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HOMOLOGY COVERINGS OF RIEMANN SURFACES

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Abstract. We extend a result on homology coverings of closed Riemann surfaces due to Maskit [1] to the class of analytically finite ones. We show that if S is an analytically finite hyperbolic Riemann surface, then its conformal structure is determined by the conformal structure of its homology cover. The homology cover of a Riemann surface S is the highest regular covering of S with an Abelian group of covering transformations. In fact, we show that the commutator subgroup of any torsion-free, finitely generated Fuchsian group of the first kind determines it uniquely.

1. The main theorem. Let S be a Riemann surface. We say that S is *analytically finite* if S is conformally equivalent to the complement of a finite number of points on a closed Riemann surface $\bar{S}$. If the genus of $\bar{S}$ is g and the number of deleted points is k, then we say that S has signature $(g, k; \infty, \dots, \infty)$.

We say that an analytically finite Riemann surface S of signature $(q, k; \infty, \dots, \infty)$ is *hyperbolic* if its universal covering surface is the hyperbolic disc. It is the case if and only if $2g-2+k>0$.

A Riemann surface $\hat{S}$ is an *Abelian cover* of S if there exists a regular covering $\pi\colon \hat{S}\to S$ with an Abelian group of deck transformations. The *homology covering* of S, $\pi\colon \tilde{S}\to S$, is the highest Abelian covering of S, that is, it is the covering corresponding to the commutator subgroup of the fundamental group $\Pi_1(S)$ of S.

THEOREM. *Let S_1 and S_2 be analytically finite hyperbolic Riemann surfaces of signature $(g_1, k_1; \infty, \dots, \infty)$ and $(g_2, k_2; \infty, \dots, \infty)$, respectively. Suppose S_1 and S_2 have conformally equivalent homology covering surfaces. Then S_1 and S_2 are conformally equivalent.*

REMARKS. The above theorem, for the class of closed hyperbolic Riemann surfaces, was proved by Maskit in [1]. Since the homology cover of a Riemann surface S, obtained by deleting one point in a closed Riemann surface $\bar{S}$, is the homology cover of $\bar{S}$ minus the orbit of a (suitable) point, the above theorem for $k_1=k_2=1$ is an easy consequence of Maskit's result.

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2. Proof of the theorem. We may assume S_1 and S_2 to have the same homology covering $\tilde{S}$. Let Γ_m be the group of deck transformations for $\tilde{S}$ covering S_m; that is, $\tilde{S}/\Gamma_m = S_m$. It is well known that $\tilde{S}$ is conformally equivalent to none of the following Riemann surfaces: (i) the Riemann sphere, (ii) the complex plane, (iii) the complex plane with one point deleted, (iv) a torus, (v) the unit disk, and (vi) a ring domain. Hence the full group of conformal automorphisms of $\tilde{S}$ acts discontinuously on $\tilde{S}$; in particular, the group Γ of conformal automorphisms of $\tilde{S}$ generated by Γ_1 and Γ_2 acts discontinuously on $\tilde{S}$. Set $S = \tilde{S}/\Gamma$.

Since S_1 and S_2 are analytically finite Riemann surfaces, they have finite hyperbolic area. Hence they are finite sheeted coverings of S; that is, both Γ_1 and Γ_2 are of finite index in Γ. It then follows that $\bar{\Gamma} = \Gamma_1 \cap \Gamma_2$ is of finite index in Γ_1, Γ_2 and Γ.

Let $\bar{S} = \tilde{S}/(\Gamma_1 \cap \Gamma_2)$. We have the following commutative diagram of coverings:

$$\begin{tikzcd}
 & \tilde{S} \arrow[d] & \\
 & \bar{S} \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\
S_1 \arrow[dr] & & S_2 \arrow[dl] \\
 & S &
\end{tikzcd}$$

The covering $q_m\colon \bar{S} \to S_m$ is a regular (possibly branched) covering with an Abelian group of deck transformations $G_m = \Gamma_m/\bar{\Gamma}$. Note that $q_m\colon \bar{S} \to S_m$ can be extended to a holomorphic mapping of the Riemann surface $\bar{S} \cup \{\text{the punctures of } \bar{S}\}$ onto the Riemann surface $S_m \cup \{\text{the punctures of } S_m\}$.

Observe that a loop w on $\bar{S}$ lifts to a loop on $\tilde{S}$ if and only if its projection to S_m is homologically trivial. Let $(q_m)_*\colon H_1(\bar{S}) \to H_1(S_m)$ be the induced map on homology with complex coefficients. The above observation says that $(q_1)_*$ and $(q_2)_*$ have the same kernel. Since q_m is a covering map, the map $(q_m)_*$ is a surjection. The fact that $(q_1)_*$ and $(q_2)_*$ have the same kernel now implies that $H_1(S_1)$ and $H_1(S_2)$ have the same dimension; that is,

$$(*) \qquad \begin{aligned}
2g_1 + k_1 - 1 &= 2g_2 + k_2 - 1\,, &&\text{if } k_1 > 0 \text{ and } k_2 > 0\\
2g_1 &= 2g_2 + k_2 - 1\,, &&\text{if } k_1 = 0 \text{ and } k_2 > 0\\
2g_1 + k_1 - 1 &= 2g_2\,, &&\text{if } k_1 > 0 \text{ and } k_2 = 0\\
2g_1 &= 2g_2\,, &&\text{if } k_1 = 0 \text{ and } k_2 = 0\,.
\end{aligned}$$

If X is an analytically finite (hyperbolic) Riemann surface of signature $(g, k; \infty, \dots, \infty)$, we denote by $H_1^{1,0}(X, \boldsymbol{C})$ the complex vector space of holomorphic

1-forms on X with poles of order at most one at the punctures. The Riemann-Roch theorem implies that the complex dimension of this space is:

$$\dim H_1^{1,0}(X, \mathbf{C}) = \begin{cases} g+k-1 & \text{if } k>0 \\ g & \text{if } k=0 . \end{cases}$$

We have natural linear injections $q_m^*: H_1^{1,0}(S_m, \mathbf{C}) \to H_1^{1,0}(\bar{S}, \mathbf{C})$. We can regard the image $q_m^*(H_1^{1,0}(S_m, \mathbf{C}))$ as those differentials in $H_1^{1,0}(\bar{S}, \mathbf{C})$ which are invariant under G_m. Clearly, $q_m^*(H_1^{1,0}(S_m, \mathbf{C}))$ is orthogonal to the kernel of $(q_m)_*$, which we denote by $\ker((q_m)_*)$. The lemma below shows that the subspace of $H_1^{1,0}(\bar{S}, \mathbf{C})$ which is orthogonal to $\ker((q_m)_*)$ is exactly $q_m^*(H_1^{1,0}(S_m, \mathbf{C}))$. Since $\ker((q_1)_*) = \ker((q_2)_*)$, we obtain $q_1^*(H_1^{1,0}(S_1, \mathbf{C})) = q_2^*(H_1^{1,0}(S_2, \mathbf{C}))$. In particular, a form w in $H_1^{1,0}(\bar{S}, \mathbf{C})$ is G_1-invariant if and only if it is G_2-invariant, and these spaces have the same dimension; that is:

$$(**) \qquad \begin{aligned} g_1+k_1-1 &= g_2+k_2-1 , && \text{if } k_1>0 \text{ and } k_2>0 \\ g_1 &= g_2+k_2-1 , && \text{if } k_1=0 \text{ and } k_2>0 \\ g_1+k_1-1 &= g_2 , && \text{if } k_1>0 \text{ and } k_2=0 \\ g_1 &= g_2 , && \text{if } k_1=0 \text{ and } k_2=0 . \end{aligned}$$

Now, $(*)$ and $(**)$ imply

$$(***) \qquad \begin{aligned} g_1 = g_2 \text{ and } k_1 &= k_2 , && \text{if } k_1>0 \text{ and } k_2>0 \\ g_1 &= g_2 , && \text{if } k_1=0 \text{ and } k_2=1 \\ g_1 &= g_2 , && \text{if } k_1=1 \text{ and } k_2=0 \\ g_1 &= g_2 , && \text{if } k_1=0 \text{ and } k_2=0 . \end{aligned}$$

As observed before, if S is a closed Riemann surface and $\tilde{S}$ is its homology cover, then the homology cover of $S-\{p\}$ is $\tilde{S}-\{\text{orbit}(q)\}$, where $\pi(q)=p$, $\pi: \tilde{S} \to S$ is the homology covering map and orbit(q) is the orbit of q under the group of deck transformations. In particular, the homology cover of a closed Riemann surface and the homology cover of a 1-punctured surface cannot be conformally equivalent. As a consequence, $g_1=g_2$ and $k_1=k_2$.

Let G be the group of conformal automorphisms of $\bar{S}$ generated by G_1 and G_2. Then $S=\bar{S}/G$. Since every form in $H_1^{1,0}(\bar{S}, \mathbf{C})$ is G-invariant if and only if it is G_m-invariant, S must have signature $(g, k; \infty, \ldots, \infty)$ with $g+k-1=g_1+k_1-1=g_2+k_2-1$, if $k_1>0$; or $g=g_1=g_2$, otherwise.

Since $k_1 \geq k$ and $g_1 \geq g$ (by area arguments), we must have $k_1=k$ and $g_1=g$. In particular, S_m is conformally equivalent to S. Then S_1 and S_2 are conformally equivalent. This finishes the proof of our theorem.

Now we proceed to establish and prove the lemma we needed above.

LEMMA. *Let* $p\colon \bar{X}\to X$ *be a covering* (*possibly branched*) *between analytically finite Riemann surfaces. Then*

$$p^*(H_1^{1,0}(X; \boldsymbol{C}))=\{w\in H_1^{1,0}(\bar{X}, \boldsymbol{C});\ w\in(\ker(p)_*)^{\perp}\}\ .$$

PROOF. Since $\bar{X}$ and X are analytically finite Riemann surfaces, p is a finite sheeted covering. Clearly, $p^*(H_1^{1,0}(X; \boldsymbol{C}))\subset\{w\in H_1^{1,0}(\bar{X}, \boldsymbol{C});\ w\in(\ker(p)_*)^{\perp}\}$.

To show the opposite inclusion, we need to recall some trivial facts.

FACT 1. If X is an analytically finite Riemann surface of signature $(g, k; \infty, \ldots, \infty)$, then every form $w\in H_1^{1,0}(X, \boldsymbol{C})$ is uniquely determined by $\int_{\alpha_i} w$, $\int_{\delta_j} w$, $i=1, \ldots, g$, $j=1, \ldots, k-1$, where the α_i's are g homologically independent disjoint simple loops on the closed Riemann surface (of genus g) $X'=X\cup\{\text{punctures of } X\}$ (also disjoint from the punctures), the δ_j's are small simple loops around $k-1$ of the punctures and all the above loops are disjoint. Figure 1 shows loops α_i and δ_j in signature $(3, 3; \infty, \infty, \infty)$.

FACT 2. Let X be an analytically finite Riemann surface of signature $(g, k; \infty, \ldots, \infty)$. For any set of simple loops α_i, δ_j, $i=1, \ldots, g$, $j=1, \ldots, k-1$, as in Fact 1, there exist disjoint simple loops β_i such that $\{\alpha_i, \beta_i\}_{i=1,\ldots,g}$ form a canonical basis for the homotopy of X' and such that $\beta_i\cap\delta_j=\varnothing$, for all i and j (see Figure 2). In

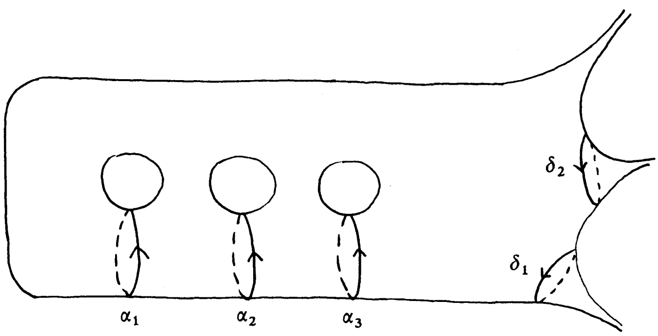

FIGURE 1. The loops α's and δ's in a surface of signature $(3, 3; \infty, \infty, \infty)$.

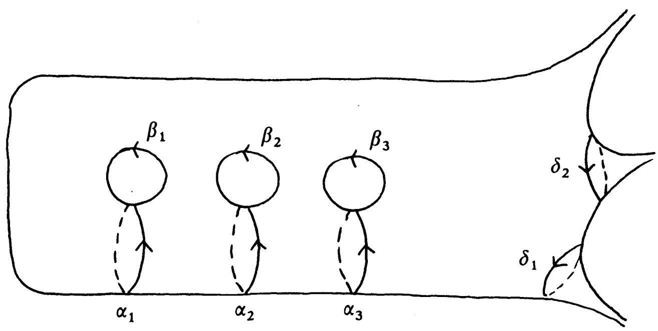

FIGURE 2. The loops α's, β's and δ's in a surface of signature $(3, 3; \infty, \infty, \infty)$.

this case, $H_1(X, \boldsymbol{Z}) = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \delta_1, \ldots, \delta_{k-1} \rangle$.

We need some notation and definitions. (1) We call a set of simple loops, as above, a canonical set. (2) Let us denote by $(g(\bar{X}), k(\bar{X}); \infty, \ldots, \infty)$ and $(g(X), k(X); \infty, \ldots, \infty)$ the signatures of $\bar{X}$ and X, respectively.

FACT 3. On $\bar{X}$ and X there exist canonical sets $\{\bar{\alpha}_i, \bar{\beta}_i, \bar{\delta}_j\}$ and $\{\alpha_n, \beta_n, \delta_l\}$ respectively, such that

$$(1) \quad p_*(\bar{\alpha}_i) \in \langle \alpha_n, \delta_l; n=1, \ldots, g(X); l=1, \ldots, k(X)-1 \rangle ,$$

$$(2) \quad p_*(\bar{\delta}_j) \in \langle \alpha_n, \delta_l; n=1, \ldots, g(X); l=1, \ldots, k(X)-1 \rangle .$$

PROOF (Fact 3). Construct a canonical set $\{\alpha_n, \beta_n, \delta_l\}$ on X. Since p is a finite sheeted covering, for any simple loop η on X there exists a positive integer m such that η^m lifts to a loop on $\bar{X}$. In particular, $p^{-1}(\eta)$ consists of a finite number of disjoint simple loops. Consider the family of simple loops $p^{-1}(\alpha_1), \ldots, p^{-1}(\alpha_{g(X)})$. In this family there exist a maximal set of disjoint simple loops which are homologically independent on the closed Riemann surface Y obtained by adding the punctures to $\bar{X}$. Now we complete them to a set of homologically independent disjoint simple loops on Y. Now consider small simple loops around $k(\bar{X})-1$ of the punctures of $\bar{X}$. We can make these loops to be disjoint from the above ones. These loops project to loops with homology in $\langle \alpha_n, \delta_l; n=1, \ldots, g(X); l=1, \ldots, k(X)-1 \rangle$. Now use Fact 2 to get a family of loops as desired.

Let $\bar{V}$ be the complex vector space generated by $\bar{\alpha}_i, \bar{\delta}_j$; $1 \le i \le g(\bar{X})$, $1 \le j \le k(\bar{X})-1$, and let V be the complex vector space generated by α_n, δ_l; $1 \le n \le g(X)$, $1 \le l \le k(X)-1$. Then Fact 3 implies that a linear map $p_* : \bar{V} \to V$ is well defined.

Fact 1 says that its dual map is

$$p^* : H_1^{1,0}(X, \boldsymbol{C}) \to H_1^{1,0}(\bar{X}, \boldsymbol{C}) .$$

Now the lemma follows from the fact that $(\ker p_*)^{\perp} = p^*(H_1^{1,0}(X, \boldsymbol{C}))$. □

We obtain the following result from the proof of our theorem:

COROLLARY. *Let G and H be finitely generated torsion-free Fuchsian groups of the first kind. Assume these groups to have the same commutator subgroup, that is,* $[G, G] = [H, H]$. *Then* $G = H$.

REFERENCES

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