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FACTORIZATION OF COMPACT COMPLEX 3-FOLDS WHICH ADMIT CERTAIN PROJECTIVE STRUCTURES

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A complex manifold X, $\dim X=3$, is of Class L, if, by definition, X contains a subdomain which is biholomorphic to a neighborhood of a projective line in a complex projective space of dimension three. In [Ka2][Ka3], we have defined complex analytic connected sum (which was called "connecting operation") of manifolds of Class L. In this paper, we shall consider how to factorize a compact manifold of Class L into prime ones. To describe our results, we introduce Klein combination of manifolds of Class L, which is a generalization of complex analytic connected sum. Our first result is that, *if a compact manifold of Class* L *is of Schottky type, then it is a Klein combination of Blanchard manifolds and* L-*Hopf manifolds* (Theorem A) (see § 1 for the definitions). This result is an analogue of Kulkarni's [Ku]. We shall prove some properties of L-Hopf manifolds (Theorem B, § 4) and give a rough classification of Blanchard manifolds (Theorem C, § 5). There are many manifolds of Schottky type. In fact, we see that *a complex analytic connected sum of Blanchard manifolds and* L-*Hopf manifolds is of Schottky type* (Theorem D).

Our work is motivated and strongly influenced by that of Kulkarni [Ku]. Theorem A and its proof is an analogue of his Theorem 6.3 and its proof.

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CONTENTS

1. Definitions and Statements of Results.

Let Ω be a subdomain in a complex

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projective space of dimension three, which is denoted by $\boldsymbol{P}^3$. Let Γ be a subgroup of $PGL(4, \boldsymbol{C})$ acting freely and properly discontinuously on Ω. We shall denote by $\Gamma\backslash\Omega$ the quotient space of Ω by the action of Γ. A compact manifold of Class L is called a *(P)-manifold* if the manifold is of the form $\Gamma\backslash\Omega$ and if Ω is *simply connected.*

DEFINITION 1.1. A compact manifold X of Class L is of *Schottky type* if X is represented as a quotient space $\Gamma\backslash\Omega$, where Ω is a subdomain in $\boldsymbol{P}^3$ such that any connected component of the complement $\Lambda := \boldsymbol{P}^3 - \Omega$ consists of a single projective line, and that Γ is a group of holomorphic automorphisms of Ω whose action is properly discontinuous and free.

Let us define two kinds of Schottky type manifolds, L-*Hopf manifolds* and *Blanchard manifolds.*

A *Blanchard manifold* is a compact complex manifold whose universal covering is biholomorphic to $\boldsymbol{P}^3 - \{\text{a single line}\}$. For more information, see §5.

An L-*Hopf manifold* (Hopf-like manifold of Class L) is a compact complex manifold whose universal covering is biholomorphic to $\boldsymbol{P}^3 - \{\text{two lines without intersection}\}$. An L-Hopf manifold is said to be *primary* if its fundamental group is infinite cyclic. For more information, see §4.

In the following, we shall use the term "complex analytic connected sum", which is the same as the "connecting operation" introduced in [Ka2], [Ka3]. The term "connected sum" will also be used, if there is not chance of confusion with the standard connected sum in differential topology.

Klein combination of Class L manifolds is a generalization of complex analytic connected sum, which is defined as follows. Let X_ν, $\nu = 1, 2$, be manifolds of Class L. Let Σ be a connected and simply connected smooth real hypersurface in $\boldsymbol{P}^3$, and W a tubular neighborhood of Σ. Let W'_1 and W'_2 be the connected components of $\boldsymbol{P}^3 - \Sigma$. Put $W_1 = W'_1 \cup W$ and $W_2 = W'_2 \cup W$. Suppose that there are open embeddings $j_\nu\colon W_\nu \to X_\nu$. Then the Klein combination $\mathrm{Kl}(X_1, X_2, j_1, j_2, \Sigma)$ of X_1 and X_2 is the union $X_1^\# \cup X_2^\#$, $X_\nu^\# = X_\nu - j_\nu(W_\nu - W)$, where $j_1(x) \in j_1(W)$, $x \in W$, is identified with $j_2(x) \in j_2(W)$ (see Figure 1). Note that we can define the Klein combination for any Σ and any pair X_1, X_2 of Class L manifolds, provided that both W_1 and W_2 are of Class L. For a sequence of manifolds $X_1, X_2, \cdots, X_s$ of Class L, we can consider their Klein combination inductively as $Y_k = \mathrm{Kl}(Y_{k-1}, X_k, j_{k-1}, j_k, \Sigma_{k-1})$, $k \geqq 2$, and $Y_1 = X_1$. If, in particular, W'_1 is biholomorphic to the domain

$$U = \{[z_0 : z_1 : z_2 : z_3] = \boldsymbol{P}^3 : |z_0|^2 + |z_1|^2 < |z_2|^2 + |z_3|^2\}\,,$$

and if Σ is CR-isomorphic to ∂U, then the Klein combination $\mathrm{Kl}(X_1, X_2, j_1, j_2, \partial U)$ is nothing but a complex analytic connected sum of X_1 and X_2, which we denote by $\mathrm{Sum}(X_1, X_2, j_1, j_2)$. When the explicit expression for the embeddings $j_\nu\colon U \to X_\nu$ is not necessary, we abbreviate $\mathrm{Sum}(X_1, X_2, j_1, j_2)$ as $\mathrm{Sum}(X_1, X_2)$. Note that the complex structure of $\mathrm{Sum}(X_1, X_2, j_1, j_2)$ depends not only on X_1, X_2 but also on the choice of

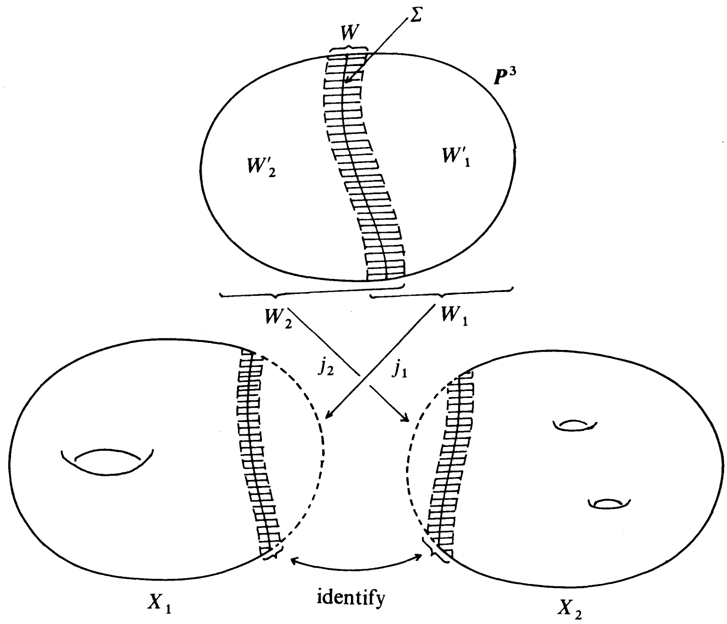

FIGURE 1.

j_1, j_2. See for example [Y].

A manifold X of Class L is said to be *prime* if $X = \mathrm{Sum}\ (X_1, X_2, j_1, j_2)$ for some manifolds X_1, X_2 of Class L implies one of X_1, X_2 is $\boldsymbol{P}^3$.

By a *line* l, we shall mean a non-singular rational curve in a manifold of Class L which has a tubular neighborhood W and a biholomorphic map j: $W \to U$ such that $j(l)$ is a projective line in $\boldsymbol{P}^3$.

For groups $G_1, \cdots, G_s$, we denote by $G_1 * G_2 * \cdots * G_s$ their free product.

Now we shall state our results.

THEOREM A. *Let $X = \Gamma \backslash \Omega$ be a compact manifold of Schottky type. Assume that Ω is simply connected and Γ is torsion free. Then Γ can be written as a free product of subgroups*

$$\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_r * \Gamma_{r+1} * \cdots * \Gamma_s,$$

where r, $0 \leqq r \leqq s$, is an integer such that

(i) *each Γ_i, $1 \leqq i \leqq r$, is an infinite cyclic group,*

(ii) *each Γ_i, $r < i \leqq s$, contains a rank 4 free abelian subgroup of finite index,*

(iii) *X is a Klein combination of r times primary* L-*Hopf manifolds and $s-r$ times Blanchard manifolds.*

THEOREM B. *Any* L-*Hopf manifold admits a primary* L-*Hopf manifold as a finite* *unramified covering. An* L-*Hopf manifold is primary if and only if its fundamental group is torsion free. Any primary* L-*Hopf manifold is biholomorphic to* M_g *(for the definition of* M_g*, see* § 4).

THEOREM C. *Let* $\Gamma\backslash\Omega$ *be any Blanchard manifold. Then* Γ *is torsion free and contains an abelian subgroup* Γ_1 *of rank* 4 *with* $[\Gamma:\Gamma_1]<+\infty$. *Moreover we can choose* Γ_1 *so that it is conjugate in* $PGL(4, \boldsymbol{C})$ *to a subgroup of either*

(A)
$$\left\{\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix};\quad a, b, c \in \boldsymbol{C}\right\}$$

or

(B)
$$\left\{\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};\quad a, b, c, d \in \boldsymbol{C}\right\}.$$

If Γ_1 *is conjugate to a subgroup of* (B), *then any element* $g \in \Gamma_1$ *except the identity satisfies* $\operatorname{rank}(I-g)=2$.

THEOREM D. *Suppose that* X *is a complex analytic connected sum of several copies of* L-*Hopf manifolds and Blanchard manifolds. Then* X *is a* (*P*)-*manifold of Schottky type.*

Both Blanchard manifolds and L-Hopf manifolds are prime. But Theorem A does not tell us that a manifold of Schottky type is a connected sum of several prime manifolds of Class L. In fact, there is an example of Schottky type manifolds whose fundamental group is a free group on two generators but not biholomorphic to a connected sum of L-Hopf manifolds. It might be true that a Schottky type manifold is a complex analytic deformation of a connected sum of Blanchard manifolds and L-Hopf manifolds. But for the moment, we cannot prove this assersion.

NOTATION AND SIMPLE REMARKS. Let S be a subspace of a topological space T. Then $[S]_T$ denotes the closure of S in T. The interior of S is denoted by $(S)_T$. The boundary of S is defined to be $[S]_T-(S)_T$ and denoted by ∂S_T. By $[S]$ (resp. ∂S), we shall mean $[S]_{\boldsymbol{P}^3}$ (resp. $\partial S_{\boldsymbol{P}^3}$).

An n-cell is denoted by B^n. An n-standard sphere is denoted by S^n.

For a number $\varepsilon \geqq 1$, we put

$$U_\varepsilon=\{[z_0:z_1:z_2:z_3]\in \boldsymbol{P}^3 : |z_0|^2+|z_1|^2<\varepsilon(|z_2|^2+|z_3|^2)\},$$

and $N_\varepsilon = U_\varepsilon - [U_{1/\varepsilon}]$. For $\varepsilon = 1$, we simply denote $U = U_1$. We remark that U_ε is biholomophic to U for any ε. It is easy to show that $[U_\varepsilon]$ is diffeomorphic to $S^2 \times B^4$ and that $\partial[U_\varepsilon]$ is diffeomorphic to $S^2 \times S^3$.

2. Topological preparation. The following proposition says that a (P)-manifold can be represented as a Klein combination of simpler ones. Essentially, this is a result in topology. Our proof is an imitation of the argument of Hempel [H, pp. 60–62 and pp. 66–67].

PROPOSITION 2.1. *Suppose that $X = \Gamma \backslash \Omega$ is a (P)-manifold. Suppose further that Γ is isomorphic to the free product of two groups G_1 and G_2 as an abstract group. Then we have the following:*

(i) *There are two (P)-manifolds $X_\nu = \Gamma_\nu \backslash \Omega_\nu$, $\nu = 1, 2$, such that $\Gamma_\nu \cong G_\nu$.*

(ii) *There are connected subdomains Y_ν in X with $X = [Y_1]_X \cup [Y_2]_X$, and with connected, simply connected common boundaries $\partial[Y_1]_X$ and $\partial[Y_2]_X$, $\partial[Y_1]_X = \partial[Y_2]_X$.*

(iii) *There are embeddings $j_\nu : [Y_\nu]_X \to X_\nu$ such that the induced morphisms $j_{\nu *} : \pi_1([Y_\nu]_X) \to \pi_1(X_\nu)$ are isomorphisms.*

(iv) *The union $(X_1 - j_1(Y_1)) \cup (X_2 - j_2(Y_2))$ is biholomorphic to $\boldsymbol{P}^3$, if the boundary $\partial[j_1(Y_1)]_X$ is identified with $\partial[j_2(Y_2)]_X$ by $j_2 \circ j_1^{-1}$.*

PROOF. For a subcomplex A in a complex, let Int A denote the interior and δA the boundary in the sence of simplicial complexes. Choose simplicial complexes C_ν, $\nu = 1, 2$, with $\pi_1(C_\nu) \cong G_\nu$ and $\pi_q(C_\nu) = 0$ for $q \geqq 2$. Join a point of C_1 with a point of C_2 by a 1-simplex A to form a complex $C = C_1 \cup A \cup C_2$. Since $\pi_1(C) \cong G_1 * G_2$ and $\pi_q(C) = 0$ for $q \geqq 2$, we can construct a continuous mapping $f : X \to C$ such that the induced map $f_* : \pi_1(X) \to \pi_1(C)$ is an isomorphism. Take a point $0 \in \operatorname{Int} A$ and put $Z = f^{-1}(0)$. Modifying f within the homotopy class, we may assume that Z is a finite union of smooth real hypersurfaces.

LEMMA 2.2. *The map f can be chosen so that $Z = f^{-1}(0)$ is simply connected.*

PROOF. Put

$$f_{[0]} = f \quad \text{and} \quad Z_{[0]} = Z. \tag{2.3}$$

Suppose that $Z_{[0]}$ contains a non-simply connected component. Let γ be a loop in $Z_{[0]}$ which represents a generator of $\pi_1(Z_{[0]})$. Since $f_* : \pi_1(X) \to \pi_1(C)$ is bijective, and since $f_{[0]}(Z_{[0]}) = \{0\}$, γ is homotopic to 0 in X. Since $\dim_{\boldsymbol{R}} X = 6 > 5$, we can choose a continuous embedding $h : (B^2, \delta B^2) \to (X, Z_{[0]})$ with $h(\delta B^2) = \gamma$, where B^2 denotes the 2-dimensional ball. We may further assume that $h(B^2)$ intersects $Z_{[0]}$ transversely. Then $h^{-1}(Z_{[0]})$ consists of a finite number of disjoint simple closed curves containing δB^2 as a connected component. Let E be a 2-cell in B^2 such that $E \cap h^{-1}(Z_{[0]}) = \delta E$ and Z' be the component of $Z_{[0]}$ containing $h(\delta E)$. Let $D^6 = B^2 \times B^4$ be a small regular neighborhood of $h(E^2)$ in X such that $N = D^6 \cap Z'^5 \cong S^1 \times B^4$ is a tubular neighborhood of $h(\delta E^2) \cong S^1$ (see

[H, p. 7] for the definition of regular neighborhoods). Let $T=\delta D-N$ and choose $E'\cong B^2\times S^3$ properly embedded in D with $\delta E'=\delta T$ (see Figure 2). We define $f_1: X\to C$

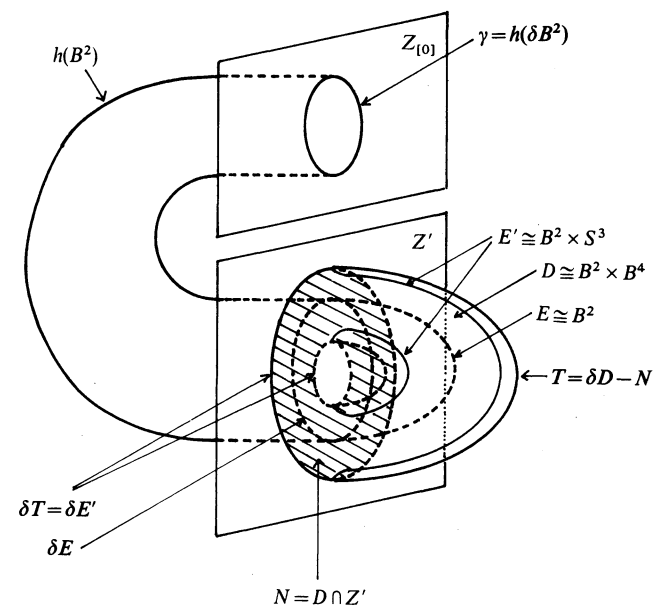

FIGURE 2.

as follows. Put

$$f_1\,|\,(X-(\operatorname{Int} D\cup \operatorname{Int} N))=f\,|\,(X-(\operatorname{Int} D\cup \operatorname{Int} N)).$$

We can extend $f_1\,|\,\delta E'$ to the map which sends E' to the point $0\in A$. Since $\pi_q(C)=0$ for all $q\geqq 2$, we can extend f_1 over $D-E'$ in such a way that $f_1^{-1}(0)\cap D=E'$. Then $f_1^{-1}(0)=(f_{[0]}^{-1}(0)-N)\cup E'$. Put $Z_1=f_1^{-1}(0)$. Then the number of connected components of $h^{-1}(Z_1)$ decreases by one. By van Kampen's theorem, there is a natural surjection $\pi_1(Z_{[0]})\to\pi_1(Z_1)$. Again choosing a 2-cell E_1 in B^2 such that $E_1\cap h^{-1}(Z_1)=\delta E_1$, we can modify f_1 and Z_1 to obtain f_2, $Z_2=f_2^{-1}(0)$ and a natural surjection $\pi_1(Z_1)\to\pi_1(Z_2)$. Continuing this step as many times as the number of connected components of $h^{-1}(Z_{[0]})$, we obtain $f_{[1]}$ and $Z_{[1]}:=f_{[1]}^{-1}(0)$ such that $h^{-1}(Z_{[1]})$ is empty. At the same time we have a surjection $\pi_1(Z_{[0]})\to\pi_1(Z_{[1]})$. Note that the element of $\pi_1(Z_{[0]})$ represented by γ vanishes in $Z_{[1]}$. Replacing $f_{[0]}$ and $Z_{[0]}$ of (2.3) by $f_{[1]}$ and $Z_{[1]}$ respectively, we repeat the argument from (2.3). Continuing this process, we have a sequence of surjections

$$\pi_1(Z)=\pi_1(Z_{[0]})\to\pi_1(Z_{[1]})\to\pi_1(Z_{[2]})\to\cdots.$$

In each step, at least one of the images of the generators of $\pi_1(Z)$ is mapped to 0. Since each map in the sequence is surjective and since $\pi_1(Z)$ is finitely generated, we see that $\pi_1(Z_{[n]})=0$ for a sufficiently large n. This proves the lemma. ■

LEMMA 2.3. *The map f of Lemma 2.2 can be chosen so that $Z=f^{-1}(0)$ is connected and simply connected.*

PROOF. By Lemma 2.2, we may assume that every component of Z is simply connected. Suppose that Z is not connected. Then there is a path $\beta: [0, 1]\to X$ such that $\beta(0)$ and $\beta(1)$ lie in different components of Z. Now $f\circ\beta$ is a loop in X and since $f_*: \pi_1(X)\to\pi_1(C)$ is surjective, there is a loop γ based at $\beta(1)$ with the relation of homotopy classes $[f\circ\gamma]=[f\circ\beta]^{-1}$. Then $\alpha=\beta\circ\gamma$ is a path satisfying (i) $\alpha(0)$ and $\alpha(1)$ are in different components of $f^{-1}(0)$, and (ii) the homotopy class $[f\circ\alpha]$ is trivial in $\pi_1(C)$. We may assume that α is a simple path which crosses Z transversely at each point of $\alpha((0, 1))$. Of all such paths satisfying the above conditions, we assume that $\#\{\alpha^{-1}(Z)\}$ is minimal. We must have $\alpha((0, 1))\cap Z=\varnothing$. For, if not, we can write $\alpha=\alpha_1\cdot\alpha_2\cdots\alpha_k$ $(k\geqq 2)$ where for each i, $\alpha_i((0, 1))\cap Z=\varnothing$ and $\{\alpha_i(0), \alpha_i(1)\}\subset Z$. Then $[f\cdot\alpha_1]\cdot[f\cdot\alpha_2]\cdots[f\cdot\alpha_k]$ is a representation of the identity element as an alternating product in the free product $\Gamma_1 * \Gamma_2$. Thus for each i, $[f\cdot\alpha_i]=1$ holds. If $\alpha_i(0)$ and $\alpha_i(1)$ lie in the same component of Z, we could reduce $\#\{\alpha^{-1}(Z)\}$. If $\alpha_i(0)$ and $\alpha_i(1)$ do not lie in the same component of Z, we contradict our minimality assumption. Thus we have $\alpha((0, 1))\cap Z=\varnothing$. Let Z_j, $j=0, 1$, be the components of Z containing $\alpha(j)$. Let N be a small regular neighborhood of $\alpha([0, 1])$ such that $N\cap Z_j=D_j$ is a spanning 5-cell of N and $N\cap Z=D_0\cup D_1$. Let B be the difference of spheres in δN bounded by $\delta D_0\cup\delta D_1$. Push $\operatorname{Int} B$ slightly into $\operatorname{Int} N$ to obtain a difference of spheres B' with $\delta B=\delta B'$ and $B\cup B'$ the boundary of $T\cong B^2\times S^4$ in N. We define a map $f_1: X\to C$ as follows. Put $f_1|(X-P)=f|(X-P)$ and $f_1(B')=0$, where $P=\operatorname{Int} N\cup\operatorname{Int} D_0\cup\operatorname{Int} D_1$. Since $[f\cdot\alpha]=1$, we can extend f_1 across a 2-cell $B^2\times\{q\}$, where $q\in S^4$. Now it remains to extend f_1 across the remaining two open 6-cells; this can be done since $\pi_5(C_v)=0$. The extension can be so chosen that $f_1^{-1}(0)\cap N=B'$. Thus $f_1^{-1}(0)=(f^{-1}(0)-(D_0\cup D_1))\cup B'$ is simply connected and has one less component than $f^{-1}(0)$. The proof is completed by induction. ■

Lemma 2.4. *The map f of Lemma 2.3 can be so chosen that both of the two connected components of the complement $X-f^{-1}(0)$ contain lines.*

PROOF. Suppose that we are given a line l in X. First we consider the case $l\cap(X-f^{-1}(0))=\varnothing$. In this case we can choose another line l' near l so that l and l' are in the same connected component. Now we are going to modify f so that $f^{-1}(0)$ separates these two lines. Let V be a small tubular neighborhood of l, which does not intersect l'. Let $\alpha:[0, 1]\to X$ be a path connecting a point $\alpha(0)$ on $f^{-1}(0)$ and a point $\alpha(1)$ on ∂V_X satisfying $\alpha((0, 1))\cap(l'\cup[V]_X\cup f^{-1}(0))=\varnothing$. Let N be a regular neighborhood of $\alpha([0, 1])$ such that $D_0=N\cap f^{-1}(0)\cong B^5$ and $D_1=N\cap\partial V_X\cong B^5$. Put $M=(\partial V_X-\operatorname{Int} D_1)\cup\delta N$, which is diffeomorphic to $S^2\times S^3$ with a 5-cell deleted and $\delta M=\delta D_0$. Push $\operatorname{Int} M$ slightly into $\operatorname{Int}(V\cup N)$ to obtain a real 5-manifold M' with $\delta M'=\delta M$ embedded properly in $\operatorname{Int}(V\cup N)$. We define a map $f_1: X\to C$ as follows. Put

$f_1 \mid (X-P)=f \mid (X-P)$ and $f_1(M')=0$, where $P=\mathrm{Int}(V\cup N)\cup \mathrm{Int}\, D_0$. Now it remains to extend f_1 across the remaining two open sets, the set $W_1 \cong B^1 \times (S^2 \times S^3 - B^5)$ bounded by M and M', and the set $W_2 \cong B^4 \times S^2$ bounded by M' and D_0. Since $\pi_2(C)=0$ and $\pi_3(C)=0$, f_1 can be extended across W_1 and W_2 so that $f_1^{-1}(0)\cap (V\cup N)=M'$ and $f_1^{-1}(0)=(f^{-1}(0)-D_0)\cup M'$. Thus $f_1: X\to C$ separates l and l'. Next we shall consider the case where the given line l intersects $f^{-1}(0)$. In this case, using the method employed in the proof of Lemma 2.2, we can modify f so that $f^{-1}(0)$ does not intersect l. Thus we have proved the lemma. ■

We insert here a gereral remark on *fundamental regions*. Suppose that a group Γ acts on a differentiable manifold Ω and that the action is free and properly discontinous. A closed subset F in Ω is a *fundamental region* for the group Γ if

(1) $\mathrm{Int}\, F$ is connected and $F=[\mathrm{Int}\, F]_\Omega$;

(2) Not two distinct points of $\mathrm{Int}\, F$ belong to the same Γ-orbit;

(3) Every Γ-orbit intersects F.

Assume that the quotient manifold $X=\Gamma\backslash\Omega$ is compact. Fix a triangulation of X such that each simplex is evenly covered by the natural projection $\Omega\to X$. Lifting this triangulation to Ω, we get a triangulation of Ω. We can construct easily a fundamental region for Γ as a connected finite subcomplex. Assume further that a simply connected subcomplex S is given in X. Let $\tilde{S}$ be a lift of S in Ω. Then the above fundamental region can be so chosen that the interior contains $\tilde{S}$.

Now we go back to the proof of Proposition 2.1. By Lemma 2.4, there is a continuous mapping $f: X\to C$ such that

(v) $Z=f^{-1}(0)$ is a connected, simply connected real 5-manifold,

(vi) the complement $X-f^{-1}(0)$ has two connected components Y_1 and Y_2,

(vii) Y_ν contains lines and $\pi_1(Y_\nu)\cong G_\nu$, $\nu=1, 2$.

Then it is clear that Y_1 and Y_2 satisfies (ii). Let $i_\nu: [Y_\nu]_X\to X$ be the natural inclusion and $\Gamma_\nu=\mathrm{Im}(\pi_1(Y_\nu)\to\pi_1(X))$. Then $\Gamma\cong\Gamma_1 * \Gamma_2$ holds. Let $p: \Omega\to X$ be the canonical projection. Fix a connected component $\tilde{Z}$ of $p^{-1}(Z)$. Then there is a fundamental region F in Ω with respect to Γ such that $\mathrm{Int}\, F$ contains $\tilde{Z}$ as a closed hypersurface. Let $F_\nu^\#$, $\nu=1, 2$, be the component of $F-\tilde{Z}$ such that $p(F_\nu^\#)\subset Y_\nu$. Obviously, the complement $\boldsymbol{P}^3-\tilde{Z}$ has two connected components. Let K_ν, $\nu=1, 2$, denote the connected component of $\boldsymbol{P}^3-\tilde{Z}$ which contains $F_\mu^\#$, $\mu\neq\nu$. Put $F_\nu=K_\nu\cup F$ and $\Omega_\nu=\bigcup_{\gamma\in\Gamma_\nu}\gamma(F_\nu)$. Then we see that $X_\nu=\Gamma_\nu\backslash\Omega_\nu$, $\nu=1, 2$, is a (P)-manifold. By construction, F_ν is a fundamental region of Γ_ν in Ω_ν. The quotient of the set $\Omega_\nu^\#=\bigcup_{\gamma\in\Gamma_\nu}\gamma(F_\nu^\#)$ in X_ν is biholomorphic to $[Y_\nu]_X$ whose boundary considered in X_ν is isomorphic to Z. Thus it is clear that (iii) and (iv) hold. ■

3. Properties of Γ of Schottky type manifolds. Let $[z_0 : z_1 : z_2 : z_3]$ be a standard system of coordinates on $\boldsymbol{P}^3$. We fix the notation as follows.

points e_j : $e_0=[1:0:0:0]$ $e_1=[0:1:0:0]$ $e_2=[0:0:1:0]$ $e_3=[0:0:0:1]$,

lines l_{jk} : $z_j=z_k=0$ $j, k=0, 1, 2, 3$, $j<k$,

planes H_j : $z_j=0$ $j=0, 1, 2, 3$.

Suppose that a line l in $\boldsymbol{P}^3$ is given by the equations

$$
\begin{aligned}
&a_0z_0+a_1z_1+a_2z_2+a_3z_3=0\,,\\
&b_0z_0+b_1z_1+b_2z_2+b_3z_3=0\,.
\end{aligned}
\tag{3.1}
$$

Then the Plücker coordinates

$$[\xi_0(l):\xi_1(l):\xi_2(l):\xi_3(l):\xi_4(l):\xi_5(l)]\in \boldsymbol{P}^5$$

of l are given by

$$
\begin{aligned}
&\xi_0(l)=\det\begin{pmatrix}a_0 & a_1\\ b_0 & b_1\end{pmatrix}, \quad \xi_1(l)=\det\begin{pmatrix}a_0 & a_2\\ b_0 & b_2\end{pmatrix}, \quad \xi_2(l)=\det\begin{pmatrix}a_0 & a_3\\ b_0 & b_3\end{pmatrix},\\
&\xi_3(l)=\det\begin{pmatrix}a_1 & a_2\\ b_1 & b_2\end{pmatrix}, \quad \xi_4(l)=\det\begin{pmatrix}a_1 & a_3\\ b_1 & b_3\end{pmatrix}, \quad \xi_5(l)=\det\begin{pmatrix}a_2 & a_3\\ b_2 & b_3\end{pmatrix}.
\end{aligned}
\tag{3.2}
$$

In terms of these coordinates, we regard the Grassmann manifold Gr(4, 2) as a hypersurface in $\boldsymbol{P}^5$. We denote by $\hat{l}$ the corresponding point on Gr(4, 2). let $\gamma: SL(4, \boldsymbol{C})\to PGL(4, \boldsymbol{C})$ denote the canonical projection. Let

$$
M=\begin{pmatrix}
\alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03}\\
0 & \alpha_{11} & \alpha_{12} & \alpha_{13}\\
0 & 0 & \alpha_{22} & \alpha_{23}\\
0 & 0 & 0 & \alpha_{33}
\end{pmatrix},
\tag{3.3}
$$

be an element of $SL(4, \boldsymbol{C})$. The Plücker coordinates $[\xi'_0:\xi'_1:\xi'_2:\xi'_3:\xi'_4:\xi'_5]$ of the line $l'=\gamma(M)^{-1}l$ are given by

$$
\begin{aligned}
&\xi'_0=\alpha_{00}\alpha_{11}\xi_0\\
&\xi'_1=\alpha_{00}\alpha_{12}\xi_0+\alpha_{00}\alpha_{22}\xi_1\\
&\xi'_2=\alpha_{00}\alpha_{13}\xi_0+\alpha_{00}\alpha_{23}\xi_1+\alpha_{00}\alpha_{33}\xi_2\\
&\xi'_3=(\alpha_{01}\alpha_{12}-\alpha_{02}\alpha_{11})\xi_0+\alpha_{01}\alpha_{22}\xi_1+\alpha_{11}\alpha_{22}\xi_3\\
&\xi'_4=(\alpha_{01}\alpha_{13}-\alpha_{03}\alpha_{11})\xi_0+\alpha_{01}\alpha_{23}\xi_1+\alpha_{01}\alpha_{33}\xi_2+\alpha_{11}\alpha_{23}\xi_3+\alpha_{11}\alpha_{33}\xi_4\\
&\xi'_5=(\alpha_{02}\alpha_{13}-\alpha_{03}\alpha_{12})\xi_0+(\alpha_{02}\alpha_{23}-\alpha_{03}\alpha_{22})\xi_1+\alpha_{02}\alpha_{33}\xi_2\\
&\qquad+(\alpha_{12}\alpha_{23}-\alpha_{13}\alpha_{22})\xi_3+\alpha_{12}\alpha_{33}\xi_4+\alpha_{22}\alpha_{33}\xi_5\,,
\end{aligned}
\tag{3.4}
$$

where $\xi_\nu=\xi_\nu(l)$, $\nu=0, 1, \cdots, 5$.

Let $\{g_\nu : \nu=1, 2, \cdots, r\}$ be a set of generators of Γ, and M_ν a representative of g_ν in $SL(4, \boldsymbol{C})$. Denote by $\tilde{\Gamma}$ the subgroup of $SL(4, \boldsymbol{C})$ generated by M_ν, $\nu=1, 2, \cdots, r$. For an elment $M \in SL(4, \boldsymbol{C})$, we write the Jordan canonical form as

$$J(M)=\begin{pmatrix} \alpha_0 & \varepsilon_0 & 0 & 0 \\ 0 & \alpha_1 & \varepsilon_1 & 0 \\ 0 & 0 & \alpha_2 & \varepsilon_2 \\ 0 & 0 & 0 & \alpha_3 \end{pmatrix}, \tag{3.5}$$

where

$$|\alpha_\nu| \leqq |\alpha_{\nu+1}|, \tag{3.6}$$

$$(\alpha_{\nu+1}-\alpha_\nu)\varepsilon_\nu=0, \tag{3.7}$$

$$\varepsilon_\nu=0 \quad \text{or} \quad 1, \tag{3.8}$$

for $\nu=0, 1, 2$.

In the following throught this section, we assume that $\Gamma\backslash\Omega$ is a manifold of Schottky type.

LEMMA 3.9. *For any $M \in \tilde{\Gamma}$, we have either*

$$|\alpha_0| \leqq |\alpha_1| < |\alpha_2| \leqq |\alpha_3|, \tag{3.10}$$

or

$$|\alpha_0| = |\alpha_1| = |\alpha_2| = |\alpha_3|. \tag{3.11}$$

PROOF. Take any $M \in \tilde{\Gamma}$, and fix it. Taking a suitable system of homogeneous coordinates on $\boldsymbol{P}^3$, we can assume $M=J(M)$, where $J(M)$ is the Jordan canonical form (3.5) satisfying (3.6), (3.7) and (3.8). Suppose that M satisfies neither (3.10) nor (3.11). Then M or M^{-1} satisfies

$$|\alpha_0| < |\alpha_1| = |\alpha_2| = |\alpha_3| \tag{3.12}$$

or

$$|\alpha_0| < |\alpha_1| = |\alpha_2| < |\alpha_3|. \tag{3.13}$$

Replacing M with M^{-1} if necessary, we may assume that M is one of the following.

$$\begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_3 \end{pmatrix}, \tag{3.14}$$

$$(3.15)\qquad \begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ 0 & \alpha_1 & 1 & 0 \\ 0 & 0 & \alpha_1 & 0 \\ 0 & 0 & 0 & \alpha_3 \end{pmatrix},$$

$$(3.16)\qquad \begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ 0 & \alpha_1 & 1 & 0 \\ 0 & 0 & \alpha_1 & 1 \\ 0 & 0 & 0 & \alpha_1 \end{pmatrix}.$$

SUBLEMMA 3.17. *In the case* (3.12), *M is not of the form* (3.14).

PROOF. Suppose that M is of the form (3.14). Take any line l in Ω and a small compact subset K in Ω which contains a neighborhood of the point $l\cap H_0$. Then, since $|\alpha_1|=|\alpha_2|=|\alpha_3|$, and since H_0 is $\gamma(M)$-invariant, we see that the set $\{n\in \boldsymbol{Z}: \gamma(M)^n(K)\cap K\neq\varnothing\}$ is infinite. Hence the action of the infinite subgroup $\langle\gamma(M)\rangle$ on Ω is *not* properly discontinuous. This is a contradiction. ∎

SUBLEMMA 3.18. *In the case* (3.12), *M is not of the form* (3.15).

PROOF. Suppose that M is of the form (3.15). Let l be a line in Ω such that $l\cap(\bigcup l_{ij})=\varnothing$ and put $\xi_\nu=\xi_\nu(l)$, $\nu=0,\cdots,5$. By (3.4), the Plücker coordinates $[\xi_0^{(-n)}:\xi_1^{(-n)}:\xi_2^{(-n)}:\xi_3^{(-n)}:\xi_4^{(-n)}:\xi_5^{(-n)}]$ of $\gamma(M)^{-n}(l)$ are

$$\xi_0^{(-n)}=\alpha_0^n\alpha_1^n\xi_0,\quad \xi_1^{(-n)}=n\alpha_0^n\alpha_1^{n-1}\xi_0+\alpha_0^n\alpha_1^n\xi_1,\quad \xi_2^{(-n)}=\alpha_0^n\alpha_3^n\xi_2,$$
$$\xi_3^{(-n)}=\alpha_1^{2n}\xi_3,\qquad \xi_4^{(-n)}=\alpha_1^n\alpha_3^n\xi_4,\qquad \xi_5^{(-n)}=n\alpha_1^{n-1}\alpha_3^n\xi_4+\alpha_1^n\alpha_3^n\xi_5.$$

Since $\xi_0\xi_1\xi_2\xi_3\xi_4\xi_5\neq 0$, we see easily that $\lim_{n\to+\infty}\gamma(M)^n(l)=l_{02}$ and $\lim_{n\to-\infty}\gamma(M)^n(l)=l_{23}$. This implies that $l_{02}\cup l_{23}$ is contained in Λ. Since any connected component of Λ is a single line, this is impossible. ∎

SUBLEMMA 3.19. *In the case* (3.12), *M is not of the form* (3.16).

PROOF. Suppose that M is of the form (3.16). We let l be a line in Ω such that $l\cap(\bigcup l_{ij})=\varnothing$ and put $\xi_\nu=\xi_\nu(l)$, $\nu=0,\cdots,5$. By (3.4) the Plücker coordinates $[\xi_0^{(-n)}:\xi_1^{(-n)}:\xi_2^{(-n)}:\xi_3^{(-n)}:\xi_4^{(-n)}:\xi_5^{(-n)}]$ of $\gamma(M)^{-n}(l)$ are

$$\xi_0^{(-n)}=\alpha_0^n\alpha_1^n\xi_0,$$
$$\xi_1^{(-n)}=n\alpha_0^n\alpha_1^{n-1}\xi_0+\alpha_0^n\alpha_1^n\xi_1,$$
$$\xi_2^{(-n)}=(1/2)(n^2-n)\alpha_0^n\alpha_1^{n-2}\xi_0+n\alpha_0^n\alpha_1^{n-1}\xi_1+\alpha_0^n\alpha_1^n\xi_2,$$
$$\xi_3^{(-n)}=\alpha_1^{2n}\xi_3,$$

$$\xi_4^{(-n)} = n\alpha_1^{2n-1}\xi_3 + \alpha_1^{2n}\xi_4 ,$$

$$\xi_5^{(-n)} = (1/2)(n^2+n)\alpha_1^{2n-2}\xi_3 + n\alpha_1^{2n-1}\xi_4 + \alpha_1^{2n}\xi_5 .$$

Since $\xi_0\xi_1\xi_2\xi_3\xi_4\xi_5 \neq 0$, we see easily that $\lim_{n\to+\infty}\gamma(M)^n(l) = l_{03}$ and $\lim_{n\to-\infty}\gamma(M)^n(l) = l_{23}$. This implies that $l_{03} \cup l_{23}$ is contained in Λ. Since any connected component of Λ is a single line, this is impossible. ∎

Next we consider the case (3.13).

SUBLEMMA 3.20. *In the case* (3.13), *M is not of the form* (3.14).

PROOF. Suppose that M is of the form (3.14). Let l' and l'' be distinct two lines in Ω. Put $\xi'_\nu = \xi_\nu(l')$ and $\xi''_\nu = \xi_\nu(l'')$, $\nu = 0, \cdots, 5$. We can choose these two lines so that the condition $\xi'_0\xi''_1 - \xi'_1\xi''_0 \neq 0$ is satisfied. There is a sequence $\{n_j\}_{j=0}^{+\infty}$ of positive integers with $\lim n_j = +\infty$ and $\lim(\alpha_1/\alpha_2)^{n_j} = 1$. Put $l'_\infty = \lim \gamma(M)^{n_j}(l')$ and $l''_\infty = \lim \gamma(M)^{n_j}(l'')$. By (3.4), the Plücker coordinates $[\xi_0^{(-n)} : \xi_1^{(-n)} : \xi_2^{(-n)} : \xi_3^{(-n)} : \xi_4^{(-n)} : \xi_5^{(-n)}]$ of $\gamma(M)^{-n}(l')$ are

$$\xi_0^{(-n)} = \alpha_0^n\alpha_1^n\xi'_0 , \qquad \xi_1^{(-n)} = \alpha_0^n\alpha_2^n\xi'_1 , \qquad \xi_2^{(-n)} = \alpha_0^n\alpha_3^n\xi'_2 ,$$

$$\xi_3^{(-n)} = \alpha_1^n\alpha_2^n\xi'_3 , \qquad \xi_4^{(-n)} = \alpha_1^n\alpha_3^n\xi'_4 , \qquad \xi_5^{(-n)} = \alpha_2^n\alpha_3^n\xi'_5 .$$

Hence we have

$$[\xi_0(l'_\infty) : \xi_1(l'_\infty) : \xi_2(l'_\infty) : \xi_3(l'_\infty) : \xi_4(l'_\infty) : \xi_5(l'_\infty)] = [\xi'_0 : \xi'_1 : 0 : 0 : 0 : 0] .$$

Combining this with the similar calculation for l''_∞, we see that the condition $\xi'_0\xi''_1 - \xi'_1\xi''_0 \neq 0$ implies that the limit lines l'_∞ and l''_∞ intersect transversely. Since $l'_\infty \cup l''_\infty \subset \Lambda$, this is a contradiction. ∎

SUBLEMMA 3.21. *In the case* (3.13), *M is not of the form* (3.15).

The proof is the same as that of Sublemma 3.18. The following sublemma is trivial.

SUBLEMMA 3.22. *In the case* (3.13), *M is not of the form* (3.16).

The proof of Lemma 3.9 is now clear from Sublemmas 3.17–3.22.

LEMMA 3.23. *Suppose that $M \in \tilde{\Gamma}$ satisfies* (3.11). *If M is of infinite order, then its Jordan canonical form J(M) is of the form*

$$(3.24) \qquad \begin{pmatrix} \alpha_0 & 1 & 0 & 0 \\ 0 & \alpha_0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 1 \\ 0 & 0 & 0 & \alpha_2 \end{pmatrix},$$

or

$$(3.25)\qquad \begin{pmatrix} \alpha_0 & 1 & 0 & 0 \\ 0 & \alpha_0 & 1 & 0 \\ 0 & 0 & \alpha_0 & 1 \\ 0 & 0 & 0 & \alpha_0 \end{pmatrix}.$$

PROOF. Taking a suitable system of homogeneous coordinates on $\boldsymbol{P}^3$, we may assume that M is (3.24), (3.25) or one of the following:

$$(3.26)\qquad \begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_3 \end{pmatrix},$$

$$(3.27)\qquad \begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 1 \\ 0 & 0 & 0 & \alpha_2 \end{pmatrix},$$

$$(3.28)\qquad \begin{pmatrix} \alpha_0 & 0 & 0 & 0 \\ 0 & \alpha_1 & 1 & 0 \\ 0 & 0 & \alpha_1 & 1 \\ 0 & 0 & 0 & \alpha_1 \end{pmatrix}.$$

Suppose that M is of the form (3.26) or (3.27). Then, among l_{ij}, there are distinct two lines $l_{i_\lambda j_\lambda}$, $\lambda=1, 2$, with $l_{i_1 j_1} \cap l_{i_2 j_2} \neq \varnothing$ on which $\gamma(M)$ acts as a projective transformation defined by a diagonal matrix. There are sequences $\{n_{\lambda\nu}\}_{\nu=1}^{\infty}$, $\lambda=1, 2$, of positive integers with $\lim_{\nu\to\infty} n_{\lambda\nu}=+\infty$ such that $\lim_{\nu\to\infty} \gamma(M)^{n_{\lambda\nu}} \mid l_{i_\lambda j_\lambda}=1$. This implies $\Omega \cup l_{i_\lambda j_\lambda}=\varnothing$, since the action of $\langle\gamma(M)\rangle$ on $\Omega \cap l_{i_\lambda j_\lambda}$ must be properly discontinuous. Thus $l_{i_\lambda j_\lambda}$, $\lambda=1, 2$, are contained in Λ, a contradiction. Hence M is neither of the forms (3.26) and (3.27). Suppose that M is of the form (3.28). Then M acts on l_{23}. By the same reason as above, l_{23} is contained in Λ, since there is a sequence $\{n_\lambda\}_{\lambda=1}^{\infty}$ of positive integers with $\lim_{\lambda\to\infty} n_\lambda=\infty$ such that $\lim_{\lambda\to\infty}(\alpha_0/\alpha_1)^{n_\lambda}=1$. Choose a line l in Ω such that $l \cap (\bigcup l_{ij})=\varnothing$. Put $\xi_\nu=\xi_\nu(l)$, $\nu=0, 1, \cdots, 5$. Then the Plücker coordinates of the limit line $l_\infty := \lim_{n\to\infty} \gamma(M)^n(l)$ are given by $[0:0:\xi_0:0:0:\xi_3]$ (cf. the calculation in the proof of Sublemma 3.19). From $\xi_0 \neq 0$, it follows that $l_\infty \cap l_{23}=\{e_1\}$. Since $l_\infty \cup l_{23} \subset \Lambda$, this is a contradiction. This completes the proof of the lemma. ∎

Combining Lemmas 3.9 and 3.23, we have the following:

PROPOSITION 3.29. *Let $X=\Gamma\backslash\Omega$ be a manifold of Schottky type. Let $\tilde{\Gamma}$ be a subgroup of $SL(4, \boldsymbol{C})$ such that $\gamma\,|\,\tilde{\Gamma}: \tilde{\Gamma}\to\Gamma$ is surjective, where $\gamma: SL(4, \boldsymbol{C})\to PGL(4)$ is the canonical projection. Then, for any $M\in\tilde{\Gamma}$ of infinite order, the Jordan canonical form $J(M)$ of M is one of the following*:

Type I

$$\begin{pmatrix} \alpha_0 & \varepsilon_0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & \varepsilon_2 \\ 0 & 0 & 0 & \alpha_3 \end{pmatrix},$$

where $|\alpha_0|\leqq|\alpha_1|<|\alpha_2|\leqq|\alpha_3|$, *and* $(\alpha_0-\alpha_1)\varepsilon_0=(\alpha_2-\alpha_3)\varepsilon_2=0$.

Type II

$$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & \beta \end{pmatrix},$$

where $|\alpha|=|\beta|$.

Type III

$$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & \alpha \end{pmatrix}.$$

4. L-Hopf manifolds (Hopf-like manifolds of Class L)

DEFINITION 4.1. A compact complex manifold is called an L-*Hopf manifold* if its universal covering Ω is a subdomain of $\boldsymbol{P}^3$ such that the complement $\Lambda:=\boldsymbol{P}^3-\Omega$, called the *limit set*, consists of two projective lines without intersection. An L-Hopf manifold is said to be *primary* if its fundamental group is infinite cyclic.

It is easy to check that an L-Hopf manifold is of Class L. Therefore an L-Hopf manifold is of Schottky type.

PROPOSITION 4.2. *A* (*P*)*-manifold $\Gamma\backslash\Omega$ of Schottky type is an* L-*Hopf manifold if and only if its fundamental group Γ contains an infinite cyclic group of finite index.*

PROOF. A theorem of Hopf [Ho, Satz Va] says that Ω has two ends if and only if

Γ contains an infinite cyclic subgroup of finite index. Therefore we see that Γ contains an infinite cyclic subgroup of finite index if and only if the limit set consists of two lines without intersection, i.e., $\Gamma\backslash\Omega$ is an L-Hopf manifold. ■

We shall use the notation of § 3. We may assume that the two lines in the limit set Λ are l_{01} and l_{23}. For any $h\in\Gamma$, we have $h(l_{01})=l_{01}$ and $h(l_{23})=l_{23}$. Indeed, if $h(l_{01})=l_{23}$ and $h(l_{23})=l_{01}$, then h is represented by a matrix

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \qquad B, C\in GL(2, \boldsymbol{C}).$$

Then we can find a non-zero vector $z''\in\boldsymbol{C}^2$ such that $CBz''=\lambda z''$ for some $\lambda\in\boldsymbol{C}-\{0\}$. Then the point $z={}^t[Bz'', \mu z'']\in\Omega$ is fixed by h, where $\mu^2=\lambda$. This is a contradiction. Hence both l_{01} and l_{23} are Γ-invariant.

An element $g\in\Gamma$ is called a *contraction* if $g^n(U\cup\partial U)$ converges to the line l_{01}. The group Γ contains a contraction. To prove this we borrow an argument of Kodaira [Ko2, p. 695]. Note that there is an element $g\in\Gamma$ such that $g(\partial U)\cap\partial U=\varnothing$. Since g leaves the line l_{01} invariant, either $g(U\cup\partial U)\subset U$ or $U\cup\partial U\subset g(U)$ holds. Replacing g with g^{-1} if necessary, we may assume that $g(U\cup\partial U)\subset U$ holds. Then we have $g^n(U\cup\partial U)\subset g^{n-1}(U), n=1, 2, 3, \cdots$. We have to show that $\bigcap_n g^n(U\cup\partial U)=l_{01}$. Suppose that $z\notin l_{01}$ is a point on the boundary of $\bigcap_n g^n(U\cup\partial U)$ and let W be a small neighborhood of z. It is clear that W is not contained in $g^n(U\cup\partial U)$ for a sufficiently large n, while z is an interior point of W. Hence W meets $g^n(U\cup\partial U)$ for all sufficiently large n. This contradicts the proper discontinuity of G. By the subsequent argument of Kodaira [Ko2, p. 695], we can also show that, if $g\in\Gamma$ is a contraction, then there exists a positive integer n such that g^n belongs to the center of Γ.

Now every element $h\in\Gamma$ has a representative of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad A, D\in GL(2, \boldsymbol{C}).$$

We define

$$e(h)=\det(AD^{-1}). \tag{4.3}$$

It is easy to see that g is a contraction if and only if

$$\begin{aligned} &\mathrm{Max}\{\text{absolute values of the eigenvalues of } A\} \\ &<\mathrm{Min}\{\text{absolute values of the eigenvalues of } D\}. \end{aligned} \tag{4.4}$$

LEMMA 4.5. *An element $h\in\Gamma$ is a contraction if and only if $|e(h)|<1$.*

PROOF. If h is a contraction, then we have $|e(h)|<1$ by (4.4). Conversely, suppose that h satisfies $|e(h)|<1$ while (4.4) is not satisfied. Choose a contraction g in Γ. Then the infinite cyclic subgroups $\langle g\rangle$ and $\langle h\rangle$ generated by g and h, respectively, have only the identity in common. This contradicts the fact that the index of $\langle g\rangle$ in Γ is finite. ∎

PROPOSITION 4.6. *The fundamental group of an* L-*Hopf manifold is a semi-direct product of a finite group and an infinite cyclic group.*

PROOF. Define a group homomorphism $\rho: \Gamma \to \boldsymbol{R}$ by

$$\rho(g) = -\log|e(g)| \qquad (g \in \Gamma).$$

Let g_1 be a contraction. Then the index d of the infinite cyclic group $\langle \rho(g_1)\rangle$ generated by $\rho(g_1)$ is finite. Hence $d^{-1}\rho(g_1)$ is a minimum positive element of $\rho(\Gamma)$. Let $g_0 \in \Gamma$ be an element such that $\rho(g_0) = d^{-1}\rho(g_1)$. Then we have the semi-direct product decomposition $\Gamma \approx \langle g_0\rangle \cdot \operatorname{Ker}\rho$. ∎

As we have seen above, a primary L-Hopf manifold is biholomorphic to the manifold M_g defined as follows (see [Ka4] for more general characterization of L-Hopf manifolds, where the arguments are carried over without the assumption that Ω is a subdomain in $\boldsymbol{P}^3$).

Fix a standard system of homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ on $\boldsymbol{P}^3$. Let l_{01} and l_{23} be the two lines defined by $z_0 = z_1 = 0$ and $z_2 = z_3 = 0$, respectively. Let $g \in PGL(4, \boldsymbol{C})$ be the automorphism of $\boldsymbol{P}^3 - (l_{01} \cup l_{23})$ defined by the 4×4 matrix

$$\begin{pmatrix} \alpha_0 & \lambda_0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & \lambda_2 \\ 0 & 0 & 0 & \alpha_3 \end{pmatrix}, \tag{4.7}$$

with the conditions

$$(\alpha_0 - \alpha_1)\lambda_1 = (\alpha_2 - \alpha_3)\lambda_2 = 0, \tag{4.8}$$

and

$$0 < |\alpha_0| \leqq |\alpha_1| < |\alpha_2| \leqq |\alpha_3|. \tag{4.9}$$

Let $\langle g\rangle$ denote the infinite cyclic subgroup in $PGL(4, \boldsymbol{C})$ generated by g. Then M_g is defined to be the quotient space $(\boldsymbol{P}^3 - (l_{01} \cup l_{23}))/\langle g\rangle$.

Thus we have easily:

THEOREM B. *Any* L-*Hopf manifold admits a primary* L-*Hopf manifold as a finite unramified covering. An* L-*Hopf manifolds is primary if and only if its fundamental group is torsion free. Any primary* L-*Hopf manifold is biholomorphic to* M_g *where* $g \in PGL(4, \boldsymbol{C})$ *is of the form* (4.7) *and satisfies the conditions* (4.8) *and* (4.9).

L-Hopf manifolds with torsions are found among the twistor spaces over compact conformally flat non-primary Hopf surfaces (cf. [Ka1]).

5. Blanchard manifolds. In this section, we shall carry out a rough classification of Blanchard manifolds. We use the notation in § 3.

DEFINITION 5.1. A compact complex manifold is called a *Blanchard manifold* if its universal covering Ω is a subdomain in $\boldsymbol{P}^3$ such that the complement $\Lambda := \boldsymbol{P}^3 - \Omega$, called the *limit set*, consists of a single projective line.

It is easy to check that a Blanchard manifold is of Class L. Therefore a Blanchard manifold is of Schottky type.

LEMMA 5.2. *Let $K \geqq 2$ be a positive constant. Let G be a subgroup of $GL(2, \boldsymbol{C})$ such that $|\mathrm{trace}(g)| \leqq K$ for all elements g in G. If G contains an element which is not conjugate in $GL(2, \boldsymbol{C})$ to a diagonal matrix, then G is conjugate in $GL(2, \boldsymbol{C})$ to a subgroup which consists of upper triangular matrices.*

PROOF. Replacing G by a conjugate subgroup in $GL(2, \boldsymbol{C})$ if necessary, we can assume that G contains

$$g_0 = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \qquad \text{where} \quad |\alpha| \geqq 1 .$$

Since $|\mathrm{trace}(g_0^n)| \leqq K$ for $n \to +\infty$, we have $|\alpha| = 1$. Let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an arbitrary element in G. Since $|\mathrm{trace}(g_0^n g)| = |\alpha^n a + n\alpha^{n-1} c + \alpha^n d| \leqq K$ for $n \to +\infty$, we infer $c = 0$. Therefore G is contained in the upper triangular subgroup of $GL(2, \boldsymbol{C})$. ■

In what follows in this section, $X = \Gamma \backslash \Omega$ always denotes a Blanchard manifold. The complement $\boldsymbol{P}^3 - \Omega$, indicated by l, is a single line by definition.

LEMMA 5.3. *Γ is torsion free.*

PROOF. If $g \in \Gamma - \{\mathrm{id}\}$ is of finite order, we can easily find a fixed point outside l as an intersection of three g-invariant planes, a contradiction. ■

PROPOSITION 5.4. *The Jordan canonical form of a representative M of any element of $\Gamma - \{\mathrm{id}\}$ is either of Type* II *or Type* III *in Proposition* 3.29.

PROOF. By Lemma 5.3, every element of $\Gamma - \{\mathrm{id}\}$ is of infinite order. Suppose that $M \in \tilde{\Gamma}$ is of Type I. Choose a line l in Ω such that $l \cap (\bigcup l_{ij}) = \varnothing$. Then we have $\lim_{n \to \infty} \gamma(M)^n(l) = l_{01}$ and $\lim_{n \to \infty} \gamma(M)^{-n}(l) = l_{23}$ (cf. L-Hopf manifolds case, § 4). Hence $l_{01} \cup l_{23}$ is contained in Λ, a contradiction. ■

In what follows in this section, we say that an element $g \in \Gamma - \{\mathrm{id}\}$ is of *Type* II (resp. *Type* III), if g is represented by a 4×4 matrix which is conjugate to a matrix of Type II (resp. type III).

PROPOSITION 5.5. *There is an abelian subgroup Γ_1 of Γ such that $[\Gamma : \Gamma_1]$ is finite.*

Our proof proceeds by a series of lemmas. Choose a system of homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ on $\boldsymbol{P}^3$ such that l is given by $l_{23} = \{z_2 = z_3 = 0\}$. Let G be the group defined by

$$\left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in SL(4, \boldsymbol{C}) : A, D \in GL(2, \boldsymbol{C}), B \in M(2, \boldsymbol{C}) \right\}.$$

Let $\psi : G \to GL(2, \boldsymbol{C}) \times GL(2, \boldsymbol{C})$ be the homomorphism defined by $\psi = (\psi_1, \psi_2)$, where

$$\psi_1\left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\right) = A \quad \text{and} \quad \psi_2\left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\right) = D.$$

Then $\tilde{\Gamma}$ is a subgroup of G.

LEMMA 5.6. *There is a subgroup $\tilde{\Gamma}_1$ of $\tilde{\Gamma}$ with $[\tilde{\Gamma} : \tilde{\Gamma}_1] < +\infty$ such that $\psi_2(\tilde{\Gamma}_1)$ is conjugate in $GL(2, \boldsymbol{C})$ to a subgroup which consists of upper triangular matrices.*

PROOF. We assume that

(5.7) *for any subgroup $\tilde{\Gamma}_1$ of $\tilde{\Gamma}$ with $[\tilde{\Gamma} : \tilde{\Gamma}_1] < +\infty$, any conjugate of the image group $\psi_2(\tilde{\Gamma}_1)$ in $GL(2, \boldsymbol{C})$ cannot be contained in the set of upper triangular matrices.*

The lemma will be verified, if we derive a contradiction.

Step 1. Pick any element $g \in \tilde{\Gamma}$ of infinite order. Suppose that both $\psi_1(g)$ and $\psi_2(g)$ are *not* conjugate to diagonal matrices in $GL(2, \boldsymbol{C})$. Then by Proposition 5.4, neither $\psi_1(g)$ nor $\psi_2(g)$ is conjugate to a diagonal matrix, and there are complex numbers α, β with $|\alpha| = |\beta| = 1$ such that $\psi_1(g)$ (resp. $\psi_2(g)$) is conjugate to

$$\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \quad \left(\text{resp. } \begin{pmatrix} \beta & 1 \\ 0 & \beta \end{pmatrix}\right).$$

Then, by Lemma 5.2, both $\operatorname{Im} \psi_1$ and $\operatorname{Im} \psi_2$ are conjugate to subgroups which consist of upper triangular matrices, a contradiction.

Step 2. Pick any element $g \in \tilde{\Gamma}$ of infinite order. By Step 1, we may assume that both $\psi_1(g)$ and $\psi_2(g)$ are conjugate to diagonal matrices in $GL(2, \boldsymbol{C})$. Thus by Proposition 5.4, we infer that both $\psi_1(g)$ and $\psi_2(g)$ are conjugate to

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad |\alpha| = |\beta| = 1.$$

Since $(\alpha\beta)^2=1$, it follows that $\beta=\bar{\alpha}$ or $\beta=-\bar{\alpha}$. In particular, we see that the equality $\det\psi_1(g)=\det\psi_2(g)=\pm 1$ holds for all $g\in\tilde{\Gamma}$. Taking a subgroup $\tilde{\Gamma}_1$ of $\tilde{\Gamma}$ with $[\tilde{\Gamma}:\tilde{\Gamma}_1]\leqq 2$, we may assume that the equality $\det\psi_1(g)=\det\psi_2(g)=1$ holds for all $g\in\tilde{\Gamma}_1$.

Step 3. By Step 2, we may assume that,

(5.8) *for all* $g\in\tilde{\Gamma}_1$, *both* $\psi_1(g)$ *and* $\psi_2(g)$ *are conjugate to*

$$\begin{pmatrix}\alpha(g) & 0\\ 0 & \overline{\alpha(g)}\end{pmatrix}, \qquad |\alpha(g)|=1 .$$

In particular,

(5.9) $\quad \mathrm{trace}(\psi_\nu(g))\,,\quad \nu=1,2\,,$ *are real for all* $g\in\tilde{\Gamma}_1$.

Moreover, by (5.7),

(5.10) *both* $\psi_1(\tilde{\Gamma}_1)$ *and* $\psi_2(\tilde{\Gamma}_1)$ *are infinite groups*.

Indeed, this is obvious for $\psi_2(\tilde{\Gamma}_1)$. If $\psi_1(\tilde{\Gamma}_1)$ is finite, then there is a subgroup $\tilde{\Gamma}_2$ of finite index in $\tilde{\Gamma}_1$ such that $\psi_1(\tilde{\Gamma}_2)$ is trivial. Since the set of eigenvalues of $\psi_1(g)$ coincides with that of $\psi_2(g)$, we see that $\psi_2(\tilde{\Gamma}_2)$ is also trivial. This implies that $\tilde{\Gamma}_2$ is conjugate to a subgroup which consists of upper triangular matrices, a contradiction. Hence $\psi_1(\tilde{\Gamma}_1)$ is also infinite.

SUBLEMMA 5.11. *The group* $\psi_2(\tilde{\Gamma}_1)$ *is a subgroup of either* $SU(2)$ *or* $SU(1,1)$.

PROOF. The infinite group $\psi_2(\tilde{\Gamma}_1)$ contains an element h_1 of infinite order by a theorem of Burnside. By a suitable change of a system of homogeneous coordinates on $\boldsymbol{P}^3$ preserving $l=l_{23}$, we may assume that h_1 is of the form

$$\begin{pmatrix}\alpha_1 & 0\\ 0 & \bar{\alpha}_1\end{pmatrix},$$

where α_1, $|\alpha_1|=1$, is *not* roots of unity. If $\psi_2(\tilde{\Gamma}_1)$ contains an element h_2 of the form

$$\begin{pmatrix}a & 0\\ c & \bar{a}\end{pmatrix}, \qquad c\neq 0\,,$$

then

$$h_2^{-1}h_1^{-1}h_2h_1=\begin{pmatrix}1 & 0\\ ac(\alpha_1^2-1) & 1\end{pmatrix}.$$

Therefore, by (5.8), we have $\alpha_1=\pm 1$, a contradiction. Thus we conclude that $\psi_2(\tilde{\Gamma}_1)$ contains no elements of the form

$$\begin{pmatrix}a & 0\\ c & \bar{a}\end{pmatrix}, \qquad c\neq 0\,.$$

Similarly, $\psi_2(\tilde{\Gamma}_1)$ contains no elements of the form

$$\begin{pmatrix} a & b \\ 0 & \bar{a} \end{pmatrix}, \qquad b \neq 0 .$$

Let

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be any element of $\psi_2(\tilde{\Gamma}_1)$. By (5.9), trace$(h_1^n h) \in \boldsymbol{R}$ for all $n \in \boldsymbol{Z}$. Hence $a = \bar{d}$ follows. Let

$$h' = \begin{pmatrix} a' & b' \\ c' & \bar{a}' \end{pmatrix}$$

be another element. Then we have

$$bc' = \bar{c}\bar{b}' . \tag{5.12}$$

If $bb' \neq 0$, then $\bar{b}'^{-1}c' = \overline{\bar{b}^{-1}c}$. Set $h' = h$. Then we see that $\bar{b}^{-1}c$ is real. Moreover, the value $\bar{b}^{-1}c$ does not depend on the elements of $\psi_2(\tilde{\Gamma}_1)$. Put $\rho = -\bar{b}^{-1}c \neq 0$. Then every element of $\psi_2(\tilde{\Gamma}_1)$ is of the form

$$\begin{pmatrix} a & b \\ -\rho\bar{b} & \bar{a} \end{pmatrix}.$$

If $\rho > 0$, then put

$$[z'_0 : z'_1 : z'_2 : z'_3] = [z_0 : z_1 : \rho^{1/2} z_2 : z_3] .$$

Then we see that $\psi_2(\tilde{\Gamma}_1)$ is a subgroup of $SU(2)$. If $\rho < 0$, then put

$$[z'_0 : z'_1 : z'_2 : z'_3] = [z_0 : z_1 : \sqrt{-1}\,|\rho|^{1/2} z_2 : z_3] .$$

Then we see that $\psi_2(\tilde{\Gamma}_1)$ is a subgroup of $SU(1, 1)$. This proves Sublemma 5.11. ∎

By the same argument, we have:

SUBLEMMA 5.13. *The group $\psi_1(\tilde{\Gamma}_1)$ is a subgroup of either $SU(2)$ or $SU(1, 1)$.*

Step 4. We assume that $\tilde{\Gamma}_1$ is torsion free, replacing $\tilde{\Gamma}_1$ with its torsion free subgroup of finite index, if necessary. This is possible by a theorem of Selberg.

SUBLEMMA 5.14. *There is a system of homogeneous coordinates on $\boldsymbol{P}^3$ such that $l = l_{23}$ and that $\psi_1(g) = \psi_2(g)$ for all $g \in \tilde{\Gamma}_1$.*

PROOF. Fix $g_1 \in \tilde{\Gamma}_1$ such that $\psi_2(g_1)$ is of infinite order. Choose coordinates on $\boldsymbol{P}^3$ such that $l = l_{23}$ and that $\psi_1(g_1) = \psi_2(g_1)$. By (5.8) and (5.9), we can write g_1 as

$$g_1=\begin{pmatrix} A & B \\ 0 & A \end{pmatrix}\in SL(4, \boldsymbol{C}),$$

with

$$A=\begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}, \qquad |\alpha|=1,$$

where α is not a root of unity. Take any $g\in\tilde{\Gamma}_1$. Then, by Sublemmas 5.11 and 5.13, we can write g as

$$g=\begin{pmatrix} P & Q \\ 0 & R \end{pmatrix}\in SL(4, \boldsymbol{C}),$$

$$P=\begin{pmatrix} p_1 & p_2 \\ \rho_1\bar{p}_2 & \bar{p}_1 \end{pmatrix}, \qquad R=\begin{pmatrix} r_1 & r_2 \\ \rho_2\bar{r}_2 & \bar{r}_1 \end{pmatrix},$$

where,

$$\rho_\nu=\begin{cases} -1 & \text{if } \psi_\nu(\tilde{\Gamma}_1)\subset SU(2), \\ 1 & \text{if } \psi_\nu(\tilde{\Gamma}_1)\subset SU(1,1). \end{cases}$$

For any $n\in\boldsymbol{Z}$, we have

$$A^nP=\begin{pmatrix} \alpha^n p_1 & \alpha^n p_2 \\ \rho_1\bar{\alpha}^n\bar{p}_2 & \bar{\alpha}^n\bar{p}_1 \end{pmatrix} \quad\text{and}\quad A^nR=\begin{pmatrix} \alpha^n r_1 & \alpha^n r_2 \\ \rho_2\bar{\alpha}^n\bar{r}_2 & \bar{\alpha}^n\bar{r}_1 \end{pmatrix}.$$

Since $\psi_1(g_1^ng)$ and $\psi_2(g_1^ng)$ are conjugate to each other, we have $\mathrm{tr}(\psi_1(g_1^ng))=\mathrm{tr}(\psi_2(g_1^ng))$. Hence $\mathrm{Re}(\alpha^n(p_1-r_1))=0$ for all $n\in\boldsymbol{Z}$. Since α is not a root of unity, we have

$$p_1=r_1. \tag{5.15}$$

Now we claim that

(5.16) *if either $\psi_1(g)$ or $\psi_2(g)$ is a diagonal matrix, then both are diagonal matrices*.

Indeed, if $P=\psi_1(g)$ is a diagonal matrix, then $1=|p_1|=|r_1|$ by (5.15). Then $r_2=0$ follows from $1=\det(\psi_2(g))=|r_1|^2-\rho_2|r_2|^2$. Then (5.16) is verified.

Therefore, by the assumption (5.7), there is $g_2\in\tilde{\Gamma}_1$ such that neither $\psi_1(g_2)$ nor $\psi_2(g_2)$ is a diagonal matrix. Put

$$g_2=\begin{pmatrix} p & q & * & * \\ \rho_1\bar{q} & \bar{p} & * & * \\ 0 & 0 & p & r \\ 0 & 0 & \rho_2\bar{r} & \bar{p} \end{pmatrix}, \qquad qr\neq 0,$$

and let

$$g=\begin{pmatrix} s & t & * & * \\ \rho_1\bar{t} & \bar{s} & * & * \\ 0 & 0 & s & u \\ 0 & 0 & \rho_2\bar{u} & \bar{s} \end{pmatrix}$$

be any element of $\tilde{\Gamma}_1$. Then, applying (5.15) to g_2g, we have $\rho_1 q\bar{t}=\rho_2 r\bar{u}$. In particular, letting $g=g_2$, we have $\rho_1|q|^2=\rho_2|r|^2$. Hence the equalities $\rho_1=\rho_2$ and $|q|=|r|$ hold. Put $\rho\bar{r}=\bar{q}$, where $|\rho|=1$. Then, for any $g\in\tilde{\Gamma}_1$, we have

$$g=\begin{pmatrix} s & t & * & * \\ \rho_1\bar{t} & \bar{s} & * & * \\ 0 & 0 & s & \rho t \\ 0 & 0 & \rho_1\overline{\rho t} & \bar{s} \end{pmatrix}.$$

Letting

$$T=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \bar{\lambda} \end{pmatrix},$$

where $\lambda^2=\rho$, we have $\psi_1(T^{-1}gT)=\psi_2(T^{-1}gT)$ for any $g\in\tilde{\Gamma}_1$. This proves Sublemma 5.14. ∎

Step 5. We fix a system of homogeneous coordinates on $\boldsymbol{P}^3$ as in Sublemma 5.14. By this sublemma, we see that $\psi_1(\tilde{\Gamma}_1)=\psi_2(\tilde{\Gamma}_1)$. Put $K=\psi_1(\tilde{\Gamma}_1)=\psi_2(\tilde{\Gamma}_1)$. By Sublemmas 5.12 and 5.13, K is a subgroup of either $SU(2)$ or $SU(1, 1)$. In this step, we consider the case $K\subset SU(2)$.

SUBLEMMA 5.15. *If $K\subset SU(2)$, then K contains an abelian subgroup K_0 of finite index.*

PROOF. The following argument is due to Wolf [W, pp. 100–102] (see also Charlap [C]). Let

$$\begin{aligned} G&=\left\{\begin{pmatrix} P & Q \\ 0 & P\end{pmatrix} : P\in SU(2),\ Q\in M(2, \boldsymbol{C})\right\} \\ &=\left\{\begin{pmatrix} P & 0 \\ 0 & P\end{pmatrix}\begin{pmatrix} I & T \\ 0 & I\end{pmatrix} : P\in SU(2),\ T\in M(2, \boldsymbol{C})\right\} \end{aligned}$$

and $\varphi: G\to SU(2)$ the homomorphism defined by

$$\varphi\left(\begin{pmatrix} P & Q \\ 0 & P \end{pmatrix}\right)=P\,.$$

To prove the sublemma, we shall use the following four facts.

(5.16) There is a neighborhood V of 1 in $SU(2)$ such that, if g, $h\in V$ and $[g,[g,h]]=1$, then $[g,h]=1$.

(5.17) There is a neighborhood V' of 1 in $SU(2)$ such that, whenever g, $h\in V'$,

$$[g,h]\,,\quad [g,[g,h]]\,,\quad [g,[g,[g,h]]]\,,\cdots$$

is a sequence in V' which converges to 1.

(5.18) Any neighborhood of 1 in $SU(2)$ contains a neighborhood V'' such that $gV''g^{-1}=V''$ for all $g\in SU(2)$.

(5.19) The identity component of the closure of K in $SU(2)$ is abelian.

For the proofs of (5.16), (5.17) and (5.18), see [W, pp. 100–101]. We shall give a proof of (5.19). Our proof is essentially a copy of [C, pp. 12–14]. Suppose that W is a neighborhood of 1 in $SU(2)$ satisfying the conditions on V, V', V'' in (5.16), (5.17) and (5.18). Let $g_1, g_2\in\tilde{\Gamma}_1$ with $\varphi(g_1)\in W$, and define $g_{i+1}=[g_1, g_i]$ for $i\geqq 2$. Write

$$g_i=\begin{pmatrix} P_i & Q_i \\ 0 & P_i \end{pmatrix}$$

with $P_i=\varphi(g_i)\in SU(2)$ and $Q_i\in M(2, \boldsymbol{C})$. Then

$$g_{i+1}=\begin{pmatrix} P_{i+1} & Q_{i+1} \\ 0 & P_{i+1} \end{pmatrix},$$

where

$$P_{i+1}=[P_1, P_i]$$

and

$$\begin{aligned} Q_{i+1}= &-P_1P_iP_1^{-1}P_i^{-1}Q_iP_i^{-1}-P_1P_iP_1^{-1}Q_1P_1^{-1}P_i^{-1} \\ &+P_1Q_iP_1^{-1}P_i^{-1}+Q_1P_iP_1^{-1}P_i^{-1}\,. \end{aligned}$$

Taking the norms, we obtain

$$\begin{aligned} |Q_{i+1}| &\leqq |-P_1P_iP_1^{-1}P_i^{-1}Q_iP_i^{-1}+P_1Q_iP_1^{-1}P_i^{-1}| \\ &\quad +|-P_1P_iP_1^{-1}Q_1P_1^{-1}P_i^{-1}+Q_1P_iP_1^{-1}P_i^{-1}| \\ &\leqq |Q_iP_1^{-1}-Q_i|+|Q_i-P_iP_1^{-1}P_i^{-1}Q_i|+|Q_1-P_1P_iP_1^{-1}Q_1|+|-Q_1+Q_1P_i| \\ &\leqq 2|1-P_1^{-1}||Q_i|+2|1-P_i||Q_1|\leqq 2|1-P_1||Q_i|+2|1-P_i||Q_1|\,. \end{aligned}$$

If $|1-P_1|\leqq 1/4$, then $|Q_{i+1}|\leqq(1/2)|Q_i|+2|1-P_i||Q_1|$. By (5.17), $\lim_{i\to+\infty}|1-P_i|=0$. Hence, for a given $\varepsilon>0$, there is an integer $n>0$ such that $2|1-P_i||Q_1|<\varepsilon$ for all $i\geqq n$. Therefore, if $i\geqq n$, we have $|Q_{i+1}|\leqq(1/2)|Q_i|+\varepsilon$. Hence, for $i\geqq n$, $k\geqq 0$, the inequality

$$|Q_{i+k}|\leqq(1/2)^k|Q_i|+\varepsilon\textstyle\sum_{l=0}^{k-1}(1/2)^l\leqq(1/2)^k|Q_i|+2\varepsilon$$

holds. Thus we have $\lim_{k\to+\infty}|Q_k|=0$. Therefore

$$g_k\to\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix},\qquad \text{as}\quad k\to+\infty\,.$$

Since $\tilde{\Gamma}_1$ is a discrete subgroup in G, this implies that $g_k=\mathrm{id}$ for a sufficiently large k. Then, by (5.16), we have $1=P_k=P_{k-1}=\cdots=P_3=[P_1, P_2]$. Since P_1 and P_2 were arbitrary in $K\cap W$, we see that $K\cap W$ is abelian, and hence so is $[K\cap W]_W$. Therefore the identity component $\bar{K}_0$ of $\bar{K}:=[K]_{SU(2)}$ is abelian. Thus (5.19) is proved.

Since $\bar{K}$ is compact, we see that the index $[\bar{K}:\bar{K}_0]$ is finite. This implies the sublemma. ■

By this sublemma, taking a suitable conjugate of K_0 in $SU(2)$, we may assume that K_0 consists of diagonal matrices. This contradicts the assumption (5.7). Thus we conclude that $\psi_1(\tilde{\Gamma}_1)=\psi_2(\tilde{\Gamma}_1)$ *cannot be contained in* $SU(2)$.

Step 6. In the final step, we consider the case where $K=\psi_1(\tilde{\Gamma}_1)=\psi_2(\tilde{\Gamma}_1)$ is a subgroup of $SU(1, 1)$.

SUBLEMMA 5.20. *If* $K\subset SU(1, 1)$, *then taking a suitable conjugate of* K *in* $SU(1, 1)$, *we may assume that* K *consists of diagonal matrices.*

PROOF. By (5.10) and a theorem of Burnside, K contains an element of infinite order. By (5.8), g may be assumed to be of the form

$$\begin{pmatrix} \alpha & 0\\ 0 & \bar{\alpha}\end{pmatrix},\qquad |\alpha|=1\,,$$

where α is *not* a root of unity. Let $h\in K$ be any element. Put

$$h=\begin{pmatrix} p & q\\ \bar{q} & \bar{p}\end{pmatrix}.$$

By (5.8), we have $|\mathrm{Re}\,\alpha^n p|\leqq 1$ for any $n\in\boldsymbol{Z}$. Since α is not a root of unity, this implies that $|p|\leqq 1$. Since $1=\det h=|p|^2-|q|^2$, we obtain $q=0$. Thus h is a diagonal matrix. This proves Sublemma 5.20. ■

By Sublemma 5.20 and the assumption (5.7), we conclude that $\psi_1(\tilde{\Gamma}_1)=\psi_2(\tilde{\Gamma}_1)$ *cannot be contained in* $SU(1, 1)$.

Thus the assumption (5.7) leads to contradictions in all cases. Hence Lemma 5.6

is proved. ■

We quote here results on compact complex surfaces. The following theorem is a part of [Ko 1, Theorem 19].

THEOREM 5.21 (Kodaira). *Let $S=G\backslash C^2$ be a compact complex surface, where G is a properly discontinuous group of holomorphic automorphisms without fixed points of C^2. If the canonical bundle of S is trivial, and if the fundamental group of S does not contain any abelian subgroup of finite index, then G is a nilpotent group generated by four elements g_1, g_2, g_3 and g_4 with relations $g_\lambda g_\mu = g_\mu g_\lambda$ for $(\lambda, \mu) \neq (3.4)$ and $g_3 g_4 = g_2^m g_4 g_3$, where m is a fixed non-zero integer. Moreover, with respect to a suitable system of coordinates on C^2, the four generators are represented by affine transformations of the following form*:

$$g_\nu = \begin{pmatrix} 1 & \bar{\alpha}_\nu & \beta_\nu \\ 0 & 1 & \alpha_\nu \\ 0 & 0 & 1 \end{pmatrix},$$

where the α_ν and β_ν are complex numbers such that
(i) $\alpha_1 = \alpha_2 = 0$,
(ii) α_3, α_4 *are linearly independent over* $\boldsymbol{R}$,
(iii) β_1, β_2 *are linearly independent over* $\boldsymbol{R}$, *and*
(iv) $\bar{\alpha}_3\alpha_4 - \bar{\alpha}_4\alpha_3 = m\beta_2 \neq 0$.

The following result is due to Suwa.

THEOREM 5.22 [Su, p. 245, Corollary]. *Let $S=G\backslash C^2$ be a compact complex surface, where G is a properly discontinuous group of affine transformations without fixed points of C^2. Then G contains a nilpotent subgroup G_1 of finite index such that, by a suitable linear change of coordinates on C^2, the linear part of G_1 consists of upper triangular matrices.*

Now we shall prove:

LEMMA 5.23. *There is a nilpotent subgroup $\tilde{\Gamma}_1$ of $\tilde{\Gamma}$ such that $[\tilde{\Gamma}:\tilde{\Gamma}_1]$ is finite. Moreover, by a suitable choice of homogeneous coordinates on $\boldsymbol{P}^3$, all elements of $\tilde{\Gamma}_1$ can be expressed as upper triangular unipotent matrices.*

PROOF. By Lemma 5.6, we can choose a system of homogeneous coordinates on $\boldsymbol{P}^3$ such that $\psi_2(\tilde{\Gamma}_1)$ consists of upper triangular matrices. Then the plane H_3 is $\gamma(\tilde{\Gamma}_1)$-invariant. The quotient $(H_3 - l_{23})/\gamma(\tilde{\Gamma}_1)$ is a compact non-singular surface. Since $H_3 - l_{23} \cong C^2$, it follows from Theorem 5.22 that, by a suitable linear change of coordinates on $H_3 - l_{23}$, the linear part of all elements of $\gamma(\tilde{\Gamma}_1) \mid H_3$ is represented by upper triangular matrices, and that $\gamma(\tilde{\Gamma}_1)$ contains a nilpotent subgroup of finite index.

By Proposition 5.4, we see that $\gamma^{-1}(\gamma(\tilde{\Gamma}_1))$ contains a desired subgroup. ∎

LEMMA 5.24. *The group $\tilde{\Gamma}_1$ of Lemma 5.23 contains an abelian subgroup of finite index.*

PROOF. Since any member of Γ_1 can be represented by an upper triangular unipotent matrix, the canonical mapping $\gamma\,|\,\tilde{\Gamma}_1 : \tilde{\Gamma}_1 \to \Gamma_1$ is an isomorphism. Suppose that $\Gamma_1 \cong \tilde{\Gamma}_1$ does not contain any abelian subgroup of finite index. Let Γ_H denote the group whose elements are the restrictions to H_3 of elements of Γ_1. In view of Theorem 5.21, there is a biholomorphic mapping $\Phi=(\varphi, \psi): \boldsymbol{C}^2 \to H_3 - l_{23}$ such that

$$\Phi(g_\nu(w_1, w_2)) = h_\nu(\Phi(w_1, w_2)) \tag{5.25}$$

for $\nu = 1, 2, 3, 4$, where the h_ν are the generators of Γ_H corresponding to g_ν. By Lemma 5.23, we can express h_ν in the form

$$h_\nu \begin{cases} u'_1 = u_1 + a_\nu u_2 + b_\nu \\ u'_2 = \qquad\quad u_2 + c_\nu\,, \end{cases}$$

where $u_1 = z_0/z_2$ and $u_2 = z_1/z_2$. The equality (5.25) is then written as

$$\varphi(w_1 + \bar{\alpha}_\nu w_2 + \beta_\nu,\ w_2 + \alpha_\nu) = \varphi(w_1, w_2) + a_\nu \psi(w_1, w_2) + b_\nu \tag{5.26}$$

$$\psi(w_1 + \bar{\alpha}_\nu w_2 + \beta_\nu,\ w_2 + \alpha_\nu) = \qquad\qquad \psi(w_1, w_2) + c_\nu\,. \tag{5.27}$$

From (5.27), we have

$$\psi = p_1 w_1 + p_2 w_2^2 + p_3 w_2 + p_4\,, \qquad p_\mu \in \boldsymbol{C}\,,$$

and

$$p_1 \bar{\alpha}_\nu + 2p_2 \alpha_\nu = 0 \tag{5.28}$$

$$p_1 \beta_\nu + p_2 \alpha_\nu^2 + p_3 \alpha_\nu = c_\nu \tag{5.29}$$

for all ν. Then equality $p_1 = p_2 = 0$ follows from the condition (iv) and (5.28). Hence we have

$$\psi = p_3 w_2 + p_4\,, \qquad p_3 \neq 0\,, \tag{5.30}$$

where,

$$p_3 \alpha_\nu = c_\nu\,. \tag{5.31}$$

It follows from (5.26) that

$$\varphi = q_1 w_1 + q_2 w_2^2 + q_3 w_2 + q_4\,, \qquad q_\mu \in \boldsymbol{C}\,,$$

with the relations

(5.32)
$$p_3 a_\nu = q_1 \bar{\alpha}_\nu + 2q_2\alpha_\nu\,,$$
$$p_4 a_\nu + b_\nu = q_1\beta_\nu + q_2\alpha_\nu^2 + q_3\alpha_\nu\,.$$

Since Φ is biholomorphic, $q_1 \neq 0$ holds. If $\alpha_\nu = 0$, then $a_\nu = c_\nu = 0$ follows from (5.31), (5.32) and $p_3 \neq 0$. Thus by the condition (i) we have

(5.33)
$$a_1 = c_1 = a_2 = c_2 = 0\,.$$

Since the mapping $\Gamma_1 \to \Gamma_H$ is bijective, there is a unique $\hat{h}_\nu \in \Gamma_1$ corresponding to h_ν for each ν. Note that the $\hat{h}_\nu$ satisfy the relations $\hat{h}_\lambda\hat{h}_\mu = \hat{h}_\mu\hat{h}_\lambda$ for $(\lambda, \mu) \neq (3, 4)$ and $\hat{h}_3\hat{h}_4 = \hat{h}_2^m\hat{h}_4\hat{h}_3$. Suppose that $\hat{h}_\nu$ is represented by

$$H_\nu = \begin{pmatrix} 1 & a_\nu & b_\nu & r_\nu \\ 0 & 1 & c_\nu & s_\nu \\ 0 & 0 & 1 & t_\nu \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Proposition 5.4 and (5.33), we infer that H_1 and H_2 are of Type II. Hence

(5.34)
$$t_1 = t_2 = 0\,,$$

and

(5.35)
$$b_1 b_2 s_1 s_2 \neq 0\,.$$

By the relations $\hat{h}_\lambda\hat{h}_\mu = \hat{h}_\mu\hat{h}_\lambda$ for $(\lambda, \mu) \neq (3, 4)$ and $\hat{h}_3\hat{h}_4 = \hat{h}_2^m\hat{h}_4\hat{h}_3$, and by (5.33) and (5.34), we obtain the following equations:

(5.36) $\quad a_3 s_1 = t_3 b_1 \qquad a_4 s_1 = t_4 b_1$

(5.37) $\quad a_3 s_2 = t_3 b_2 \qquad a_4 s_2 = t_4 b_2$

(5.38) $\quad a_3 c_4 = a_4 c_3 + m b_2$

(5.39) $\quad c_3 t_4 = c_4 t_3 + m s_2\,.$

If $a_3 \neq 0$, then $t_3 \neq 0$ and $b_1/s_1 = b_2/s_2 = a_3/t_3$ follows from (5.35), (5.36) and (5.37). Hence $c_4 t_3 = c_3 t_4 + m s_2$ follows from (5.36) and (5.38). Therefore we have $s_2 = 0$ by (5.39). This contradicts (5.35). Thus $a_3 = 0$ and hence $t_3 = 0$. Similarly, $a_4 = t_4 = 0$ holds. Combining these with (5.33) and (5.34), we see that Γ_H is abelian. This contradicts the assumption. ■

The proof of Proposition 5.5 is now clear by Lemma 5.24.

PROPOSITION 5.40. *Suppose that Γ is abelian. Then, with respect to a certain system of homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ on $\boldsymbol{P}^3$ satisfying $l = \{z_2 = z_3 = 0\}$, Γ is represented by a subgroup $\tilde{\Gamma}$ of either*

(A)
$$\left\{\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} ;\ a, b, c \in \boldsymbol{C}\right\}$$

or

(B)
$$\left\{\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ;\ a, b, c, d \in \boldsymbol{C}\right\}.$$

If $\tilde{\Gamma}$ is in (B), *then any element $g \in \Gamma$ except the identity satisfies* $\mathrm{rank}(I-g)=2$. *In any case, the rank of Γ is* 4.

PROOF. By Proposition 5.4, every element of $\Gamma-\{\mathrm{id}\}$ is either of Type II or Type III. Since Γ is abelian, and since Γ leaves the line l invariant, there is a system of homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ on $\boldsymbol{P}^3$ such that

(5.41) $$l=\{z_2=z_3=0\}\ .$$

Put $\tilde{S}=\{z_3=0\}-l$, which is biholomorphic to $\boldsymbol{C}^2$. Note that the restriction $\Gamma \to \Gamma \mid \tilde{S}$ is bijective and that $(\Gamma \mid \tilde{S})\backslash\tilde{S}$ is a complex torus of dimension 2, where $\Gamma \mid \tilde{S}=\{g \mid \tilde{S} : g \in \Gamma\}$. Hence we see that $\mathrm{rank}\,\Gamma=4$ and that

(5.42) every element of Γ is represented by an upper triangular unipotent 4×4 matrix.

First suppose that Γ contains no elements of Type III. Let g be any element of $\Gamma-\{\mathrm{id}\}$ and let

(5.43) $$G=\begin{pmatrix} 1 & a_1 & b_1 & c \\ 0 & 1 & a_2 & b_2 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

be a representative of g. Since $(I-G)^2=0$, we have

(5.44) $$a_1a_2=a_2a_3=a_1b_2+a_3b_1=0\ .$$

Suppose that $a_3 \neq 0$. Then $a_2=0$ follows from (5.44). Moreover, $[a_1 : b_2 : a_3 : 0]$ is a fixed point of G outside l. This is absurd. Hence we obtain $a_3=0$. By $\mathrm{rank}(I-G)=2$, $a_1=0$ follows from (5.44). Thus we are in the case (B). Next suppose that Γ contains an element g of Type III. Let G be a representative of $g \in \Gamma-\{\mathrm{id}\}$ of the form (5.43). Then we have $a_1a_2a_3 \neq 0$. Replace Γ with $\tau^{-1}\Gamma\tau$, where τ is represented by

$$\begin{pmatrix} a_1a_2a_3 & a_1b_2+a_3b_1 & c & 0 \\ 0 & a_2a_3 & b_2 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $\tau^{-1}g\tau$ is represented by

$$J=\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let

$$H=\begin{pmatrix} 1 & p & q & r \\ 0 & 1 & s & t \\ 0 & 0 & 1 & u \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

be a representative of an arbitrary element $h \in \Gamma - \{\mathrm{id}\}$. Since Γ is abelian, we have easily $HJ=JH$. From this equation it follows that $p=s=u$ and $q=t$. Thus we are in the case (A). ∎

Combining Lemma 5.3, Propositions 5.5 and 5.40, we have the following theorem, which gives a (rough) classification of Blanchard manifolds up to finite unramified coverings.

THEOREM C. *Let $\Gamma\backslash\Omega$ be any Blanchard manifold. Then Γ is torsion free and contains an abelian subgroup Γ_1 of rank* 4 *with $[\Gamma:\Gamma_1] < +\infty$. Moreover we can choose Γ_1 so that it is conjugate in $PGL(4, \boldsymbol{C})$ to a subgroup of either* (A) *or* (B) *in Proposition* 5.40.

In the following, a Blanchard manifolds is said to be of *type* A (resp. *type* B) if its fundamental group contains an abelian subgroup of finite index which is conjugate to a subgroup of (A) but not (B) (resp. a subgroup of (B)).

EXAMPLE 1. First we shall give an example of Blanchard manifolds of type A. Let $\tilde{\Gamma}$ be a subgroup of $SL(4, \boldsymbol{C})$ generated by $G_1=I+N$, $G_2=I+iN$, $G_3=I+N^2$ and $G_4=I+iN^2$, where $i=\sqrt{-1}$, and

$$N=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Put $\Gamma=\gamma(\tilde{\Gamma})$, $l=\{z_2=z_3=0\}$ and $\Omega=\boldsymbol{P}^3-l$. Then $\Gamma\backslash\Omega$ is a (P)-manifold of Schottky type. A proof of this fact will be given in Appendix.

EXAMPLE 2. Next example is a Blanchard manifold of type B, which is classical. We define a group of automorphisms of $\boldsymbol{P}^3-l_{23}$ as follows. Let A_j, $j=1, 2, 3, 4$, be the elements of $GL(2, \boldsymbol{C})$ satisfying $\det(\sum_{j=1}^{4} r_j A_j)\neq 0$ for all $(r_j)\in \boldsymbol{R}^4-\{(0, 0, 0, 0)\}$. It is not difficult to find such matrices. Define $G_j\in GL(4, \boldsymbol{C})$ by the 4×4 matrix

$$\begin{pmatrix} I & A_j \\ 0 & I \end{pmatrix},$$

where I is the 2×2 identify matrix. Let $\tilde{\Gamma}$ be the abelian subgroup generated by the four elements G_j, $j=1, 2, 3, 4$. Put $\Gamma=\gamma(\tilde{\Gamma})$ and $\Omega=\boldsymbol{P}^3-l_{23}$. Then $\Gamma\backslash\Omega$ is a (P)-manifold of Schottky type, which is a classical Blanchard manifold defined in [B].

REMARK. Blanchard manifolds of type (A) and type (B) are not biholomorphic to each other. Indeed, if they were biholomorphic, the representation of their fundamental groups in $PGL(4, \boldsymbol{C})$ defined by their flat projective structures must be conjugate to each other in $PGL(4, \boldsymbol{C})$, since a manifold of Class L admits only a unique flat projective structure [Ka3].

6. Proof of Theorem A. Our proof goes along almost the same line as Kulkarni's [Ku, p. 266]. Let $X=\Gamma\backslash\Omega$ be a compact manifold of Schottky type. Assume that Ω is simply connected and Γ is torsion free. By a theorem of Hopf[Ho, Satz I], the cardinality of the ends of Ω is one, two or that of a continuum. Suppose that Ω has an uncountable number of ends. Since $[\Omega]=\boldsymbol{P}^3$ and since Γ is finitely generated, we can apply a theorem of Kulkarni [Ku, Theorem 5.1], and see that Γ has an uncountable number of ends as an abstract group. Hence by a theorem of Stallings [St], Γ can be written as a free product of two proper subgroups, $\Gamma=\Gamma_1 * \Gamma_2$. By Proposition 2.1, X is a Klein combination of two manifolds, $X_\nu=\Gamma_\nu\backslash\Omega_\nu$, $\nu=1, 2$, of Schottky type. Since $[\Omega_\nu]=\boldsymbol{P}^3$ and since the Γ_ν are torsion free and finitely generated, Γ_1(resp. Γ_2) can be written again as a free product of proper subgroups $\Gamma_1=\Gamma_3 * \Gamma_4$ (resp. $\Gamma_2=\Gamma_3 * \Gamma_4$), when Ω_1(resp. Ω_2) has an uncountable number of ends. Grushko's theorem says that, if a group G is a free product of groups G_1 and G_2, then the minimal number of the generators for G is the sum of the corresponding numbers for G_1 and G_2. Hence the above process of factoring Γ terminates in a finite number of steps. Thus Γ is written as

$$\Gamma=\Gamma_1*\Gamma_2*\cdots*\Gamma_r*\Gamma_{r+1}*\cdots*\Gamma_s,$$

where r, $0\leqq r\leqq s$, is an integer such that

(i) each Γ_i, $1\leqq i\leqq r$, has two ends,

(ii) each Γ_i, $r<i\leqq s$, has one end,

(iii) X is a Klein combination of $\Gamma_\nu\backslash\Omega_\nu$, $\nu=1, \cdots, s$.

In case (i), $\Gamma_\nu\backslash\Omega_\nu$ is a primary L-Hopf manifold by Proposition 4.6. In case (ii), $\Gamma_\nu\backslash\Omega_\nu$ is a Blanchard manifold. Hence the theorem follows from Theorems B and C. ■

7. **Proof of Theorem D.** To prove Theorem D, first we prepare elementary topological facts. Let Ω be a domain in $\boldsymbol{P}^3$ and put $\Lambda=\boldsymbol{P}^3-\Omega$. Let α be any connected component of Λ.

LEMMA 7.1. *$\boldsymbol{P}^3-\alpha$ is open and connected.*

PROOF. Since Λ is closed in $\boldsymbol{P}^3$, so is any connected component of Λ. Hence $\boldsymbol{P}^3-\alpha$ is open. Since $\boldsymbol{P}^3-\alpha$ is locally connected, any connected component of $\boldsymbol{P}^3-\alpha$ is open in $\boldsymbol{P}^3-\alpha$ and hence in $\boldsymbol{P}^3$. Let V be any connected component of $\boldsymbol{P}^3-\alpha$ which does not intersect Ω. The boundary ∂V is contained in α. Therefore $\alpha\cup V=\alpha\cup[V]$ is connected. Then, since $\alpha\cup V\subset\Lambda$ and since α is a connected component of Λ, V is contained in α. This is absurd. Hence every connected component of $\boldsymbol{P}^3-\alpha$ meets Ω. Thus $\boldsymbol{P}^3-\alpha$ is connected, since Ω is connected. ■

LEMMA 7.2. *Suppose further that Ω contains a line l_1. Then there is a system of open neighborhoods $\{A_n\}_{n\in\boldsymbol{N}}$ of α in $\boldsymbol{P}^3$ which has the following properties;*

(0) *$\boldsymbol{P}^3-A_n$ contains l_1 for all n,*

(1) *A_n is connected for all n,*

(2) *$\boldsymbol{P}^3-A_n$ is connected for all n,*

(3) *$A_n\supset A_{n+1}$ for all n, and*

(4) *$\bigcap_n A_n=\alpha$.*

PROOF. It is easy to construct a system of open neighborhoods $\{A'_n\}$ which satisfies (0), (1), (3) and (4). Denote by A^c_n the unique (closed) connected component of $\boldsymbol{P}^3-A'_n$ which contains l_1. Put $A_n=\boldsymbol{P}^3-A^c_n$. Obviously $\{A_n\}$ satisfies (0), (1) and (3). Now we shall show that $\{A_n\}$ is the desired system. First we shall prove (2). Put $\boldsymbol{P}^3-A'_n=A^c_n\cup\bigcup_\lambda L_{n\lambda}$, where the $L_{n\lambda}$ are non-empty (closed) connected components of $\boldsymbol{P}^3-A'_n$ other than A^c_n. Then we have

$$A_n=A'_n\cup\bigcup_\lambda L_{n\lambda}.$$

It suffices to show that $\partial A'_n\cap L_{n\lambda}\neq\varnothing$ holds for any λ. Suppose that $\partial A'_n\cap L_{n\lambda_0}=\varnothing$ holds for some $\lambda=\lambda_0$. Then by $A'_n\cap L_{n\lambda_0}=\varnothing$, we have $[A'_n]\cap L_{n\lambda_0}=\varnothing$. Let L be the connected component of $P^3-[A'_n]$ such that $L_{n\lambda_0}\subset L$. From the relation $L_{n\lambda_0}\subset L\subset P^3-A'_n$, we infer that $L_{n\lambda_0}=L$. Since $L_{n\lambda_0}$ is closed in P^3, so is L. On the other hand, since $P^3-[A'_n]$ is locally connected, L is open in $P^3-[A'_n]$ and hence in P^3. Therefore we conclude that L is empty by $L\neq P^3$. This is a contradiction. Thus (2) is verified. Next we shall prove (4). Put $A=\bigcap_n A_n$. It suffices to show that $A\subset\alpha$. Suppose that there is a point $x\in A-\alpha$. By Lemma 7.1, $P^3-\alpha$ is pathwise connected, because so is a connected open set of a manifold. Let C be a path in $P^3-\alpha$ joining x with a point $y\in l_1$. By $\bigcap_n A'_n=\alpha$, it follows that there is an integer n_0 such that $x\in A_n-A'_n$ for all $n\geqq n_0$. Note that $C\cap A'_n=\varnothing$ for $n\geqq n_0$ holds, since otherwise C would be contained in A^c_n, and consequently $x\in A^c_n$. This is absurd. Let $z_n\in C\cap A'_n$. Choosing a suitable subsequence, we may assume that $\lim_{n\to\infty}z_n=z\in C$ exists. Since $z_n\in A'_n$ and $\bigcap_n A'_n=\alpha$, this implies that $z\in\alpha$. Hence $z\in C\cap\alpha\subset(P^3-\alpha)\cap\alpha=\varnothing$, a contradiction. This proves (4). Thus the lemma is proved. ■

For a subset W of P^3, we denote by $\hat{W}$ the subset in the Grassmann manifold $\mathrm{Gr}:=\mathrm{Gr}(4, 2)$ which parametrizes lines in W. Similarly, we denote by $\hat{l}$ the point in Gr which corresponds to a line l in P^3. The next lemma is a key to the proof of Theorem 7.6, from which Theorem D follows immediately.

LEMMA 7.3. *Let $X=\Gamma\backslash\Omega$ be a (P)-manifold and α a connected component of the limit set Λ. Then α is a line if the following conditions* (i) *and* (ii) *are satisfied.*

(i) *There is a compact subset K in Ω which has the following properties.*

  (i-a) *Through any point in K, there passes a line contained in K.*

  (i-b) *For any point $v\in\partial\alpha$ and for any neighborhood V of v on P^3, there is an element $g\in\Gamma$ such that $V\cap g(K)\neq\varnothing$.*

(ii) *There are subdomains W_1, $W_{1\varepsilon}$, $W_1\Subset W_{1\varepsilon}$ in P^3, and a sequence $\{g_j\}$ of distinct elements of Γ which have the following properties;*

  (ii-a) *W_1 and $W_{1\varepsilon}$ are biholomorphic to U,*

  (ii-b) *some neighborhood of $[W_{1\varepsilon}-W_1]$ is contained in Ω,*

  (ii-c) *$\alpha\subset g_j(W_1)\subset g_j(W_{1\varepsilon})\subset W_1$ and $g_{j+1}(W_1)\subset g_{j+1}(W_{1\varepsilon})\subset g_j(W_1)\subset g_j(W_{1\varepsilon})$ for all j.*

PROOF. Let v be any point on $\partial\alpha$ and $\{V_j\}_{j=1}^{\infty}$ be a system of open neighborhood of v in P^3 such that $V_j\supset V_{j+1}$ and $\bigcap_j V_j=\{v\}$. By (i-b), for any j, there is an $h_j\in\Gamma$ such that $V_j\cap h_j(K)\neq\varnothing$. Therefore by (i-a) there is a line l'_j in K such that $V_j\cap h'_j(l'_j)\neq\varnothing$. Since the action of Γ on Ω is properly discontinuous, we can choose a subsequence of $\{l'_j\}$ such that $\{l_j\}$, $l_j=h'_j(l'_j)$, converges to a line l_∞ in Λ. Obviously, we have $v\in l_\infty\cap\partial\alpha\subset l_\infty\cap\alpha$. Hence $l_\infty\subset\alpha$. This implies that, *for any point of $\partial\alpha$, there is a line passing through the point.* Therefore to prove the lemma it suffices to show that $\hat{\alpha}$ is a single point. From the argument above it follows in particular that *$\hat{\alpha}$ is not empty.* In Sublemma 7.5 below, we shall show that $\hat{\alpha}$ is indeed a single point.

Each g_j induces an injective holomorphic mapping $\hat{g}_j$: $\hat{W}_{1\varepsilon}\to\hat{W}_1$. Since $\hat{W}_1$ is biholomorphic to a bounded domain, $\{\hat{g}_j\}$ forms a normal family. Taking a convergent subsequence, we assume that $\{\hat{g}_j\}$ itself converges to a holomorphic mapping $\hat{g}_\infty: \hat{W}_{1\varepsilon} \to \hat{W}_1$ uniformly on any compact subset of $\hat{W}_{1\varepsilon}$.

SUBLEMMA 7.4. $\hat{\alpha} = \hat{g}_\infty(\hat{W}_{1\varepsilon})$.

PROOF. First we shall show $\hat{\alpha} \subset \hat{g}_\infty(\hat{W}_{1\varepsilon})$. Let $\hat{l} \in \hat{\alpha}$ be any point. Since $l \subset g_j(W_1)$, for any j, there is a line $l_j \subset W_1$ such that $l = g_j(l_j)$. We can choose a subsequence $\{g'_j\}$ of $\{g_j\}$ such that the corresponding subsequence $\{\hat{l}'_j\}$ of $\{\hat{l}_j\}$ converges to a point in $[\hat{W}_1]_{\mathrm{Gr}}$. Put $\hat{l}_0 = \lim_j \hat{l}'_j$. Then, since the convergence $\hat{g}_j \to \hat{g}_\infty$ is uniform on $[\hat{W}_1]_{\mathrm{Gr}}$, we have $\hat{l} = \lim_j \hat{g}'_j(\hat{l}'_j) = \lim_j \hat{g}'_j(\hat{l}_0) = \hat{g}_\infty(\hat{l}_0)$. Hence $\hat{l} \in \hat{g}_\infty([\hat{W}_1]_{\mathrm{Gr}}) \subset \hat{g}_\infty(\hat{W}_{1\varepsilon})$. Thus we obtain $\hat{\alpha} \subset \hat{g}_\infty(\hat{W}_{1\varepsilon})$. Conversely, we shall show $\hat{\alpha} \supset \hat{g}_\infty(\hat{W}_{1\varepsilon})$. Put $T = W_{1\varepsilon} - [W_1]$, which is a subdomain in Ω by (ii-b). Take any line l in T. Since the action of Γ on Ω is properly discontinuous, we see that the limit line l_∞, $\hat{l}_\infty := \hat{g}_\infty(\hat{l}) = \lim_j \hat{g}_j(\hat{l})$, does not intersect Ω, i.e., $l_\infty \subset \Lambda$. Let l' be another line in T. There is a path $\hat{C}$ in $\hat{T}$ which joins $\hat{l}$ and $\hat{l}'$. Since the action of Γ on Ω is properly discontinuous, $\hat{g}_\infty(\hat{C}) \subset \hat{\Lambda}$ holds. Since $\hat{C}$ is connected, there is a connected component β of Λ such that both $\hat{g}_\infty(\hat{l})$ and $\hat{g}_\infty(\hat{l}')$ are on the same $\hat{\beta}$. Therefore we have

$$\hat{g}_\infty(\hat{T}) \subset \hat{\beta}\,.$$

Now we claim $\hat{g}_\infty(\hat{W}_{1\varepsilon}) \subset \hat{\beta}$. By Lemma 7.2, there is a system of neighborhoods $\{B_n\}_{n \in \boldsymbol{N}}$ of β in $\boldsymbol{P}^3$ which has the following properties;

(1) B_n is connected for all n,
(2) $\boldsymbol{P}^3 - B_n$ is connected for all n,
(3) $B_n \supset B_{n+1}$ for all n, and
(4) $\bigcap B_n = \beta$.

Put $T' = [W_{1\delta} - W_{1\delta'}]$, where $1 < \delta' < \delta < \varepsilon$. Since $\hat{g}_j \to \hat{g}_\infty$ is uniformly convergent on $\hat{T}'$, we see by the above argument that, for any $n > 0$, there is an integer j_n such that $g_j(T') \subset B_n$ for all $j > j_n$. Then $g_j(W_1) \subset B_n$ follows for $j > j_n$, since $g_j(W_1) \subset W_1$ and since $\boldsymbol{P}^3 - B_n$ is connected. This implies that $\bigcap_{j \geqq 0} g_j(W_1) \subset \beta$. Thus we have $\hat{g}_\infty(\hat{W}_1) \subset \hat{\beta}$. This together with $\hat{g}_\infty(\hat{T}) \subset \hat{\beta}$ verifies the claim. Since $\hat{\alpha} \subset \hat{g}_\infty(\hat{W}_{1\varepsilon})$ as shown above and since $\hat{\alpha}$ is not empty, there is a line in α which is parametrized by a point of $\hat{g}_\infty(\hat{W}_{1\varepsilon})$. Hence $\alpha \cap \beta \neq \varnothing$, i.e., $\alpha = \beta$. This implies $\hat{g}_\infty(\hat{W}_{1\varepsilon}) \subset \hat{\alpha}$. Thus the sublemma is proved. ∎

SUBLEMMA 7.5. *$\hat{\alpha}$ is a point.*

PROOF. Since α is compact, the corresponding set $\hat{\alpha}$ is compact. Therefore, by Sublemma 7.4, the holomorphic mapping $\hat{g}_\infty$ has a compact image in $\hat{W}_1$. Since $\hat{W}_1$ is biholomorphic to a bounded domain, $\hat{g}_\infty$ is a constant mapping. Consequently, $\hat{\alpha}$ is a single point. ∎

Clearly Lemma 7.3 follows from Sublemma 7.5.

PROPOSITION 7.6. *A Klein combination of (P)-manifolds is a (P)-manifold.*

PROOF. Let $X_1=\Gamma_1\backslash\Omega_1$ and $X_2=\Gamma_2\backslash\Omega_2$ be (*P*)-manifolds and $X=\mathrm{Kl}(X_1, X_2, j_1, j_2, \Sigma)$ the Klein combination of them. By the definition of the Klein combination, there is a tubular neighborhood W of Σ such that the mappings j_ν are holomorphic open embeddings of $W_\nu=W\cup W'_\nu$ into X_ν, where the W'_ν are the connected components of $\boldsymbol{P}^3-\Sigma$. The manifold X is the union $X_1^\sharp\cup X_2^\sharp$, $X_\nu^\sharp=X_\nu-j_\nu(W_\nu-W)$, where $j_1(x)\in j_1(W)$, $x\in W$, is identified with $j_2(x)\in j_2(W)$. Let $\tilde{j}_\nu$: $W_\nu\to\Omega_\nu\subset\boldsymbol{P}^3$ be a lift of j_ν. Note that $\tilde{j}_\nu$ extends to an element of $PGL(4, \boldsymbol{C})$ [Ka3, Lemma 3.2]. Put $\tilde{W}_\nu=\tilde{j}_\nu(W_\nu)$ and $\tilde{\Sigma}_\nu=\tilde{j}_\nu(\Sigma)$. Let $\tilde{F}_\nu$ be a fundamental region for Γ_ν in Ω_ν which contains $\tilde{W}_\nu$. By $\tilde{j}_\nu^{-1}$, we regard $\tilde{F}_\nu$ as a subset in $\boldsymbol{P}^3$ which contains W_ν and $\tilde{\Sigma}_\nu$ as Σ. Put $\tilde{F}=(\tilde{F}_1-W'_1)\cup(\tilde{F}_2-W'_2)$ and $\Omega=\bigcup_{g\in\Gamma}g(\tilde{F})$, where Γ is a subgroup of $PGL(4, \boldsymbol{C})$ generated by $\tilde{j}_\nu^{-1}\Gamma_\nu\tilde{j}_\nu$, $\nu=1, 2$. Then it is easy to see that Ω is the universal convering of X, $\tilde{F}$ is a fundamental region for Γ and that Γ is isomorphic to the free product of Γ_1 and Γ_2 (cf. [Ma, p. 302]). Thus X is a (*P*)-manifold. ■

THEOREM 7.7. *Suppose that $X_1=\Gamma_1\backslash\Omega_1$ and $X_2=\Gamma_2\backslash\Omega_2$ are (P)-manifolds. Then $X=\mathrm{Sum}(X_1, X_2, j_1, j_2)$ is a (P)-manifold of Schottky type if and only if both X_1 and X_2 are of Schottky type.*

PROOF. The "only if" part follows from the fact that every connected component of Λ_ν is a connected component of Λ. The rest of this section is devoted to the proof of the "if" part. Suppose that X_1 and X_2 are (*P*)-manifolds of Schottky type and X is represented by $X=\Gamma\backslash\Omega$. In view of Proposition 7.6, it is enough to show that any connected component α of $\Lambda=\boldsymbol{P}^3-\Omega$ is a line. Put $\Sigma=\partial U$. Then $N_\varepsilon=U_\varepsilon-[U_{1/\varepsilon}]$ is a tubular neighborhood of Σ in $\boldsymbol{P}^3$. Let W_1 and W_2 be the connected component of $\boldsymbol{P}^3-\Sigma$. Put $W_{\nu\varepsilon}=W_\nu\cup N_\varepsilon$. By choosing a suitable $\varepsilon>1$, we can form the manifold X as the union $X_{1\varepsilon}^\sharp\cup X_{2\varepsilon}^\sharp$, where $X_{\nu\varepsilon}^\sharp=X_\nu-j_\nu(W_{\nu\varepsilon}-N_\varepsilon)$, and, for $x\in N_\varepsilon$, $j_1(x)\in j_1(N_\varepsilon)$ is identified with $j_2(x)\in j_2(N_\varepsilon)$. Let $p: \Omega\to X$ be the covering projection. By our construction of X, Ω contains the hypersurface Σ.

For $K=\Sigma$, the condition (i-a) of Lemma 7.3 is satisfied.

Now we shall construct a sequence of distinct elements of Γ satisfying the condition (ii-c). Coose a fundamental region F for Γ in Ω so that F contains N_ε. The set $F_\nu=F\cup W_\nu$ is a fundamental region in Ω_ν for Γ_ν. Let $\Omega_\nu^\sharp$ be the connected component of $p^{-1}(X_\nu^\sharp)$ such that $\partial\Omega_\nu^\sharp$ contains K as a connected component, where $X_\nu^\sharp=X_\nu-j_\nu([W_\nu])$. Note that $\Omega_1^\sharp\subset W_2$ and $\Omega_2^\sharp\subset W_1$. We have

$$\boldsymbol{P}^3-\Omega_\nu^\sharp=\Lambda_\nu\cup\bigcup_{g\in\Gamma_\nu}g([W_\nu]) \qquad \nu=1, 2, \tag{7.8}$$

where the right-hand side is a disjoint union. Let α be a connected component of Λ. If α is contained in $g(\Lambda_1)$ or $g(\Lambda_2)$ for some $g\in\Gamma$, then α is a line, since both X_1 and X_2 are of Schottky type. Thus we assume that α *is contained in neither $g(\Lambda_1)$ nor $g(\Lambda_2)$ for any $g\in\Gamma$*. Since $\alpha\cap[N_\varepsilon]=\varnothing$, we may assume $\alpha\subset W_1-[N_\varepsilon]$ without loss of generality. By (7.8) togehter with $\alpha\cap\Omega_2^\sharp=\varnothing$ and $\alpha\cap\Lambda_2=\varnothing$, there is an element $g'_1\in\Gamma_2$ such that

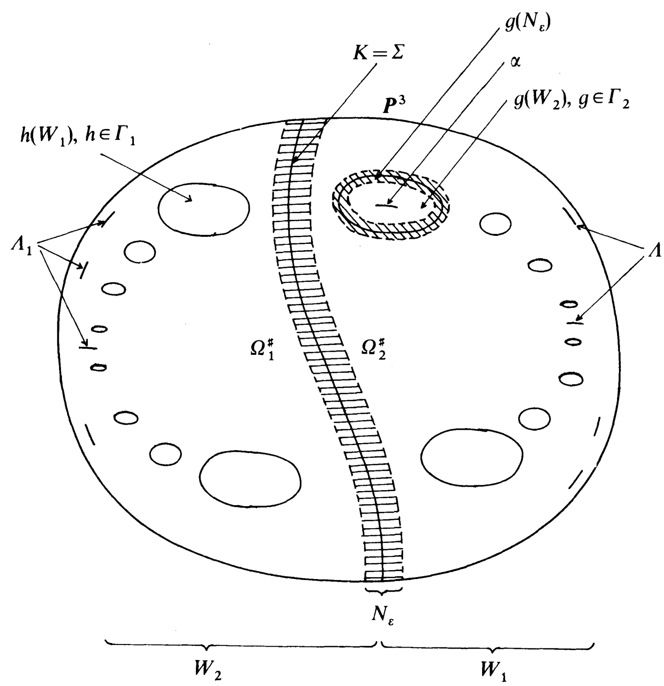

FIGURE 3.

$\alpha\subset g_1'([W_2])$. By (7.8) together with $\alpha\cap g_1'(\Omega_1^\#)=\varnothing$ and $\alpha\cap g_1'(\Lambda_1)=\varnothing$, there is an element $g_1''\in\Gamma_1$ such that $\alpha\subset g_1'g_1''([W_1])$. Put $g_1=g_1'g_1''$. Obviously $g_1([W_{1\varepsilon}])\subset W_1$. By (7.8) together with $\alpha\cap g_1(\Omega_2^\#)=\varnothing$ and $\alpha\cap g_1(\Lambda_2)=\varnothing$, there is an element $g_2'\in\Gamma_2$ such that $\alpha\subset g_1g_2'([W_2])$. By (7.8) together with $\alpha\cap g_1g_2'(\Omega_1^\#)=\varnothing$ and $\alpha\cap g_1g_2'(\Lambda_1)=\varnothing$, there is an element $g_2''\in\Gamma_1$ such that $\alpha\subset g_1g_2'g_2''([W_1])$. Put $g_2=g_1g_2'g_2''$. Obviously $g_2([W_{1\varepsilon}])\subset g_1(W_1)\subset g_1([W_{1\varepsilon}])\subset W_1$. Continuing this process, we obtain a sequence $\{g_j\}_{j=1}^\infty$ of distinct elements of Γ which satisfies the condition (ii-c).

To apply Lemma 7.3, it remains to check the condition (i-b). Take a point v on $\partial\alpha$ and its spherical open neighborhood B in P^3 with the center v. We claim that $B\cap(\bigcup_{g\in\Gamma}g(K))\neq\varnothing$. To verify this, assuming the equality $B\cap(\bigcup_{g\in\Gamma}g(K))=\varnothing$, we derive a contradiction. Let V be a connected component of $B\cap\Omega$. Then V is open. Since the image set $p(V)$ in X is connected and does not intersect $\Sigma=p(K)$, $p(V)$ is contained either in $X_1^\#$ or $X_2^\#$. We may assume $p(V)\subset X_1^\#$ without loss of generality. Then there is an element $g\in\Gamma$ such that $V\subset g(\Omega_1^\#)$. Replacing $\Omega_1^\#$ with another suitable connected component of $p^{-1}(X_1^\#)$ if necessary, we may assume that $g=1$, i.e.,

$$V\subset\Omega_1^\#=P^3-\left(\Lambda_1\cup\bigcup_{g\in\Gamma_1}g([W_1])\right). \tag{7.9}$$

If $\partial V_B\subset\Omega$, then $v\in B\subset\Omega$, a contradiction. Hence $\partial V_B\not\subset\Omega$. Take any point $x\in\partial V_B-\Omega$.

Suppose that $x \in \Omega_1^\#$. Since $\Omega_1^\#$ is an open set, there is a connected neighborhood V' of x such that $V' \subset B \cap \Omega_1^\# \subset B \cap \Omega$. Since $x \in V' - V$ and since $V \cap V' \neq \varnothing$, this contradicts the fact that V is a connected component of $B \cap \Omega$. Hence x is *not* contained in $\Omega_1^\#$. Since $V \subset \Omega_1^\#$, we have $x \in \partial V \subset [\Omega_1^\#]$. Note that $\partial \Lambda_1 = \Lambda_1$ holds by the assumption that X_1 is of Schottky type. Hence it follows from (7.9) that

$$x \in \partial \Omega_1^\# = \partial\left(\Lambda_1 \cup \bigcup_{g \in \Gamma_1} g([W_1])\right) \subset \Lambda_1 \cup \partial\left(\bigcup_{g \in \Gamma_1} g([W_1])\right). \tag{7.10}$$

Suppose that $x \in \partial(\bigcup_{g \in \Gamma_1} g([W_1]))$. If $x \in \Omega_1$, then we can choose a system of relatively compact neighborhoods $\{V_k\}$, $k = 1, 2, \cdots$, of x in Ω_1 such that $V_k \supset V_{k+1}$ and $\bigcap_k V_k = \{x\}$. For any k, there is $g_k \in \Gamma_1$ such that $V_k \cap g_k([W_1]) \neq \varnothing$. Since $[W_1]$ is a compact subset contained in Ω_1, and since the action of Γ_1 on Ω_1 is properly discontinuous, the set $\{g_k : k = 1, 2, \cdots\}$ is a finite set. Hence we have $x \in g_{k'}([W_1])$ for some k'. Since $x \notin \Omega_1^\#$, this implies $x \in g_{k'}(K)$, a contradiction. Hence we have $x \notin \Omega_1$, i.e., $x \in \Lambda_1$. Thus from (7.10), $x \in \Lambda_1$ follows in any case. Since x is an arbitrary point in $\partial V_B - \Omega$, we have $\partial V_B - \Omega \subset \Lambda_1$. Since $[\Omega_1] = \boldsymbol{P}^3$ because of the assumption that X_1 is of Schottky type, this inclusion implies that the connected component V is dense in $B \cap \Omega$. Hence $B \cap \Omega$ is connected. Now replace V with $B \cap \Omega$ and repeat the above argument. Then we obtain the inclusion relation $\partial(B \cap \Omega)_B - \Omega \subset \Lambda_1$. Hence we have

$$v \in \partial\alpha \cap B \subset (\partial\Omega) \cap B - \Omega \subset \partial(B \cap \Omega)_B - \Omega \subset \Lambda_1 .$$

Since v is an arbitrary point on $\partial\alpha$, we see that $\partial\alpha \subset \Lambda_1$. Then $\alpha \subset \Lambda_1$ follows easily. Since we have assumed that α is contained in neither $g(\Lambda_1)$ nor $g(\Lambda_2)$ for any $g \in \Gamma$, this is a contradiction. Hence the condition (i-b) is verified.

Thus the "if" part of the proposition follows from Lemma 7.3. ■

As a corollary we have:

THEOREM D. *Suppose that X is a complex analytic connected sum of several copies of* L-*Hopf manifolds and Blanchard manifolds. Then X is a* (*P*)-*manifold of Schottky type.*

Appendix. We shall prove that the pair (Ω, Γ) in Example 1, §5, defines a (P)-manifold. It suffices to show that the action of Γ on Ω is free and properly discontinuous and that the quotient space is compact.

Any element $G \in \tilde{\Gamma}$ is given by $G = G_1^m G_2^n G_3^p G_4^q$ with $m, n, p, q \in \boldsymbol{Z}$. For $a = (a_0, a_1, a_2, a_3) \in \boldsymbol{C}^4$, put $Ga = (a'_0, a'_1, a'_2, a'_3)$. Then we have

$$\begin{aligned} a'_0 = a_0 &+ (m+in)a_1 + (m(m-1)/2 - n(n-1)/2 - imn + p + iq)a_2 \\ &+ (m(m-1)(m-2)/6 - mn(n-1)/2 + i(m(m-1)n/2 - n(n-1)(n-2)/6) \\ &+ mp - nq + i(mq+np))a_3 , \end{aligned} \tag{A.1}$$

$$a'_1 = a_1 + (m+in)a_2 + (m(m-1)/2 - n(n-1)/2 - imn + p + iq)a_3 , \tag{A.2}$$

(A.3) $\quad a'_2 = a_2 + (m+in)a_3$,

(A.4) $\quad a'_3 = a_3$.

It is easy to show the following:

LEMMA A.5. *The action of Γ on Ω is free.*

LEMMA A.6. *The action of Γ on Ω is properly discontinuous.*

PROOF. It is easy to see that for any compact subset K in Ω, there is a positive number M such that K is contained in the set

$$K' = \{[z_0 : z_1 : z_2 : z_3] \in \boldsymbol{P}^3 : |z_0| + |z_1| \leqq M(|z_2| + |z_3|)\} .$$

Put $\Lambda = \{g \in \Gamma : g(K') \cap K' \neq \varnothing\}$. It suffices to show that Λ is a finite set. Suppose that $\{g_\nu\}$, $\nu = 1, 2, 3, \cdots$, is a sequence of elements of Λ. Let $\{a_\nu\} \subset K'$ be a sequence of points such that $g_\nu(a_\nu) \in K'$. Choosing a subsequence of $\{g_\nu\}$, we may assume that $\{a_\nu\}$ converges to a point $a = [\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3]$ in K'. Note that $(\alpha_2, \alpha_3) \neq (0, 0)$. If $\alpha_3 \neq 0$, then we may assume $\alpha_3 = 1$. Then there is an integer ν_0 such that $a_\nu = [\alpha_0^{(\nu)} : \alpha_1^{(\nu)} : \alpha_2^{(\nu)} : 1]$ holds for all $\nu \geqq \nu_0$ and that $\lim_{\nu\to\infty} \alpha_j^\nu = \alpha_j$, $j = 0, 1, 2$. Then it follows easily from the relations (A.1), $\cdots$, (A.4) that $g_\nu = g_{\nu+1} = \cdots$ hold for all $\nu \geqq \nu_0$. The argument is the same for the case $\alpha_3 = 0$ and $\alpha_2 \neq 0$. ∎

LEMMA A.7. *The quotient space $\Gamma \backslash \Omega$ is compact.*

PROOF. It is enough to show that, for any point $\alpha = [\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3]$ in Ω, the orbit $\Gamma\alpha$ intersects the compact set

(A.8) $$K = \{[z_0 : z_1 : z_2 : z_3] \in \boldsymbol{P}^3 : |z_0| + |z_1| \leqq 410(|z_2| + |z_3|)\} .$$

First we consider the case $\alpha_3 = 1$. By (A.2) and (A.3), we may assume that $|\alpha_1| \leqq 1/\sqrt{2}$ and $|\alpha_2| \leqq 1/\sqrt{2}$. We put

$$\begin{aligned} P(m, n, \alpha_1, \alpha_2) = {} & (m+in)\alpha_1 + (m(m-1)/2 - n(n-1)/2 - imn)\alpha_2 \\ & + m(m-1)(m-2)/6 - mn(n-1)/2 + i(m(m-1)n/2 - n(n-1)(n-2)/6) , \end{aligned}$$

$$Q(m, n, \alpha_2) = (m+in)\alpha_2 + m(m-1)/2 - n(n-1)/2 - imn ,$$

and

$$R(m, n, \alpha_1, \alpha_2) = P(m, n, \alpha_1, \alpha_2) - (m + in + \alpha_2)(\alpha_1 + Q(m, n, \alpha_2)) .$$

Regarding m and n as real variables, we can show the inequalities

$$\left| \frac{\partial R}{\partial m}(m, n, \alpha_1, \alpha_2) \right| \leqq (|m| + |n| + 1)^2 ,$$

and

$$\left|\frac{\partial R}{\partial n}(m, n, \alpha_1, \alpha_2)\right| \leqq (|m|+|n|+1)^2 .$$

Here we have used $|\alpha_1| \leqq 1/\sqrt{2}$ and $|\alpha_2| \leqq 1/\sqrt{2}$. It is easy to check that the mapping defined by $z = x + iy \mapsto R(x, y, \alpha_1, \alpha_2)$ is a surjection of $\boldsymbol{C}$ to itself. Hence we can find $(m_0, n_0) \in \boldsymbol{Z}^2$ such that

$$\text{(A.9)} \qquad |\alpha_0 + R(m_0, n_0, \alpha_1, \alpha_2)| \leqq 8(|m_0|+|n_0|+1)^2 .$$

Suppose that $(m_0, n_0) \neq (0, 0)$. Using $|\alpha_2| \leqq 1/\sqrt{2}$, we have the inequality

$$\text{(A.10)} \qquad |m_0|+|n_0|+1 \leqq 5\sqrt{2}\,|\alpha_2 + m_0 + in_0| .$$

Combining (A.9) and (A.10), we obtain

$$\text{(A.11)} \qquad |(\alpha_0 + R(m_0, n_0, \alpha_1, \alpha_2))/(\alpha_2 + m_0 + in_0)| \leqq 40\sqrt{2}\,(|m_0|+|n_0|+1) .$$

Put $A(p, q) = \alpha_1 + Q(m_0, n_0, \alpha_2) + p + iq$. Then by (A.11) we can choose $(p_0, q_0) \in \boldsymbol{Z}^2$ so that both inequalities

$$|A(p_0, q_0)| \leqq 40\sqrt{2}\,(|m_0|+|n_0|+1)$$

and

$$|(\alpha_0 + R(m_0, n_0, \alpha_1, \alpha_2))/(\alpha_2 + m_0 + in_0) + A(p_0, q_0)| \leqq \sqrt{2}$$

hold. Put $\alpha' = [\alpha'_0 : \alpha'_1 : \alpha'_2 : 1] = \gamma(G_1^{m_0} G_2^{n_0} G_3^{p_0} G_4^{q_0})\alpha$. Then we have

$$\begin{aligned}
|\alpha'_0| &= |\alpha_0 + P(m_0, n_0, \alpha_1, \alpha_2) + (\alpha_2 + m_0 + in_0)(p_0 + iq_0)| \\
&= |\alpha_0 + R(m_0, n_0, \alpha_1, \alpha_2) + (\alpha_2 + m_0 + in_0)A(p_0, q_0)| \leqq \sqrt{2}\,(|m_0|+|n_0|+1) \\
|\alpha'_1| &= |\alpha_1 + Q(m_0, n_0, \alpha_2) + (p_0 + iq_0)| = |A(p_0, q_0)| \leqq 40\sqrt{2}\,(|m_0|+|n_0|+1) .
\end{aligned}$$

Hence, using (A.10), we obtain

$$|\alpha'_0| + |\alpha'_1| \leqq (41\sqrt{2})(|m_0|+|n_0|+1) \leqq 410(|\alpha'_2|+1) .$$

Thus (A.8) is satisfied. If $(m_0, n_0) = (0, 0)$, then by (A.9) we have

$$|\alpha_0 - \alpha_1\alpha_2| \leqq 8 .$$

We can choose $(p_0, q_0) \in \boldsymbol{Z}^2$ so that $|\alpha_1 + p_0 + iq_0| \leqq 2$. Put $\alpha' = [\alpha'_0 : \alpha'_1 : \alpha'_2 : \alpha'_3] = \gamma(G_3^{p_0} G_4^{q_0})\alpha$. Then we have

$$\begin{aligned}
|\alpha'_0| &= |\alpha_0 + \alpha_2(p_0 + iq_0)| = |\alpha_0 - \alpha_1\alpha_2| + |\alpha_2|\,|\alpha_1 + p_0 + iq_0| \leqq 10 , \\
|\alpha'_1| &= |\alpha_1 + p_0 + iq_0| \leqq 2 .
\end{aligned}$$

Hence we obtain

$$|\alpha'_0|+|\alpha'_1|\leqq 12\leqq 410(|\alpha'_2|+1)\,.$$

Thus (A.8) is satisfied. Next consider the case $\alpha_3=0$. In this case we may assume that $\alpha_2=1$. Then we can find m_0, n_0, p_0, $q_0\in \boldsymbol{Z}$ easily such that $|\alpha'_0|\leqq 2$ and $|\alpha'_1|\leqq 2$. Hence (A.8) is satisfied. ∎

By the above three lemmas, we see that *the manifold $\Gamma\backslash\Omega$ is a (P)-manifold of Schottky type.*

REFERENCES

[B] M. A. BLANCHARD, Sur les varietes analytique complexes, Ann. Sci. Ecole Norm. Sup. 73 (1956), 157–202.

[C] L. S. CHARLAP, Bieberbach groups and flat manifolds, Universitext, Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, 1986.

[He] J. HEMPEL, 3-manifolds, Ann. Math. Studies 86, Princeton Univ. Press., 1976.

[Ho] H. HOPF, Enden offener Raume und unendliche diskontinuierliche Gruppen, Comm. Math. Helv. 16 (1943–44), 81–100.

[Ka1] MA. KATO, Compact differentiable 4-folds with quaternionic structures, Math. Ann. 248 (1980), 79–96, Erratum, Math. Ann. 283 (1989), 352.

[Ka2] MA. KATO, Examples of simply connected compact complex 3-folds, Tokyo J. Math. 5 (1982), 341–364.

[Ka3] MA. KATO, On compact complex 3-folds with lines, Japanese J. Math. 11 (1985), 1–58.

[Ka4] MA. KATO, Compact complex 3-folds with projective structures; The infinite cyclic fundamental group case, Saitama Math. J. 4 (1986), 35–49.

[K-Y] MA. KATO AND A. YAMADA, Examples of simply connected compact complex 3-folds II, Tokyo J. Math. 9 (1986), 1–28.

[Ko1] K. KODAIRA, On the structure of compact complex surfaces. I. Amer. J. Math. 86 (1964), 751–798.

[Ko2] K. KODAIRA, On the structure of compact complex surfaces. II, Amer. J. Math. 88 (1966), 682–721.

[Ku] R. S. KULKARNI, Groups with domains of discontinuity, Math. Ann. 237 (1978), 253–272.

[Ma] B. MASKIT, On Klein's combination theorem III, Ann. Math. Studies 66 (1971), 297–316.

[N] M. V. NORI, The Schottky groups in higher dimensions, Proc. Lefschetz Centen. Conf., Mexico City/Mex. 1984, Pt. I, AMS Contem. Math. 58 (1986), 195–197.

[St] J. STALLINGS, Group theory and three-dimensional manifolds, Yale Math. Monographs 4, Yale Univ. Press, 1971.

[Su] T. SUWA, Compact quotient spaces of C^2 by affine transformation groups, J. Diff. Geom. 10 (1975), 239–252.

[W] J. A. WOLF, Spaces of constant curvature, Publish or Perish, INC. Berkeley, 1977, fifth edition 1984.

[Y] A. YAMADA, Small deformations of certain compact manifolds of Class L, Tohoku Math. J. 38 (1986), 99–122.

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