 - 2q,\, t_1]\,.$$

Then by (A_2) and (4.2) again, we have for $t \in [t_1, t_3]$

$$\begin{aligned} x'(t) &\leqq \gamma x(t) + \beta \sup\nolimits_{s \in [t-q,\, t]}(-x(s)) \leqq \gamma x(t) + \beta \sup\nolimits_{s \in [t-q,\, t_1]} |x(s)| \\ &\leqq \beta r(e^{\gamma(t-t_1)} - 1) + \frac{\beta^2}{\gamma} r(e^{\gamma(t_1+q-t)} - 1)\,. \end{aligned} \tag{4.12}$$

Therefore, by (4.11) and (4.12), we have for $t \in [t_1, t_3]$,

$$x(t) \leqq \int_{t_1}^{t} x'_+(s)ds \leqq \beta r \int_{t_1}^{t} (e^{\gamma(s-t_1)} - 1)ds + \int_{t_1}^{t} \min\left\{\beta r,\, \frac{\beta^2}{\gamma} r(e^{\gamma(t_1+q-s)} - 1)\right\} ds\,.$$

Suppose (4.7) holds. It then follows from $t_3 < t_1 + q$ that

$$\begin{aligned} x(t) &< \beta r \int_{t_1}^{t_1+q} (e^{\gamma(s-t_1)} - 1)ds + \frac{\beta^2}{\gamma} r \int_{t_1}^{t_1+q} (e^{\gamma(t_1+q-s)} - 1)ds \\ &= \frac{\beta(\gamma+\beta)}{\gamma^2}(e^{\gamma q} - 1 - \gamma q)r \leqq (1-\eta-\gamma q)r\,. \end{aligned}$$

Next suppose (4.8) holds and choose $t_2 \in [t_1, t_1 + q)$ so that

$$e^{\gamma(t_1+q-t_2)} = (\gamma+\beta)/\beta\,.$$

Then we have for $t \in [t_1, t_3]$,

$$\begin{aligned} x(t) &\leqq \int_{t_1}^{t} x'_+(s)ds < \beta r \int_{t_1}^{t_1+q} (e^{\gamma(s-t_1)} - 1)ds + \int_{t_1}^{t_2} \beta r ds + \frac{\beta^2}{\gamma} r \int_{t_2}^{t_1+q} (e^{\gamma(t_1+q-s)} - 1)ds \\ &= \frac{\beta}{\gamma}\left(e^{\gamma q} - \frac{\gamma+\beta}{\gamma}\log\frac{\gamma+\beta}{\beta}\right) r \leqq (1-\eta-\gamma q)r\,. \end{aligned}$$

In either case, we have a contradiction to (4.10) at $t = t_3$, and at the same time we have

$$\int_{t_1}^{t} x'_+(s)ds \leqq (1-\eta-\gamma q)r \qquad \text{for all} \quad t\in[t_1, t_1+q].$$

The proof is now complete.

Similarly,we have a lemma for negative solutions of (4.1):

LEMMA 4.5. *Let $x(t)$ be a solution of* (4.1) *on $[t_1-2q, T]$ for some $t_1 \geqq 0$ and $T \geqq t_1+q$ such that $x(t_1)=0$ and $x(t)<0$ for all $t\in(t_1, T]$, and let $r=\sup_{s\in[t_1-2q,t_1]}|x(s)|$. Suppose that there exists $\eta\geqq 0$ such that* (4.7) *or* (4.8) *holds. Then*

$$|x(t)| \leqq \int_{t_1}^{t} x'_-(s)ds \leqq (1-\eta-\gamma q)r \qquad \textit{for all} \quad t\in[t_1, t_1+q].$$

THEOREM 4.1. *Suppose that there exist $\alpha<0$, $\beta>0$, $a(t)<0$ and $b(t)>0$ satisfying* (A_2) *and* (4.2)–(4.4), *and*

$$\frac{\beta(\beta-\alpha)}{\alpha^2}(e^{-\alpha q}-1+\alpha q) \leqq 1+\alpha q \tag{4.13}$$

or

$$-\frac{\beta}{\alpha}(e^{-\alpha q}-1)>1 \quad \textit{and} \quad -\frac{\beta}{\alpha}\left(e^{-\alpha q}+\frac{\beta-\alpha}{\alpha}\log\frac{\beta-\alpha}{\beta}\right) \leqq 1+\alpha q. \tag{4.14}$$

Then the zero solution of (4.1) *is uniformly stable.*

PROOF. As above, we let $\gamma=-\alpha$, and show that

$$|x(t; t_0, \phi)| \leqq \|\phi\|\left(1+\frac{\gamma q^2}{1-\gamma q}(\gamma+\beta)\right)e^{2(\gamma+\beta)q} \tag{4.15}$$

for all $0\leqq t_0 \leqq t$ and $\phi\in C^q(H)$. Let

$$\rho=\|\phi\|\left(1+\frac{\gamma q^2}{1-\gamma q}(\gamma+\beta)\right)e^{2(\gamma+\beta)q}$$

and suppose (4.15) is false for some solution $x(t)=x(t; t_0, \phi)$ of (4.1). Then there exists $t_5>t_0$ such that $|x(t_5)|>\rho$. By (A_2) and (4.2), $|x'(t)|\leqq(\gamma+\beta)\|x_t\|$ and hence $|x(t)|\leqq\|\phi\|e^{2(\gamma+\beta)q}$ for all $t\in[t_0-q, t_0+2q]$, which shows $t_0+2q<t_5$. Then it follows from Lemma 4.3 that there exists $t_1\in[t_0+q, t_5)$ such that $x(t_1)=0$. Choose t_2, t_3 and t_4 so that $t_0+q\leqq t_1\leqq t_2<t_3<t_4\leqq t_5$, $x(t_2)=0$, $0<|x(t)|\leqq\rho$ for all $t\in[t_2, t_3]$ and

$$|x(t)|>\rho \qquad \text{for all} \quad t\in(t_3, t_4]. \tag{4.16}$$

We may suppose that $x(t)>0$ for all $t\in(t_2, t_4]$, since the argument for the other case is similar. By letting $\eta=0$ in Lemma 4.4, we have

$$x(t) \leqq \int_{t_2}^{t_2+q} x'_+(s)ds \leqq (1-\gamma q)\rho$$

for all $t \in [t_2, t_2+q]$ and by Lemma 4.1

$$x(t_4) \leqq x(t_2+q) + \int_{t_2+q}^{t_4} x'_+(s)ds \leqq (1-\gamma q)\rho + \frac{\gamma q}{1-\gamma q}(1-\gamma q)\rho = \rho\,,$$

which contradicts (4.16). This completes the proof.

REMARK 4.1. The region S_2 of points (α, β) which satisfy the conditions of Theorem 4.1 is illustrated in Figure 2. Letting $\alpha \to 0$ in (4.13) and (4.14), we have $\beta q \leqq 3/2$.

We shall study the asymptotic stability of the zero solution of (4.1). Consider the equation (1.1) in which $-\alpha = \beta < 1/q$. Then the zero solution is uniformly stable but not asymptotically stable. So we have to add an assumption to insure the asymptotic stability for (4.1). We assume for (4.1) that

$$\liminf_{t \to \infty} (a(t) + b(t)) > 0\,. \tag{4.17}$$

Then there exist $\delta > 0$ and $T_1 > 0$ such that

$$a(t) + b(t) \geqq \delta \qquad \text{for all} \quad t \geqq T_1\,. \tag{4.18}$$

First, we consider an eventually positive solution of (4.1): Let $x(t)$ be a solution of (4.1) such that there exists $T_2 \geqq T_1$ such that

$$x(t) > 0 \qquad \text{for all} \quad t \geqq T_2\,. \tag{4.19}$$

Then by Lemma 4.1,

$$\int_{T_2}^{\infty} x'_+(s)ds < \infty\,. \tag{4.20}$$

Suppose that $\liminf_{t\to\infty} x(t) > 0$. Then there exist $\varepsilon > 0$ and $T_3 \geqq T_2$ such that

$$x(t) > \varepsilon \qquad \text{for all} \quad t \geqq T_3\,.$$

By (4.20), we can choose $T_4 \geqq T_3$ so that

$$\int_{T_4}^{\infty} x'_+(t)dt < \varepsilon\delta/\gamma\,.$$

As in the proof of Lemma 4.1, by (A_2), (4.2) and (4.18),

$$x'(t) \leqq c(t)x(t) + b(t) \sup_{s\in[-q,0]}(-x(t+s))$$
$$\leqq \gamma \int_{t-q}^{t} x'_+(s)ds - (b(t)-c(t)) \inf_{s\in[-q,0]} x(t+s) \leqq \gamma \int_{T_4}^{\infty} x'_+(s)ds - \varepsilon\delta < 0$$

for all $t \geqq T_4 + q$. Hence

$$x(t) \to -\infty \qquad \text{as} \quad t \to \infty ,$$

which contradicts (4.19). Thus we have

$$\liminf_{t\to\infty} x(t) = 0 . \tag{4.21}$$

Suppose that $\limsup_{t\to\infty} x(t) > 0$. Then by (4.21), there exist $\varepsilon > 0$ and two sequences $\{\tau_n\}$ and $\{t_n\}$ tending to ∞ such that $\tau_n < t_n < \tau_{n+1}$, $x(\tau_n) = \varepsilon/2$, $\varepsilon/2 < x(t) < \varepsilon$ for all $t \in (\tau_n, t_n)$ and $x(t_n) = \varepsilon$. Hence by (4.20)

$$\varepsilon/2 = x(t_n) - x(\tau_n) \leqq \int_{\tau_n}^{t_n} x'_+(s)ds \to 0 \qquad \text{as} \quad n \to \infty ,$$

which yields a contradiction. Thus we have

$$\lim_{t\to\infty} x(t) = 0 \tag{4.22}$$

for any eventually positive solution $x(t)$ of (4.1). In a similar way, we can show (4.22) for any eventually negative solution $x(t)$ of (4.1). In order to complete the proof of the asymptotic stability of the zero solution of (4.1), we show (4.22) for any oscillatory solution of (4.1). Suppose that

$$\frac{\beta(\gamma+\beta)}{\gamma^2}(e^{\gamma q} - 1 - \gamma q) < 1 - \gamma q$$

or

$$\frac{\beta}{\gamma}(e^{\gamma q} - 1) > 1 \quad \text{and} \quad \frac{\beta}{\gamma}\left(e^{\gamma q} - \frac{\gamma+\beta}{\gamma}\log\frac{\gamma+\beta}{\beta}\right) < 1 - \gamma q .$$

Then there exists $\eta > 0$ such that (4.7) or (4.8) holds. Let $x(t)$ be a solution of (4.1) such that there exists a sequence $\{t_n\}$ tending to ∞ with $x(t_n) = 0$ and $x(t) \neq 0$ for $t \neq t_n$. Let $r_n = \sup_{s\in[t_n-2q,t_n]} |x(s)|$. In order to prove (4.22), it suffices to show that for each n,

$$|x(t)| \leqq (1-\eta)r_n \qquad \text{for all} \quad t \in [t_n, t_{n+1}] .$$

We may assume that $x(t) > 0$ for all $t \in (t_n, t_{n+1})$, since the proof in the other case is similar. If $t_{n+1} \leqq t_n + q$, then by Lemma 4.4,

$$x(t) \leqq (1 - \eta - \gamma q)r_n \leqq (1-\eta)r_n \qquad \text{for all} \quad t \in (t_n, t_{n+1}) .$$

If $t_{n+1} > t_n + q$, then by Lemma 4.4,

$$x(t) \leqq \int_{t_n}^{t} x'_+(s)ds \leqq (1-\eta-\gamma q)r_n \qquad \text{for all} \quad t \in (t_n, t_n+q]\,,$$

and by Lemma 4.1,

$$x(t) \leqq x(t_n+q) + \int_{t_n+q}^{t} x'_+(s)ds \leqq (1-\eta-\gamma q)r_n + \frac{\gamma q}{1-\gamma q}(1-\gamma q)r_n = (1-\eta)r_n$$

for $t \in (t_n, t_{n+1})$. Thus we have the following:

THEOREM 4.2. *Suppose that there exist* $\alpha < 0$, $\beta > 0$, $a(t) < 0$ *and* $b(t) > 0$ *satisfying* (A_2), (4.2)–(4.4) *and* (4.17). *Further suppose that*

$$\frac{\beta(\beta-\alpha)}{\alpha^2}(e^{-\alpha q} - 1 + \alpha q) < 1 + \alpha q \tag{4.23}$$

or

$$-\frac{\beta}{\alpha}(e^{-\alpha q} - 1) > 1 \quad \text{and} \quad -\frac{\beta}{\alpha}\left(e^{-\alpha q} + \frac{\beta-\alpha}{\alpha}\log\frac{\beta-\alpha}{\beta}\right) < 1 + \alpha q\,. \tag{4.24}$$

Then the zero solution of (4.1) *is asymptotically stable.*

EXAMPLE 4.1. Let $a, b\colon [0, \infty) \to [0, \infty)$ and $r\colon [0, \infty) \to [0, q]$ be continuous functions such that

$$\alpha \leqq a(t) \leqq 0 \leqq b(t) \leqq \beta \quad \text{and} \quad a(t) + b(t) \geqq 0$$

for some $\alpha < 0$ and $\beta > 0$, and let $g\colon S(H) \to \boldsymbol{R}$ be a continuous function such that

$$|g(x)| \leqq |x| \quad \text{for} \quad x \in S(H) \quad \text{and} \quad xg(x) > 0 \quad \text{for} \quad x \neq 0\,.$$

Then the delay-differential equation

$$x'(t) = -a(t)g(x(t)) - b(t)x(t - r(t)) \tag{4.25}$$

satisfies (A_2) and (4.2)–(4.4). Hence it follows from Theorems 4.1 and 4.2 that

(i) if α and β satisfy (4.13) or (4.14), then the zero solution of (4.25) is uniformly stable, and

(ii) if $a(t)$ and $b(t)$ satisfy (4.17) and if α and β satisfy (4.23) or (4.24), then the zero solution of (4.25) is asymptotically stable.

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