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ON A RESULT OF GROSS AND YANG

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1. Introduction and main results. By a meromorphic function we shall always mean a meromorphic function in the complex plane. We use the usual notation of the Nevanlinna theory of meromorphic functions as explained in [1]. We use E to denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence.

For any set S and any meromorphic function h let

$$E_h(S)=\bigcup_{a\in S}\{z\,|\,h(z)-a=0\}\,,$$

where each zero of $h-a$ with multiplicity m is repeated m times in $E_h(S)$ (cf. [2]).

Gross and Yang [3] obtained the following results:

THEOREM A. *Let $S_1=\{a_1, a_2\}$ and $S_2=\{b_1, b_2\}$ be two pairs of distinct elements with $a_1+a_2=b_1+b_2$ but $a_1a_2\neq b_1b_2$. Suppose that there are two nonconstant entire functions f and g of finite order such that $E_f(S_j)=E_g(S_j)$ for $j=1, 2$. Then either $f\equiv g$, $f+g\equiv a_1+a_2$ or*

$$f(z)=\frac{c}{2}\pm\left[\frac{a_1a_2-b_1b_2}{2}e^{-p}\right]^{1/2}$$

and

$$g(z)=\frac{c}{2}\pm\left[\frac{a_1a_2-b_1b_2}{2}e^{p}\right]^{1/2},$$

where $c=a_1+a_2$ and $p(z)$ is a polynomial.

THEOREM B. *Let $S_1=\{a_1, a_2\}$ and $S_2=\{b_1, b_2\}$ be any two disjoint pairs of complex numbers with $a_1a_2\neq b_1b_2$. Suppose that there are two nonconstant entire functions f and g of finite order such that $E_f(S_j)=E_g(S_j)$ for $j=1, 2$. Then either $f(z)\equiv Ag(z)+B$ for some constants A,* B, *or*

$$f(z)=c_1+c_2e^{p(z)}\quad\text{and}\quad g(z)=c_1+c_2e^{-p(z)}$$

for some polynomial $p(z)$ and constants c_1 and c_2.

The above results of Gross and Yang, however, are not true for

$$f(z)=1-4e^{z}, \quad g(z)=1-e^{-z}, \quad S_1=\{-1,1\} \quad \text{and} \quad S_2=\{-\sqrt{3}\,i, \sqrt{3}\,i\}.$$

In this note, we prove the following theorem which is a correction of Theorem A.

THEOREM 1. *Assume that the conditions of Theorem* A *are satisfied. Then f and g must satisfy exactly one of the following relations*:

(i) $f \equiv g$,

(ii) $f+g \equiv a_1+a_2$,

(iii) $(f-c/2)\cdot(g-c/2)\equiv \pm((a_1-a_2)/2)^2$, *where* $c=a_1+a_2$. *This occurs only for* $(a_1-a_2)^2+(b_1-b_2)^2=0$.

(iv) $(f-a_j)\cdot(g-a_k)\equiv(-1)^{j+k}(a_1-a_2)^2$ *for* $j,k=1,2$. *This occurs only for* $3(a_1-a_2)^2+(b_1-b_2)^2=0$.

(v) $(f-b_j)\cdot(g-b_k)\equiv(-1)^{j+k}(b_1-b_2)^2$ *for* $j,k=1,2$. *This occurs only for* $(a_1-a_2)^2+3(b_1-b_2)^2=0$.

From Theorem 1 we immediately obtain the following:

COROLLARY. *If, in addition to the assumptions of Theorem* 1,

$$((a_1-a_2)/(b_1-b_2))^2 \neq -1,\ -3,\ -1/3\,,$$

then either $f\equiv g$ or $f+g\equiv a_1+a_2$.

Now it is natural to ask what can be said if f and g are two meromorphic functions of arbitrary growth in Theorem 1. In this direction, we have the following results.

THEOREM 2. *Let $S_1=\{a_1,a_2\}$ and $S_2=\{b_1,b_2\}$ be two pairs of distinct elements with $a_1+a_2=b_1+b_2$ but $a_1a_2\neq b_1b_2$, and let $S_3=\{\infty\}$. Suppose that f and g are two nonconstant meromorphic functions satisfying $E_f(S_j)=E_g(S_j)$ for $j=1,2,3$. Then*

$$T(r,f)=(1+o(1))T(r,g) \qquad \textit{for} \quad r \notin E\,.$$

THEOREM 3. *If, in addition to the assumptions of Theorem 2, $\delta(c/2,f)>1/5$, where $c=a_1+a_2$, then f and g must satisfy exactly one of the following relations*:

(i) $f\equiv g$,

(ii) $f+g\equiv a_1+a_2$,

(iii) $(f-c/2)\cdot(g-c/2)\equiv\pm((a_1-a_2)/2)^2$. *This occurs only for* $(a_1-a_2)^2+(b_1-b_2)^2=0$.

THEOREM 4. *If, in addition to the assumptions of Theorem* 2,

$$N\left(r,\frac{1}{f-b_1}\right)+N\left(r,\frac{1}{f-b_2}\right)=(2+o(1))T(r,f) \qquad \textit{for} \quad r\notin E$$

and $\delta(c/2,f)>0$, where $c=a_1+a_2$, then the conclusions of Theorem 3 *hold.*

2. Some lemmas. In order to prove our theorems, we need the following lemmas.

LEMMA 1. *Let $h(z)$ be a nonconstant entire function. Then*

$$T(r, h') = o(T(r, e^h)) \qquad \textit{for} \quad r \notin E.$$

PROOF. We have

$$T(r, h') \leqslant (1 + o(1)) T(r, h) \qquad \text{for} \quad r \notin E.$$

On other hand, by Clunie's result (cf. [1, p. 54]), we have $T(r, h) = o(T(r, e^h))$. Thus $T(r, h') = o(T(r, e^h))$ for $r \notin E$, which proves Lemma 1.

LEMMA 2 (cf. [4, Lemma 3]). *Let f_1, f_2 and f_3 be meromorphic functions with $f_3 \not\equiv$ constnat. Suppose that $\sum_{j=1}^{3} f_j \equiv 1$ and that*

$$\sum_{j=1}^{3} N(r, f_j) = o(T(r)) \qquad \textit{for} \quad r \notin E$$

and

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) < (\lambda + o(1)) T(r) \qquad \textit{for} \quad r \notin E,$$

where $T(r)$ denotes the maximum of $T(r, f_j)$ for $j = 1, 2, 3$ and λ is a positive constant < 1. Then either $f_1 \equiv 1$ or $f_2 \equiv 1$.

LEMMA 3 (cf. [5, Theorem 2]). *Let $p(z)$ and $q(z)$ be nonconstnat polynomials of the same degree. If $(e^{p(z)} - 1)/(e^{q(z)} - 1)$ is entire, then $p(z) = m\, q(z) + 2n\pi i$, where m, n are integers.*

3. Proof of Theorem 2. By the assumption of Theorem 2, we have two entire functions p and q such that

$$(1) \qquad \begin{aligned} (g - a_1) \cdot (g - a_2) &= e^p (f - a_1) \cdot (f - a_2), \\ (g - b_1) \cdot (g - b_2) &= e^q (f - b_1) \cdot (f - b_2). \end{aligned}$$

Let

$$(2) \qquad G(z) = (g(z) - c/2)^2, \qquad F(z) = (f(z) - c/2)^2,$$

where $c = a_1 + a_2 = b_1 + b_2$. Again let $a = ((a_1 - a_2)/2)^2$, $b = ((b_1 - b_2)/2)^2$. By the assumption of Theorem 2, we have $a \neq 0$, $b \neq 0$ and $a \neq b$. From (1) we obtain

$$(3) \qquad G - a = e^p (F - a), \qquad G - b = e^q (F - b).$$

It is easy to see from the second main theorem and our assumption that

$$T(r,G)=O(T(r,F)) \qquad \text{for} \quad r \notin E, \tag{4}$$
$$T(r,e^{p})+T(r,e^{q})=O(T(r,F)) \qquad \text{for} \quad r \notin E.$$

Suppose that $F \not\equiv G$. Then $e^{q} \not\equiv e^{p}$. Thus from (3) we obtain

$$F=\frac{be^{q}-ae^{p}+a-b}{e^{q}-e^{p}}, \qquad G=\frac{be^{-q}-ae^{-p}+a-b}{e^{-q}-e^{-p}}. \tag{5}$$

Let $\{z_n\}$ be the set of poles of F. Then from (2) and (5), $\{z_n\}$ are the roots of $(e^{q-p}-1)'=(q'-p')e^{q-p}=0$. Thus

$$N(r,F) \leqslant 2N\left(r,\frac{1}{q'-p'}\right) \leqslant 2T(r,q')+2T(r,p')+O(1).$$

By Lemma 1 and (4), we obtain

$$N(r,F)=o(T(r,F)) \qquad \text{for} \quad r \notin E, \tag{6}$$

that is,

$$N(r,f)=o(T(r,f)) \qquad \text{for} \quad r \notin E. \tag{7}$$

Let $\{z'_n\}$ be the set of roots of $F=0$. Then from (2) and (5), $\{z'_n\}$ are the roots of $(be^{q}-ae^{p}+a-b)'=e^{p}(bq'e^{q-p}-ap')=0$. Thus

$$N\left(r,\frac{1}{F}\right) \leqslant 2N\left(r,\frac{1}{bq'e^{q-p}-ap'}\right) \leqslant 2T(r,e^{q-p})+o(T(r,F)) \qquad \text{for} \quad r \notin E,$$

that is,

$$N\left(r,\frac{1}{f-c/2}\right) \leqslant N\left(r,\frac{1}{e^{q-p}-1}\right)+o(T(r,f)) \qquad \text{for} \quad r \notin E. \tag{8}$$

By the second fundamental theorem, we have

$$4\,T(r,f)<N\left(r,\frac{1}{f-a_1}\right)+N\left(r,\frac{1}{f-a_2}\right)+N\left(r,\frac{1}{f-b_1}\right)+N\left(r,\frac{1}{f-b_2}\right)+N\left(\frac{1}{f-c/2}\right)$$
$$+N(r,f)+o(T(r,f)) \qquad \text{for} \quad r \notin E.$$

Hence by (7) and (8), we obtain

$$2\,T(r,F)<N\left(r,\frac{1}{F-a}\right)+N\left(r,\frac{1}{F-b}\right)+N\left(r,\frac{1}{e^{q-p}-1}\right)+o(T(r,F)) \qquad \text{for} \quad r \notin E. \tag{9}$$

From (3) we have

$$G-F=(F-a)\cdot(e^p-1)=(F-b)\cdot(e^q-1)=\frac{e^p}{b-a}(F-a)\cdot(F-b)\cdot(e^{q-p}-1).$$

Then

$$N\left(r,\frac{1}{G-F}\right)=N\left(r,\frac{1}{F-a}\right)+N\left(r,\frac{1}{e^p-1}\right)+o(T(r,F)) \tag{10}$$

$$=N\left(r,\frac{1}{F-b}\right)+N\left(r,\frac{1}{e^q-1}\right)+o(T(r,F))$$

$$=N\left(r,\frac{1}{F-a}\right)+N\left(r,\frac{1}{F-b}\right)+N\left(r,\frac{1}{e^{q-p}-1}\right)+o(T(r,F)) \qquad \text{for} \quad r\notin E.$$

By (9) and (10) we easily obtain

$$2\,T(r,F)<N\left(r,\frac{1}{G-F}\right)+o(T(r,F))<(1+o(1))T(r,F)+T(r,G) \qquad \text{for} \quad r\notin E,$$

that is,

$$(1-o(1))T(r,F)<T(r,G) \qquad \text{for} \quad r\notin E.$$

In the same way, we have

$$(1-o(1))T(r,G)<T(r,F) \qquad \text{for} \quad r\notin E.$$

Hence

$$T(r,F)=(1+o(1))T(r,G) \qquad \text{for} \quad r\notin E,$$

which implies

$$T(r,f)=(1+o(1))T(r,g) \qquad \text{for} \quad r\notin E.$$

This completes the proof of Theorem 2.

REMARK. From the proof of Theorem 2, it is easy to see that if $F\not\equiv G$, then the following conclusions hold:

$$N\left(r,\frac{1}{G-F}\right)=(2+o(1))T(r,F) \qquad \text{for} \quad r\notin E, \tag{11}$$

$$N\left(r,\frac{1}{f-c/2}\right)=T(r,e^{q-p})+o(T(r,f)) \qquad \text{for} \quad r\notin E, \tag{12}$$

$$(1+o(1))T(r,F)\leqslant T(r,e^p)\leqslant(3/2+o(1))T(r,F) \qquad \text{for} \quad r\notin E, \tag{13}$$

$$(1+o(1))T(r,F)\leqslant T(r,e^q)\leqslant(3/2+o(1))T(r,F) \qquad \text{for} \quad r\notin E. \tag{14}$$

4. Proof of Theorem 3. In the following, we shall use the notation of the above section.

If $F\equiv G$, then either $f\equiv g$ or $f+g\equiv a_1+a_2$. Next, assume that $F\not\equiv G$. Let

$$f_1=\frac{1}{a-b}\cdot(e^q-e^p)\cdot F\,,\quad f_2=-\frac{b}{a-b}\cdot e^q\,,\quad f_3=\frac{a}{a-b}\cdot e^p$$

and denote by $T(r)$ the maximum of $T(r, f_j)$ for $j=1, 2, ,3$. From (5) we have

$$\sum_{j=1}^{3} f_j\equiv 1\,. \tag{15}$$

By (6) and (13) we obtain

$$\sum_{j=1}^{3} N(r, f_j)=o(T(r)) \qquad \text{for} \quad r\notin E\,. \tag{16}$$

Again by (2) and (12), we get

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right)=3\,N\left(r, \frac{1}{f-c/2}\right)+o(T(r)) \qquad \text{for} \quad r\notin E\,. \tag{17}$$

It is clear that

$$\begin{aligned} N\left(r, \frac{1}{f-c/2}\right)&\leqslant(1-\delta(c/2, f)+o(1))T(r, f) \\ &=\frac{1}{2}(1-\delta(c/2, f)+o(1))T(r, F) \qquad \text{for} \quad r\notin E\,. \end{aligned} \tag{18}$$

From (10) we obtain

$$N\left(r, \frac{1}{e^p-1}\right)+N\left(r, \frac{1}{e^q-1}\right)=N\left(r, \frac{1}{G-F}\right)+N\left(r, \frac{1}{e^{q-p}-1}\right)+o(T(r, F)) \qquad \text{for} \quad r\notin E\,.$$

This implies that

$$T(r, e^p)+T(r, e^q)=(2+o(1))T(r, F)+N\left(r, \frac{1}{f-c/2}\right) \qquad \text{for} \quad r\notin E\,, \tag{19}$$

by (11) and (12). Combining (18) and (19), we have

$$T(r, e^p)+T(r, e^q)\leqslant\frac{1}{2}(5-\delta(c/2, f)+o(1))T(r, F) \qquad \text{for} \quad r\notin E\,. \tag{20}$$

It follows from (17), (19) and (20) that

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right)=3(T(r, e^p)+T(r, e^q))-6T(r, F)+o(T(r)) \tag{21}$$

$$\leqslant \left(3-\frac{12}{5-\delta(c/2, f)}\right)(T(r, e^p)+T(r, e^q))+o(T(r))$$

$$\leqslant \left\{\frac{6(1-\delta(c/2, f))}{5-\delta(c/2, f)}+o(1)\right\} T(r) \qquad \text{for} \quad r \notin E .$$

Since $\delta(c/2, f) > 1/5$,

$$\frac{6(1-\delta(c/2, f))}{5-\delta(c/2, f)}=1-\frac{5\delta(c/2, f)-1}{5-\delta(c/2, f)}<1 .$$

By (13), (14) and Lemma 2, we obtain

$$\frac{1}{a-b}\cdot(e^q-e^p)\cdot F=1$$

and

$$-\frac{b}{a-b}\cdot e^q+\frac{a}{a-b}\cdot e^p=0 .$$

Thus

(22) $$e^q=\frac{a}{b}e^p$$

and

(23) $$F=be^{-p} .$$

Again by (5) and (22),

(24) $$G=(a+b)-ae^p .$$

From (2) we know that G has no simple zeros. Thus by (24) we have

$$a+b=0$$

and

(25) $$G=be^p .$$

By (23) and (25), we get $F\cdot G\equiv a^2$, which implies that

$$(f-c/2)\cdot(g-c/2)\equiv \pm((a_1-a_2)/2)^2 .$$

This completes the proof of Theorem 3.

5. **Proof of Theorem 4.** Suppose that $F\not\equiv G$. Proceeding as in the proof of

Theorem 3, we also obtain (15), (16), (17), (18), (19) and (20). By the assumption of Theorem 4, we have

$$N\left(r, \frac{1}{F-b}\right) = (1+o(1))T(r, F) \qquad \text{for} \quad r \notin E\,.$$

Again from (10) we obtain

$$T(r, e^{q}) = (1+o(1))T(r, F) \qquad \text{for} \quad r \notin E\,.$$

From this and (19), we get

$$T(r, e^{p}) = (1+o(1))T(r, F) + N\left(r, \frac{1}{f-c/2}\right) \qquad \text{for} \quad r \notin E\,. \tag{26}$$

Again by (18),

$$T(r, e^{p}) \leqslant \frac{1}{2}(3-\delta(c/2, f) + o(1))T(r, F) \qquad \text{for} \quad r \notin E\,. \tag{27}$$

It follows from (17), (26) and (27) that

$$\begin{aligned} \sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) &= 3T(r, e^{p}) - 3T(r, F) + o(T(r)) \\ &\leqslant \left(3 - \frac{6}{3-\delta(c/2, f)}\right) T(r, e^{p}) + o(T(r)) \\ &\leqslant \left\{\frac{3(1-\delta(c/2, f))}{3-\delta(c/2, f)} + o(1)\right\} T(r) \qquad \text{for} \quad r \notin E\,. \end{aligned}$$

Since $\delta(c/2, f) > 0$,

$$\frac{3(1-\delta(c/2, f))}{3-\delta(c/2, f)} = 1 - \frac{2\delta(c/2, f)}{3-\delta(c/2, f)} < 1\,.$$

Proceeding as in the proof of Theorem 3, by Lemma 2, we also have

$$(f-c/2)\cdot(g-c/2) \equiv \pm((a_1-a_2)/2)^2\,,$$

which occurs only for $(a_1-a_2)^2 + (b_1-b_2)^2 = 0$.

This completes the proof of Theorem 4.

6. Proof of Theorem 1. Suppose that $F \not\equiv G$. Since f and g are nonconstant entire functions of finite order, p and q are polynomials. From (13) and (14) we obtain $\deg p = \deg q$. If $\deg(q-p) < \deg p$, then from (12),

$$N\left(r, \frac{1}{f-c/2}\right)=o(T(r, f))$$

and hence $\delta(c/2, f)=1$. By Theorem 3, we obtain

$$(f-c/2)\cdot(g-c/2)\equiv \pm((a_1-a_2)/2)^2 ,$$

which occurs only for $(a_1-a_2)^2+(b_1-b_2)^2=0$.

Next, assume that $\deg(q-p)=\deg p$. From (5) we have

$$\frac{(F-b)e^p}{b-a}=\frac{e^p-1}{e^{q-p}-1} \tag{28}$$

and hence by Lemma 3,

$$p=m(q-p)+2n\pi i \quad \text{and} \quad q=(m+1)(q-p)+2n\pi i ,$$

where m, n are integers.

If m is positive, from (12), (19) and (28) we obtain

$$\begin{aligned} &T(r, e^p)=mT(r, e^{q-p})=(1+o(1))T(r, F) ,\\ &T(r, e^q)=(m+1)T(r, e^{q-p})=\left(1+\frac{1}{m}+o(1)\right)T(r, F) . \end{aligned} \tag{29}$$

Again by (14), we get $m\geqslant 2$. If $m\geqslant 3$, from (12) and (29), we obtain $\delta(c/2, f)=1-2/m>1/5$. By Theorem 3, $N(r, (f-c/2)^{-1})=0$, which is a contradiction. Thus $m=2$. From (28) we obtain

$$F=(b-a)\cdot\left(e^{2(p-q)}+e^{p-q}+\frac{b}{b-a}\right)=(b-a)\cdot\left[\left(e^{p-q}+\frac{1}{2}\right)^2+\left(\frac{b}{b-a}-\frac{1}{4}\right)\right]. \tag{30}$$

From (2) we know that all the zeros of F must be multiple. Thus by (30) we have $b/(b-a)=1/4$ and $F=b(2e^{p-q}+1)^2$. Hence

$$f=b_1+(b_1-b_2)e^{p-q} \qquad \text{or} \quad f=b_2+(b_2-b_1)e^{p-q} .$$

In the same way, we obtain

$$g=b_1+(b_1-b_2)e^{q-p} \qquad \text{or} \quad g=b_2+(b_2-b_1)e^{q-p} .$$

Hence, f and g must satisfy exactly one of the following relations:

$$(f-b_j)\cdot(g-b_k)\equiv(-1)^{j+k}(b_1-b_2)^2 \qquad \text{for} \quad j, k=1, 2 .$$

This occurs only for $(a_1-a_2)^2+3(b_1-b_2)^2=0$.

If m is negative, in the same manner as above, we have $m=-3$, $3a+b=0$,

$$(f-a_j)\cdot(g-a_k)\equiv(-1)^{j+k}(a_1-a_2)^2 \qquad \text{for} \quad j, k=1, 2 .$$

This occurs only for $3(a_1-a_2)^2+(b_1-b_2)^2=0$.

This completes the proof of Theorem 1.

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