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BOUNDS OF AUTOMORPHISM GROUPS OF GENUS 2 FIBRATIONS

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Abstract. For a relatively minimal fibration of genus 2, the best bounds of the orders of its automorphism group, abelian automorphism group and cyclic automorphism group are obtained as a linear function of the self-intersection number of the canonical divisor.

It is well known that the automorphism group of a surface of general type is finite and bounded by a function of K^2 (cf. [1]). Since then, several authors worked on this subject and found better upper bounds of the group. Recently Xiao [11], [12] obtained a linear bound for this group. Hence it is natural to investigate the upper bounds for particular classes of surfaces. Here we are interested in the upper bounds of various automorphism groups of surfaces with genus 2 pencils. As a first step, in the present paper, we will study the upper bounds of automorphism groups of genus 2 fibrations.

We always assume that S is a smooth projective surface over the complex number field. A genus 2 fibration is a morphism $f: S \to C$ where C is a projective curve such that a general fiber of f is a smooth curve of genus 2.

DEFINITION 0.1. An automorphism of the fibration $f: S \to C$ is a pair of automorphisms $(\tilde{\sigma}, \sigma)$ with $\tilde{\sigma} \in \mathrm{Aut}(S)$, $\sigma \in \mathrm{Aut}(C)$ such that the diagram

$$\begin{tikzcd} S \arrow[r, "\tilde{\sigma}"] \arrow[d, "f"'] & S \arrow[d, "f"] \\ C \arrow[r, "\sigma"] & C \end{tikzcd}$$

commutes.

The automorphism group of fibration f will be denoted by $\mathrm{Aut}(f)$. The main results of this paper are the following:

THEOREM 0.1. *Suppose S is a surface of general type over the complex number field with a relatively minimal genus 2 fibration $f: S \to C$. Then*

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$$|\mathrm{Aut}(f)| \leq 504K_S^2 .$$

If f is not locally trivial, then

$$|\mathrm{Aut}(f)| \leq 288K_S^2 .$$

More precisely,

$$|\mathrm{Aut}(f)| \leq \begin{cases} 126K_S^2 , & \text{if } g(C) \geq 2 ; \\ 144K_S^2 , & \text{if } g(C) = 1 ; \\ 120K_S^2 + 960 , & \text{if } g(C) = 0 . \end{cases}$$

These bounds are the best possible.

THEOREM 0.2. *Suppose S is a surface of general type over the complex number field with a relatively minimal genus 2 fibration $f: S \to C$. Then an abelian automorphism group G of f satisfies*

$$|G| \leq 12.5K_S^2 + 100 .$$

This bound is the best possible.

THEOREM 0.3. *Suppose S is a surface of general type over the complex number field with a relatively minimal genus 2 fibration $f: S \to C$. Then a cyclic automorphism group G of f satisfies*

$$|G| \leq \begin{cases} 5K_S^2, & \text{if } g(C) = 1 , \quad K_S^2 \geq 12 ; \\ 12.5K_S^2 + 90 , & \text{if } g(C) = 0 . \end{cases}$$

These bounds are the best possible.

THEOREM 0.4. *Suppose S is a minimal surface of general type over the complex number field with a genus 2 fibration $f: S \to C$ with $g(C) \geq 2$. Then a cyclic automorphism group G of f satisfies*

$$|G| \leq 5K_S^2 + 30 .$$

Theorem 0.1 will be obtained as a consequence of several propositions in Section 3. In Section 4, we discuss abelian and cyclic automorphism groups of the fibration f. The propositions proved there imply Theorems 0.2, 0.3 and 0.4. We remark that Xiao [7] has obtained a bound for abelian automorphism groups of f. Our theorem is an improvement of his. Examples are given in Section 5 to show that most of these bounds are the best possible.

1. Preliminaries. The surfaces with genus 2 pencils have been studied by many authors. The facts we need in this paper appeared mostly in [3], [6], [9], [10]. In particular, Xiao's book [10] gave a systematic description of the properties of genus 2 fibrations which are just what we need here. Unfortunately, this book has not been translated into English yet, hence it is not available for most readers. For this reason, we will recall some materials in this section.

Let $f: S \to C$ be a relatively minimal fibration of genus 2 and $\omega_{S/C} = \omega_S \otimes f^*\omega_C^\vee$ the relative canonical sheaf of f. For a sufficiently ample invertible sheaf $\mathscr{L}$ on C, the natural homomorphism $f^*(f_*\omega_{S/C} \otimes \mathscr{L}) \to \omega_{S/C} \otimes f^*\mathscr{L}$ defines a natural map Φ:

$$\begin{tikzcd} S \arrow[rr, dashed, "\Phi"] \arrow[dr, "f"'] & & P = \mathbf{P}(f_*\omega_{S/C} \otimes \mathscr{L}) \arrow[dl, "\pi"] \\ & C, & \end{tikzcd}$$

Φ is called a relative canonical map. By a succession of blow-ups, we can obtain the following commutative diagram:

$$\begin{tikzcd} \tilde{S} \arrow[rr, "\tilde{\theta}"] \arrow[d, "\rho"'] & & \tilde{P} \arrow[d, "\psi"] \\ S \arrow[rr, dashed, "\Phi"] \arrow[dr, "f"'] & & P \arrow[dl, "\pi"] \\ & C, & \end{tikzcd}$$

where ρ and ψ are composites of finitely many blow-ups, $\tilde{\theta}$ is a double cover. Then we get the branch loci $\tilde{R}$ on $\tilde{P}$ and R on P such that $\tilde{R}$ is the minimal even resolution of R (i.e., the canonical resolution of the double cover). If $\mathscr{L}$ is sufficiently ample, then all the singularities of R must be located in one of the six types 0), I), II), III), IV) and V) of singular fibers defined by Horikawa [3].

P is a relatively minimal ruled surface. We denote a section which has the least self-intersection number by C_0 with $C_0^2 = -e$. We use F to denote both the fiber of f and π.

A singular point of the branch locus is said to be *negligible* if this point itself and all its infinitely near points are double points or triple points with at least two different tangents. By the minimal even resolution, the inverse image of a negligible singular point is composed of (-2)-curves. All other singular points are said to be *non-negligible*. The singular fiber of type 0) in the classification of Horikawa is nothing else but the fiber which does not contain any non-negligible singular points.

The minimal even resolution $\psi: \tilde{P} \to P$ can be decomposed into $\tilde{\psi}: \tilde{P} \to \hat{P}$ followed by $\hat{\psi}: \hat{P} \to P$, where $\tilde{\psi}$ and $\hat{\psi}$ are composed respectively of negligible and non-negligible blow-ups. The image of $\tilde{R}$ in $\hat{P}$ is denoted by $\hat{R}$.

If we take away all the isolated vertical (-2)-curves from the reduced divisor $\hat{R}$, we get a new reduced divisor $\hat{R}_p$, which is called the *principal part* of the branch locus $\hat{R}$. Then for any fiber F of $\pi: P \to C$, the second and third *singularity index* $s_2(F)$, $s_3(F)$ of F is defined as follows:

If R has no quadruple singularities on F, then $s_3(F)$ equals the number of $(3 \to 3)$ type singularities of R on F. Otherwise $s_3(F)$ equals the number of $(3 \to 3)$ type singularities of R on F plus one. Hence $s_3(F)=0$ if and only if R has no non-negligible singularities on F.

Let $\varphi: \hat{R}_p \to C$ be the natural projection induced by $\pi \circ \hat{\psi}: \hat{P} \to C$. Then the second singularity index $s_2(F)$ of F is the ramification index of the divisor $\hat{R}_p$ on $f(F)$ with respect to the projection φ. If $\hat{R}_p$ has singularities (which must be negligible) on F, the singularity index $s_2(F)$ can be calculated as follows:

For a smooth point $p \in \hat{R}_p \cap F$, the ramification index of φ at p can be defined as that for an ordinary smooth curve. If $p \in \hat{R}_p \cap F$ is a singular point of $\hat{R}_p$, then the ramification index of φ at p is defined as the sum of ramification indices of the normalization of $\hat{R}_p$ at the pre-image of p with respect to its projection to C plus the double of the contribution to the arithmetic genus of $\hat{R}_p$ during its normalization at the singular point p. If the normalization of $\hat{R}_p$ contains an isolated vertical component E, then the contribution of E to the ramification index of φ is equal to $2g(E)-2$.

Since there are a finite number of fibers F with $s_i(F) \neq 0$, we define the i-th *singularity index* $s_i(f)$ of f to be the sum of $s_i(F)$ for all fibers, when $i=2, 3$. If we take away from the branch locus R all the fibers F with odd $s_3(F)$, we obtain a divisor R_p which is called the *principal part* of R. Suppose that

$$R_p \sim -3K_{P/C} + nF,$$

where $K_{P/C}$ is the relative canonical divisor of π and $\sim$ represents the numerical equivalence. With these definitions, the formula for the relative invariants of a genus 2 fibration can be stated as follows:

THEOREM 1.1 (Xiao [10]). *Let $f: S \to C$ be a relatively minimal fibration of genus 2. Then*

$$K_{S/C}^2 = K_S^2 - 8(g(C)-1) = \frac{1}{5}s_2(f) + \frac{7}{5}s_3(f) = 2n - s_3(f),$$

$$\chi_f = \chi(\mathcal{O}_S) - (g(C)-1) = \frac{1}{10}s_2(f) + \frac{1}{5}s_3(f) = n - s_3(f).$$

2. Local cases. We begin with a local fibration $f: S_\Delta \to \Delta$ where f is an analytic mapping onto the unit disk Δ, S_Δ is a 2-dimensional analytic smooth manifold and the fibers of f are projective curves. We assume that the fiber over the zero is singular and all the fibers over $\Delta^* = \Delta - \{0\}$ are smooth curves of genus 2.

Similarly, we have a commutative diagram:

$$\begin{tikzcd} \tilde{S}_\Delta \arrow[r, "\tilde{\theta}"] \arrow[d, "\rho"'] & \tilde{P}_\Delta \arrow[d, "\psi"] \\ S_\Delta \arrow[r, dashed, "\Phi"] \arrow[dr, "f"'] & P_\Delta \arrow[d, "\pi"] \\ & \Delta . \end{tikzcd}$$

Denote the branch locus in P_Δ by R_Δ. We also denote the horizontal part of R_Δ by R'_Δ, that is,

$$R'_\Delta = \begin{cases} R_\Delta - F_0 , & \text{if } R_\Delta \text{ contains } F_0 , \\ R_\Delta & \text{otherwise} . \end{cases}$$

Let $F_0 = \pi^{-1}(0)$, $F_t = \pi^{-1}(t)$, $t \in \Delta^*$, and $K_\Delta = \{\tilde{\sigma} \in \mathrm{Aut}(S_\Delta) \mid f \circ \tilde{\sigma} = f\}$. Any automorphism $\tilde{\sigma} \in K_\Delta$ induces an automorphism σ of P_Δ satisfying $\pi \circ \sigma = \pi$ and $\sigma(R_\Delta) = R_\Delta$. If we denote the image of K_Δ by $\bar{K}_\Delta \subseteq \mathrm{Aut}\, P_\Delta$, then

$$|K_\Delta| = 2 |\bar{K}_\Delta| .$$

Note that any finite automorphism group of $\boldsymbol{P}^1$ must be those in Table 1.

TABLE 1.

$G \subseteq \mathrm{Aut}(\boldsymbol{P}^1)$		$\lvert G \rvert$	Number of points in an orbit
Cyclic group	Z_n	n	1, n
Dihedral group	D_{2n}	$2n$	2, n, $2n$
Tetrahedral group	T_{12}	12	4, 6, 12
Octahedral group	O_{24}	24	6, 8, 12, 24
Icosahedral group	I_{60}	60	12, 20, 30, 60

For any $\sigma \in \bar{K}_\Delta$, its restriction $\sigma|_{F_t}$ to $F_t \cong \boldsymbol{P}^1$ must preserve the set of six points contained in $F_t \cap R_\Delta$. Hence $\bar{K}_\Delta$ can be isomorphic to one of the following groups O_{24}, T_{12}, D_{12}, D_6, Z_6, Z_5, D_4, Z_4, Z_3, Z_2 and $\{1\}$.

LEMMA 2.1. *If $\bar{K}_\Delta \cong O_{24}$, T_{12} or D_{12}, then F_0 is contained in R_Δ, and R_Δ has six ordinary double points on F_0. In this case, we have $s_2(F_0) = 10$ and $s_3(F_0) = 0$.*

PROOF. Since $\bar{K}_\Delta \cong O_{24}$, T_{12} or D_{12}, $R_\Delta \cap F_t$ $(t \in \Delta^*)$ consists respectively of six vertices of a regular octahedron, of six points corresponding to the centers of edges of a regular tetrahedron, or of sixth roots of unity. These six horizontal branches of R'_Δ

cannot intersect when $t \to 0$. Since R_Δ must have some singularities by assumption, F_0 is contained in R_Δ.

Since R_Δ does not contain non-negligible singularities, one has $s_3(F_0)=0$ and $R_\Delta = \hat{R}_\Delta = (\hat{R}_\Delta)_p$. On F_0, R_Δ has six ordinary double points, the contribution of each double point to the arithmetic genus of R_Δ during its normalization being equal to one. The pre-image of F_0 in the normalization of R_Δ is a smooth vertical rational curve which does not meet any other branches, so its contribution to the index $s_2(F_0)$ is equal to -2. Therefore $s_2(F_0)=2\times 6+(-2)=10$. □

We list the following useful lemmas, whose proofs are evident. Since local equations are used for calculation of singularity indices, they are given in simplified form, omitting some higher order terms. All the non-negligible singularities here are canonical, i.e., those defined by Horikawa.

LEMMA 2.2. *If $\bar{K}_\Delta \cong D_6$ and R'_Δ is not étale over Δ, then up to coordinate transformation we have*:

(1) *The equation of R'_Δ is $(x^3-t^k)(t^kx^3-1)$, $k>0$. In this case, $s_3(F_0)=0$ implies $s_2(F_0)\geq 4$.*

(2) *The equation of R'_Δ is $(x^3-1)^2-t^k(x^3+1)^2$, $k>0$. In this case, we have $s_3(F_0)=0$ and $s_2(F_0)\geq 3$.*

LEMMA 2.3. *If $\bar{K}_\Delta \cong Z_6$ and R'_Δ is not étale over Δ, then up to coordinate transformation, the equation of R'_Δ is x^6-t^k, $1\leq k\leq 3$. If $k=3$, it has a non-negligible singularity with $s_3(F_0)=1$ and $s_2(F_0)=3$. Otherwise $s_2(F_0)\geq 5$.*

LEMMA 2.4. *If $\bar{K}_\Delta \cong Z_5$ and R'_Δ is not étale over Δ, then up to coordinate transformation, we have*:

(1) *The equation of R'_Δ is $x(x^5-t^k)$, $k=1, 2$. In this case, $s_3(F_0)=0$ and $s_2(F_0)\geq 6$.*

(2) *The equation of R'_Δ is $x(t^kx^5-1)$, $k=1, 2$. In this case, $s_3(F_0)=0$ and $s_2(F_0)\geq 4$.*

LEMMA 2.5. *If $\bar{K}_\Delta \cong D_4$ and R'_Δ is not étale over Δ, then up to coordinate transformation, we have*:

(1) *The equation of R'_Δ is $(x^2-1)((x-1)^2-t^k(x+1)^2)(t^k(x-1)^2-(x+1)^2)$, $k>0$. In this case, $s_3(F_0)=0$ implies $s_2(F_0)\geq 6$.*

(2) *The equation of R'_Δ is $(x^2-1)(x^2-t^k)(t^kx^2-1)$, $k>0$. In this case, we have $s_3(F_0)=0$ and $s_2(F_0)\geq 2$.*

LEMMA 2.6. *If $\bar{K}_\Delta \cong Z_4$ and R'_Δ is not étale over Δ, then up to coordinate transformation, the equation of R'_Δ is $x(x^4-t^k)$, $k=1, 2$. In this case, we have $s_3(F_0)=0$ and $s_2(F_0)\geq 5$.*

LEMMA 2.7. *If $\bar{K}_\Delta \cong Z_3$ and R'_Δ is not étale over Δ, then up to coordinate transformation, we have*:

(1) *The equation of R'_Δ is $(x^3-t^{k_1})(t^{k_2}x^3-a(t))$, k_1, $k_2>0$, $a(0)\neq 0$. In this case, $s_3(F_0)=0$ implies $s_2(F_0)\geq 4$.*

(2) *The equation of* R'_Δ *is* $x^6+a(t)x^3+t^k$, $1\leq k\leq 3$. *In this case,* $s_3(F_0)=0$ *implies* $s_2(F_0)\geq 5$.

(3) *The equation of* R'_Δ *is* $(x^3-b-t^{k_1})(x^3-b-t^{k_2}a(t))$, k_1, $k_2>0$, $a(0)\neq 0$ *and* $b\neq 0$. *In this case, we have* $s_3(F_0)=0$ *and* $s_2(F_0)\geq 6$.

(4) *The equation of* R'_Δ *is* $(x^3-t^k)(x^3-a(t))$, $1\leq k\leq 3$, $a(0)\neq 0$. *In this case, we have* $s_3(F_0)=0$ *and* $s_2(F_0)\geq 2$.

(5) *The equation of* R'_Δ *is* $((x-b)^2-t^ka(t))(x-b\omega)^2-\omega^2t^ka(t))((x-b\omega^2)^2-\omega t^ka(t))$, $k>0$, $a(0)\neq 0$, $b\neq 0$, $\omega=\exp(2\pi i/3)$. *In this case, we have* $s_3(F_0)=0$ *and* $s_2(F_0)\geq 3$.

We summarize the results of Lemmas 2.2 through 2.7 in Table 2 where we assume that R'_Δ has only negligible singularities or ramifications on F_0.

TABLE 2.

$\bar{K}_\Delta$	$\lvert K_\Delta\rvert$	$s_2(F_0)$	$\lvert K_\Delta\rvert/s_2(F_0)$
D_6	12	≥ 3	≤ 4
Z_6	12	≥ 5	≤ 2.4
Z_5	10	≥ 4	≤ 2.5
D_4	8	≥ 2	≤ 4
Z_4	8	≥ 5	≤ 1.6
Z_3	6	≥ 2	≤ 3
Z_2	4	≥ 1	≤ 4
1	2	≥ 1	≤ 2

LEMMA 2.8. *If* R'_Δ *has only negligible singularities or ranifications on* F_0, *then* $|K_\Delta|/s_2(F_0)\leq 4$. *Moreover, if* $\bar{K}_\Delta\cong Z_6$, Z_5, Z_4 *or* $\{1\}$, *then* $|K_\Delta|/s_2(F_0)\leq 2.5$.

3. Bounds of automorphism groups.

Let $G=\mathrm{Aut}(f)$ be the automorphism group of the fibration $f: S\to C$ of genus two. Then we have an exact sequence

$$1\to K\to G\to H\to 1\ ,$$

$$(\tilde{\sigma},\sigma)\mapsto\sigma\ ,$$

where $H\subseteq\mathrm{Aut}(C)$, $K=\{(\tilde{\sigma},\mathrm{id})\in G\}=\{\tilde{\sigma}\in\mathrm{Aut}(S)\,|\,f\circ\tilde{\sigma}=f\}$. Thus $|G|=|K||H|$. The elements of H are often regarded as transformations of the fibers of f or π.

PROPOSITION 3.1. *If* $f: S\to C$ *is a relatively minimal fibration of genus* 2 *with* $g(C)\geq 2$, *then*

$$|\mathrm{Aut}(f)|\leq 504K_S^2\ .$$

PROOF. Since $|K|\leq 48$, $|H|\leq|\mathrm{Aut}(C)|\leq 84(g(C)-1)$, we have

$$|G|=|K||H|\leq 4032(g(C)-1)\ .$$

On the other hand, $K^2_{S/C} \geq 0$ and the equality holds if and only if f is locally trivial. Hence

$$K^2_S \geq 8(g-1)(g(C)-1) = 8(g(C)-1),$$

and $|G| \leq 504 K^2_S$. $\square$

PROPOSITION 3.2. *If $f: S \to C$ is a relatively minimal fibration of genus 2 with $g(C) \geq 2$ which is not locally trivial, then*

$$|\mathrm{Aut}(f)| \leq 126 K^2_S.$$

PROOF. Let R' denote the horizontal part of the branch locus R. If R' is not étale over C, then by the lemmas in Section 2, we have $|K| \leq 12$. Since $|H| \leq 84(g(C)-1) \leq 10.5 K^2_S$, $|G| \leq 12|H| \leq 126 K^2_S$.

Now assume that R' is étale. Since f is not locally trivial, we must have $K^2_{S/C} > 0$, i.e., either $s_3(f) > 0$ or $s_2(f) > 0$. So R must contain some fiber F_0. By Lemma 2.1, $s_3(F_0) = 0$ and $s_2(F_0) = 10$. Let $p = f(F_0)$, $n = |H|$. Since H is a subgroup of $\mathrm{Aut}(C)$, H determines a finite morphism $\tau: C \to X = C/H$. Denote the ramification index of $p \in C$ with respect to τ by r and the other ramification indices by r_i. Then Hurwitz's theorem implies that

$$2g(C) - 2 = n(2g(X) - 2) + n \sum \left(1 - \frac{1}{r_i}\right).$$

Since the H-orbit of the point p has n/r points, this implies that $s_2(f) \geq 10n/r$. Hence

$$\begin{aligned} K^2_S &\geq \frac{1}{5} s_2(f) + 8(g(C)-1) = \frac{2n}{r} + 4n\left[2g(X) - 2 + \sum\left(1 - \frac{1}{r_i}\right)\right] \\ &= 4n\left[2g(X) - 2 + \frac{1}{2r} + \sum\left(1 - \frac{1}{r_i}\right)\right]. \end{aligned}$$

It is not difficult to see that the expression $2g(X) - 2 + 1/2r + \sum(1 - 1/r_i)$ reaches its minimal value $2/21$ (under the condition $2g(X) - 2 + \sum(1 - 1/r_i) > 0$) when $g(X) = 0$, $r_1 = 2$, $r_2 = 3$, and $r = r_3 = 7$, that is,

$$K^2_S \geq \frac{8}{21} n = \frac{8}{21} |H|.$$

Thus

$$|G| \leq 48 |H| \leq 126 K^2_S.$$

$\square$

REMARK. It is not difficult to see that if $g(C) \geq 2$, f is not locally trivial and $|\mathrm{Aut}(f)| = 126 K^2_S$, then $|\mathrm{Aut}(C)| = 84(g(C)-1)$, $|\mathrm{Aut}(F)| = 48$ for any smooth fiber F and $\mathrm{Aut}(f) \cong \mathrm{Aut}(C) \times \mathrm{Aut}(F)$. We will give an example later. In this case, the fibration

f is of constant moduli.

LEMMA 3.1. *Let S be a surface of general type which has a relatively minimal genus 2 fibration $f: S \to C$. If the third singularity index $s_3(f) \neq 0$, then*

$$|\mathrm{Aut}(f)| \leq \frac{60}{7} r K_{S/C}^2 ,$$

where

$$r = \min_{s_3(F) \neq 0} |\mathrm{Stab}_H f(F)| ,$$

$\mathrm{Stab}_H f(F)$ *being the stabilizer of $f(F)$ in H.*

PROOF. Let F_0 be a singular fiber such that $s_3(F_0) \neq 0$ and $r = |\mathrm{Stab}_H f(F_0)|$. Then

$$K_{S/C}^2 \geq \frac{7}{5} s_3(f) \geq \frac{7 s_3(F_0)}{5r} |H| ,$$

and we get

$$|G| = |K||H| \leq \frac{r}{s_3(F_0)} \cdot \frac{60}{7} K_{S/C}^2 \leq \frac{60}{7} r K_{S/C}^2 .$$

□

LEMMA 3.2. *Let S be a surface of general type which has a relatively minimal genus 2 fibration $f: S \to C$. If the horizontal part R' of the branch locus R is not étale and has only negligible singularities or ramifications, then*

$$|\mathrm{Aut}(f)| \leq 20 r K_{S/C}^2 ,$$

where

$$r = \min\{ |\mathrm{Stab}_H f(F)| \mid F \textit{ singular fiber} \} .$$

PROOF. Let F_0 be a singular fiber with $r = |\mathrm{Stab}_H f(F_0)|$. Since here

$$K_{S/C}^2 \geq \frac{1}{5} s_2(f) \geq \frac{s_2(F_0)}{5r} |H| ,$$

we have

$$|G| = |K||H| \leq \frac{r|K|}{s_2(F_0)} \cdot 5 K_{S/C}^2 \leq 20 r K_{S/C}^2 ,$$

by Lemma 2.8. □

LEMMA 3.3. *Let S be a surface of general type which has a relatively minimal genus 2 fibration $f: S \to C$. If the horizontal part R' of the branch locus R is étale, then*

$$|\operatorname{Aut}(f)|\leq 24rK_{S/C}^2,$$

where

$$r=\min\{|\operatorname{Stab}_H f(F)|\,|\,F \text{ singular fiber}\}.$$

PROOF. Let F_0 be a singular fiber with $r=|\operatorname{Stab}_H f(F_0)|$. By assumption, we have $s_2(F_0)=10$. Hence

$$|G|=|K||H|\leq\frac{r|K|}{s_2(F_0)}\cdot 5K_{S/C}^2\leq 24rK_{S/C}^2.$$

□

Let $\bar{K}$ denote the subgroup in $\operatorname{Aut}(P)$ which is induced by K. If $\sigma\in\bar{K}$, then $\pi\circ\sigma=\pi$ and $\sigma(R)=R$. Let K_1 be a cyclic subgroup of order m of $\bar{K}$, and let $Q=P/K_1$ be the quotient surface. Then Q is a ruled surface. We have a commutative diagram:

$$\begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr, "\pi"'] & & Q \arrow[dl, "\pi'"] \\ & C. & \end{tikzcd}$$

Let C_0 and $C_\infty\sim C_0+eF$ be the reduced ramification divisors of K_1. Let C_0' be a section of π' with the least self-intersection number $C_0'^2=-e'$, and let F' be a general fiber of π'. Then $\alpha^*C_0'=mC_0$, $\alpha^*C_\infty'=mC_\infty$, $\alpha^*F'=F$ and $e'=me$. Let $D=\alpha(R')$, and let $C'=C_0'+C_\infty'$ be the branch locus. Then $C'\sim 2C_0'+e'F'\sim -K_{Q/C}$.

LEMMA 3.4. *Assume $\bar{K}\cong D_6$. If R' is not étale and has only negligible singularities or ramifications, then f has more than one H-orbits of singular fibers.*

PROOF. Let K_1 be the unique cyclic subgroup of order 3 of $\bar{K}$. There are two types of singular fibers as listed in Lemma 2.2. Let F_0 be a singular fiber. Then the local equations of D near F_0 are (1) $(x-t^k)(t^kx-1)$, $k\leq 3$, (2) $(x-1)^2-t^k(x+1)^2$, $k>0$. In Case (1), D meets C' at two points in F_0. In Case (2), D does not meet C' in F_0.

If all the singular fibers of f are of type (1), then D is an étale cover of C. This means that $a=e'$ and $C'\sim D$. Hence $DC'=0$, which is impossible because D and C' meet in F_0.

If all the singular fibers of f are of type (2), then $DC'=0$. Hence $D\sim C'$ and $D(D+K_{Q/C})=0$. This means that D is étale over C, a contradiction. □

LEMMA 3.5. *Assume $\bar{K}\cong D_4$. If R' is not étale, then f has more than one H-orbits of singular fibers. If H is cyclic and $g(C)=0$, then*

$$|\operatorname{Aut}(f)|\leq 12.5K_{S/C}^2.$$

PROOF. In this case, there are four sections in P which do not meet one another. Hence $e=0$. R' contains two of these sections denoted by C_0 and C_∞. Let K_1 be a cyclic subgroup of $\bar{K}$ with C_0 and C_∞ are ramifications. Assume that there is only one H-oribt of singular fibers. If these singular fibers are all of type (1) in Lemma 2.5, then the local equation of $D=\alpha(R'-C_0-C_\infty)$ is $(x-t^k)(t^kx-1)$, namely, D is étale. Therefore $D\sim 2C'_0$, $DC'_0=DC'_\infty=0$, a contradiction. If the singular fibers are of type (2) in Lemma 2.5, then D does not meet C'_0 and C'_∞. Hence $D\sim 2C'_0$, $D^2=0$, a contradiction. Hence there are at least two H-orbits.

Now suppose H is cyclic. Let $h=|H|$. An H-orbit is said to be *big* if it contains h fibers. If there is a big H-orbit whose singular fibers are of type (1), then $s_2(F_0)\geq 6$, so $|G|\leq(20/3)K^2_{S/C}$. If $|G|>(20/3)K^2_{S/C}$, then the singular fibers in a big H-orbit must be of type (2) with $k\leq 2$. Let F_2 and F_3 denote two fibers fixed by H. Then at least one of them is of type (1). The structure of types (1) and (2) implies that the normalization of $D=\alpha(R'-C_0-C_\infty)$ is étale with respect to π'. Hence D must be decomposed into two isomorphic sections D_1 and D_2 with $D_1\sim D_2\sim C'_0+aF'$. Since both D_1 and D_2 meet C'_0 and C'_∞, F_2 and F_3 are all singular of type (1). Since $D_1D_2=2a=kh$, we get $D_1C'_0=a=kh/2$. Hence the local equation of R' near F_2 or F_3 is $(x^2-1)((x-1)^2-t^{kh/2}(x+1)^2)(t^{kh/2}(x-1)^2-(x+1)^2)$. When $h\geq 6$, these are non-negligible singularities. If F_i $(i=2, 3)$ is a singular fiber of type I), then $s_3(F_i)=2[(kh-2)/8]+1\geq(kh-1)/4$. If F_i is of type II), then $s_3(F_i)=2[kh/8]\geq(kh-6)/4$. So

$$K^2_{S/C}\geq\frac{1}{5}\times 2\times h+\frac{7}{5}\times\frac{h-6}{4}\times 2=\frac{11}{10}h-\frac{21}{5}.$$

$$|G|=8h\leq\frac{80}{11}\left(K^2_{S/C}+\frac{21}{5}\right)<12.5K^2_{S/C}.$$

If there are more than one big H-orbits, it can be similarly shown that $|G|\leq 12.5K^2_{S/C}$. □

LEMMA 3.6. *Assume $\bar{K}\cong Z_3$. If R' is not étale and has only negligible singularities or ramifications and f has only one H-orbit of singular fibers, then*

$$|\mathrm{Aut}(f)|\leq 6rK^2_{S/C},$$

where

$$r=\min\{|\mathrm{Stab}_H f(F)|\,|\,F \text{ singular fiber}\}.$$

PROOF. Let $K_1=\bar{K}$. If the singular fibers are of types (1) or (4) in Lemma 2.7, then $D\sim 2C'_0+aF'$ is étale. $D(K_{Q/C}+D)=0$ implies $a=e'$. Hence $D(C'_0+C'_\infty)=0$, a contradiction. If the singular fiber F_0 is of type (5) with $k=1$, then D is irreducible and smooth near F_0. This implies $DC'_\infty\neq 0$, a contradiction. Therefore $s_2(F_0)\geq 5$ for any singular fiber F_0. So $|G|\leq 6rK^2_{S/C}$. □

LEMMA 3.7. *Assume $\bar{K}\cong Z_2$. If R' is not étale and f has only one H-orbit of singular fibers, then*

$$|\mathrm{Aut}(f)|\leq 5rK_{S/C}^2\,,$$

where

$$r=\min\{|\mathrm{Stab}_H f(F)|\,|\,F \text{ singular fiber}\}\,.$$

PROOF. Let F_0 be a singular fiber. $|G|>5rK_{S/C}^2$ implies $s_2(F_0)\leq 3$. We distinguish between two cases.

Case I. R' contains C_0 and C_∞. Then the local equation of R' near F_0 must be (1) $x(x^2-t)(x^2-a(t))$, $a(0)\neq 0$, $s_2(F_0)=3$, or (2) $x((x^2-a^2)^2-t)$, $a\neq 0$, $s_2(F_0)=2$. Let $D=\alpha(R'-C_0-C_\infty)\sim 2C_0'+aF'$. If all the singular fibers are of type (1), then D is étale. This is impossible. If the singular fibers are of type (2), then D is irreducible and does not meet C'. This is impossible.

Case II. R' does not contain C_0 and C_∞. Then the local equation of R' may be (1) $(x^2-t)(x^2-a(t))(x^2-b(t))$, $a(0)b(0)\neq 0$, $a(0)\neq b(0)$, $s_2(F_0)=1$; (2) $(x^2-t)(ta(t)x^2-1)(x^2-b(t))$, $a(0)b(0)\neq 0$, $s_2(F_0)=2$; (3) $((x^2-a^2)^2-t)(x^2-b(t))$, $ab(0)\neq 0$, $s_2(F_0)=2$; (4) $((x^2-a^2)^2-t)(x^2-tb(t))$, $b(0)\neq 0$, $s_2(F_0)=3$. Let $D=\alpha(R')\sim 3C_0'+aF'$. If F_0 is of type (1) or (2), then D is étale and smooth. D must be decomposed into three disjoint components. This means $e'=0$, a contradiction. If F_0 is of type (3) or (4), then D is smooth. The ramification index is $D(D+K_{Q/C})=4a-6e'=|H|/r$. Hence $DC'=2a-3e'=|H|/2r$. This is a contradiction because we have $DC'=0$ for type (3) and $DC'=|H|/r$ for type (4).

PROPOSITION 3.3. *If S is a minimal surface of general type which has a genus 2 fibration $f: S\to C$ with $g(C)=1$, then*

$$|\mathrm{Aut}(f)|\leq 144K_S^2\,.$$

PROOF. In this case, we have

$$K_S^2=K_{S/C}^2=\frac{1}{5}s_2(f)+\frac{7}{5}s_3(f)>0\,.$$

Thus either $s_3(f)>0$ or $s_2(f)>0$.

Let $j(C)$ be the j-invariant of the elliptic curve C. Let m denote the number of points contained in a smallest H-orbit of C. Since H is a finite subgroup of $\mathrm{Aut}(C)$, we have

$$m=\begin{cases}|H|/2 & \text{if } j(C)\neq 0,1728\,,\\ |H|/4 & \text{if } j(C)=1728\,,\\ |H|/6 & \text{if } j(C)=0\,.\end{cases}$$

Since $r\leq 6$, by Lemmas 3.1, 3.2 and 3.3, the conclusion is immediate. □

PROPOSITION 3.4. *If S is a surface of general type which has a relatively minimal fibration $f: S \to C$ of genus 2 with $g(C)=0$, then*

$$|\mathrm{Aut}(f)| \leq 120(K_S^2+8).$$

Moreover, we have

$$|\mathrm{Aut}(f)| \leq 48(K_S^2+8)$$

for $K_S^2 \geq 33$, and when $K_S^2 \leq 32$, there are only four exceptions.

PROOF. In this case, we have

$$K_S^2+8=K_{S/C}^2=\frac{1}{5}\, s_2(f)+\frac{7}{5}\, s_3(f)>0.$$

Hence either $s_3(f)>0$ or $s_2(f)>0$.

Case I. Assume that R' is étale over C. If $r \leq 5$, then by Lemma 3.3

$$|G| \leq 24rK_{S/C}^2 \leq 120(K_S^2+8).$$

If $r \geq 6$, then H must be a cyclic or a dihedral group. In this case, there are at most two singular fibers. Hence $K_{S/C}^2 \leq 4$ by Theorem 1.1. This means that S is not of general type [10, Theorem 4.2.5, p. 90].

Case II. Assume that R' is not étale. Then f is a fibration of variable moduli. Hence f must contain more than two singular fibers (cf. [2]). This implies $r \leq 5$. The conclusion follows from Lemmas 3.1 and 3.2.

In the preceding argument, we can see that $|G| \leq 48(K_S^2+8)$ holds if $r \leq 2$. If $|G|>48(K_S^2+8)$, we must have $r>3$. Then H is one of T_{12}, O_{24} and I_{60}.

If f has more than one H-orbit of singular fibers, then

$$\begin{aligned}\frac{K_{S/C}^2}{|G|} &\geq \frac{1}{5r}\left(\frac{s_2(F_0)}{|K|}+\frac{7s_3(F_0)}{|K|}\right)+\frac{1}{5r_1}\left(\frac{s_2(F_1)}{|K|}+\frac{7s_3(F_1)}{|K|}\right)\\ &\geq \frac{1}{25}\times\frac{1}{4}+\frac{1}{20}\times\frac{1}{4}=\frac{9}{400}>\frac{1}{48}.\end{aligned}$$

Therefore f has only one H-orbit.

If the singular fibers has non-negligible singularities, then by Lemma 3.1, $|G| \leq (60/7)rK_{S/C}^2 \leq (300/7)K_{S/C}^2 < 48K_{S/C}^2$. Suppose that the horizontal part R' of the branch locus has only negligible singularities or ramifications. Then by Lemmas 3.4, 3.5, 3.6 and 3.7, we have

$$|G| \leq 12.5rK_{S/C}^2.$$

Thus $|G|>48K_{S/C}^2$ implies that $r \geq 4$ and $\bar{K}$ is Z_6 or Z_5. If $\bar{K} \cong Z_6$, then $r=5$ and $H \cong I_{60}$. To ensure $|G|>48K_{S/C}^2$, we have $s_2(F_0)=5$, i.e., $R=R' \sim -3K_{P/C}+nF$ is a smooth irreducible divisor. As a multiple cover on C, the ramification index of R is equal to

$R(R+K_{P/C})=12n$. On the other hand, this ramification index is equal to $5\times(60/5)=60$, i.e., $n=5$. However, $2n=10=K_{S/C}^2\neq s_2(f)/5=12$, a contradiction.

If $\bar{K}\cong Z_5$, then $|G|>48K_{S/C}^2$ implies $s_2(F_0)=4$. In this case $R=R'=C_0+R_1$, where $R_1\sim 5C_0+(n+3e)F$ is an smooth irreducible divisor and $R_1C_0=0$, i.e., $n=2e$. Computing the ramification index of R_1 we get $R_1(R_1+K_{P/C})=10n=4|H|/r$. Thus $5r$ divides $|H|$, a contradiction. Hence $|G|>48(K_S^2+8)$ implies that R' is étale over C. There are only a finite number of possibilities. We list the possible fibrations with $|G|>48(K_S^2+8)$ in Table 3.

TABLE 3.

H	r	$\|G\|$	K_S^2	$\|K\|/(K_S^2+8)$	$\|K\|/K_S^2$
I_{60}	5	2880	16	120	180
I_{60}	3	2880	32	72	90
O_{24}	4	1152	4	96	288
O_{24}	3	1152	8	72	144

In Section 5 we will show the existence. □

COROLLARY 3.5. *If S is a minimal surface of general type which has a genus 2 fibration $f: S\to C$ with $g(C)=0$, then*

$$|\mathrm{Aut}(f)|\leq 288K_S^2\,.$$

PROOF. If $K_S^2\geq 2$, then $48(K_S^2+8)<288K_S^2$. By Proposition 3.4 we need only check the four exceptional examples. □

4. Abelian automorphism groups. Let $G\subseteq\mathrm{Aut}(f)$ be an abelian group. Then it is well known that $|K|\leq 12$.

PROPOSITION 4.1 (Xiao [7, Lemma 8]). *Let $f: S\to C$ be a relatively minimal fibration of genus 2 with $g(C)\geq 2$. Then an abelian automorphism group G of S satisfies*

$$|G|\leq 6K_S^2+96\,.$$

Let $\bar{G}\subseteq\mathrm{Aut}(P)$ be the induced automorphism group of a commutative group G. Then

$$1\to\bar{K}\to\bar{G}\to H\to 1\,.$$

LEMMA 4.1. *Assume that $\bar{K}\cong Z_3$ and $g(C)=0$. Let $p\in C$ be a fixed point of the cyclic group H, and let $F=\pi^{-1}(p)$. If there is a $\bar{K}|_F$-orbit containing three points in F, then*

$$s_2(F)\geq 3|H|\,.$$

PROOF. Since p is a fixed point of H, the induced action of $\bar{G}$ on F forms a commutative subgroup $\bar{G}|_F \subseteq \mathrm{Aut}(F) \cong \mathrm{Aut}(\boldsymbol{P}^1)$. Since $\bar{G}|_F$ stabilizes this $\bar{K}|_F$-orbit, we have $\bar{G}|_F = \bar{K}|_F \cong Z_3$, i.e., $H|_F = 1$. Hence the local equation of R' near F has the form $f(x^3, t^h)$ where $h = |H|$. More explicitly, the local equation of R' is (3) $(x^3 - b - t^{k_1 h} a_1(t^h))(x^3 - b - t^{k_2 h} a_2(t^h))$ or (5) $((x-b)^2 - t^{kh} a(t^h))((x - b\omega)^2 - \omega^2 t^{kh} a(t^h))((x - b\omega^2)^2 - \omega t^{kh} a(t^h))$, $b \neq 0$. Thus $s_2(F) \geq 3h = 3|H|$. □

PROPOSITION 4.2. *If S is a surface of general type which has a relatively minimal fibration $f: S \to C$ of genus 2 with $g(C) \leq 1$, then an abelian automorphism group G of f satisfies*

$$|G| \leq 12.5(K_S^2 + 8).$$

PROOF. It is well known that H must be a cyclic group or a dihedral group $D_4 \cong Z_2 \oplus Z_2$.

If $g(C) = 1$ and H does not act freely on C, then $|H| \leq 6$. Hence $|G| \leq 72 < 12.5(K_S^2 + 8)$. If $g(C) = 0$ and $H \cong D_4$, then $|G| \leq 48$ and the claim holds too. So we can assume that H is a cyclic group and that there exists a singular fiber F_0 with $|\mathrm{Stab}_H f(F_0)| = 1$.

Case I. Suppose that the horizontal part R' of the branch locus R is étale over C. Then $|G| \leq 6K_{S/C}^2$.

Case II. Suppose that R' is not étale. If there is a big H-orbit with $s_3(F_0) \neq 0$, then

$$K_{S/C}^2 \geq \frac{7}{5} s_3(f) \geq \frac{7}{5} |H|,$$

so

$$|G| \leq \frac{60}{7} K_{S/C}^2 < 12.5(K_S^2 + 8).$$

Now suppose that on the big H-orbits R' has only negligible singularities or ramifications. If $\bar{K} \cong Z_6$, Z_5, Z_4 or $\{1\}$, then by Lemma 2.8, we have

$$|G| \leq \frac{|K|}{s_2(F_0)} \cdot 5K_{S/C}^2 \leq 12.5K_{S/C}^2 \leq 12.5(K_S^2 + 8).$$

Suppose that $\bar{K} \cong D_4$, Z_3 or Z_2 and that $|G| > 12.5(K_S^2 + 8)$. Then Lemmas 3.5, 3.6 and 3.7 imply that f must have more than one H-orbits of singular fibers. To ensure $|G| > 12.5(K_S^2 + 8)$, f cannot have more than one big H-orbits. Thus we have $g(C) = 0$. Lemma 3.5 excludes the case of $\bar{K} \cong D_4$.

If $\bar{K} \cong Z_3$, then $s_2(F_0) \leq 2$. Hence F_0 must be of type (4) of Lemma 2.7 with $k = 1$. Taking $K_1 = \bar{K}$ we construct the quotient surface $Q = P/K_1$ as in §3. Then $D = \alpha(R')$ is étale near F_0. But D cannot be étale. Hence at least one of the H-stabilized fibers F_2 and F_3 is of type (2) $k = 1$ or type (5) $k = 1$. Lemma 4.1 excludes the case of type (5). Suppose one of the F_i is of type (2). Then $D \sim 2C_0' + aF'$ is irreducible and smooth. As a smooth double cover of $C\cong \boldsymbol{P}^1$, the ramification index of D is at least 2. So F_2 and F_3 are all of type (2). Then $DC'=D(D+K_{Q/C})=2(a-e')=2$, a contradiction.

If $\bar{K}\cong Z_2$, then $s_2(F_0)=1$. Hence the local equation of R' near F_0 is $(x^2-t)(x^2-a(t))(x^2-b(t))$, $a(0)b(0)\neq 0$, $a(0)\neq b(0)$. So $D=\alpha(R')$ is étale near F_0.

If F_2 and F_3 have no ramifications, then D can be decomposed into three components $D_i\sim C'_0+a_iF'$, $i=1, 2, 3$. These three components must meet one another on F_2 and F_3. So there exists at least one point on F_i where three components intersect. The local equation of R' is $(x^4+a(t)x^2+t^2)(x^2+t^2b(t))$. Since $D_i(C'_\infty-C'_0)=e'$, we have $|H|\leq 1$.

If F_2 or F_3 has ramifications, the equation of R' near F_i must be one of (1) x^6-t; (2) $(x^4-t)(t^ka(t)x^2-1)$, $a(0)\neq 0$; (3) $((x^2-a^2)^2-t)(x^2-t^k)$, $a\neq 0$; (4) $((x^2-a^2)^2-t)(x^2-b(t))$, $b(t)\neq 0$. If F_2 is of type (1), then D is irreducible and smooth. As a smooth triple cover of $C\cong \boldsymbol{P}^1$, the ramification index of D is at least 4. Hence F_3 is of type (1) as well. Let $D\sim 3C'_0+aF'$. Then $2DC'=D(D+K_{Q/C})=4$, impossible. If F_2 is of type (2), then D is smooth and cannot be irreducible. D has two components $D_1\sim 2C'_0+aF'$ and $D_2\sim 2C'_0+bF'$. By the same argument, we have $D_1C'=D_1(D_1+K_{Q/C})+2$. Hence $D_1C'_0=0$ and $D_1D_2=0$, which is impossible.

Suppose that G is a cyclic automorphism group of f. Similarly, there is an exact sequence

$$1\longrightarrow K\xrightarrow{\alpha} G\xrightarrow{\beta} H\longrightarrow 1\,,$$

where $H\subseteq \mathrm{Aut}(C)$, $K=\{(\tilde{\sigma}, \mathrm{id})\in G\}$. It is known that $|K|\leq 10$.

LEMMA 4.2. *Suppose that* $f\colon S\to C$ *is a fibration and that* G *is a cyclic automorphism group of* f. *Suppose there exists a point* $p\in C$ *such that*

(1) $\sigma|_{f^{-1}(p)}\in K|_{f^{-1}(p)}$, *for* $\sigma\in G$ *and* σ *stabilize* $f^{-1}(p)$;

(2) $K\to \mathrm{Aut}(f^{-1}(p))$ *is injective.*

Then $|K|$ *and* $|\mathrm{Stab}_H(p)|$ *are coprime.*

PROOF. Let $H_1=\mathrm{Stab}_H(p)$, $F=f^{-1}(p)$. Let $h=|H_1|$, $k=|K|$, $d=(h, k)$. Assume that σ is a generator of $\beta^{-1}(H_1)$. Then $\beta((\sigma^{k/d})^h)=1$ implies $\sigma^{hk/d}\in K$. On the other hand, since $\sigma|_F\in K|_F$ by (1), we obtain $(\sigma^{h/d})^k|_F=\mathrm{id}_F$. Thus $\sigma^{kh/d}=1$ by (2). This is impossible.

PROPOSITION 4.3. *If* S *is a surface of general type which has a relatively minimal fibration* $f\colon S\to C$ *of genus* 2 *with* $g(C)=1$. *Then a cyclic automorphism group* G *of* f *satisfies*

$$|G|\leq 5K_S^2$$

for $K_S^2\geq 12$.

PROOF. If H does not act freely on C, then $|H|\leq 6$. Hence $|G|\leq 60$ and the conclusion holds. Therefore we assume that H acts freely. So $G \cong K \times H$ and G is cyclic if and only if $(|K|, |H|)=1$. We distinguish two cases.

Case I. Suppose that the horizontal part R' of the branch locus R is étale over C. There exists a singular fiber F_0 with $|\mathrm{Stab}_H(f(F_0))|=1$. It is not difficult to show that in this case $|G| \leq 5K_S^2$.

Case II. Suppose that R' is not étale.

(a) $\bar{K} \cong Z_5$. Let F_0 be a singular fiber. The local equation of R' near F_0 is (1) $x(x^5-t^k)$ or (2) $x(t^k x^5-1)$, $k=1, 2$. We construct the quotient surface $Q=P/\bar{K}$ as in Section 3. R' must contain one of the sections C_0 and C_∞. We take this section away from R', and get a reduced divisor R_1 with $R_1 F=5$. Let $D=\alpha(R_1)$. Then $D \sim C'_0+aF'$. Since $DC'_0=0$, we have $a=e'=5e$. Thus $R_1 \sim 5C_0+5eF$ and $R_1 C_\infty=5e$. Since the intersection number of R_1 and F on the fiber F_0 is equal to $k \leq 2$, the number of singular fibers must be a multiple of 5. But $|H|$ cannot be divisible by 5, hence the singular fibers are located in different H-orbits. This means $|G| \leq 5K_S^2$.

(b) $\bar{K} \cong Z_4$. The local equation of R' near a singular fiber F_0 is $x(x^4-t^k)$, $k=1$, 2. We use the same construction as in Case (a). Then R' must contain C_0 and C_∞. Let $R_1=R'-C_0-C_\infty$ and $D=\alpha(R_1)$. Then $D \sim C'_0+e'F'$. Similarly we deduce $R_1 C_\infty=4e$. Since $|H|$ cannot be even, there are more than one singular H-orbits. So $|G| \leq 5K_S^2$.

(c) $\bar{K} \cong Z_3$. If f has only one H-orbit of singular fibers and if $|G|>5K_S^2$, then $s_2(F_0)=5$, namely, the local equations of R' is $x^6+a(t)x^3+t$. Constructing the quotient surface $Q=P/\bar{K}$, we see that $D=\alpha(R') \sim 2C'_0+aF'$ is a smooth irreducible curve and $r \neq |H|$. Since $DC'_0=0$ and $DC'_\infty=|H|$, we get $a=e'=3e=|H|$, i.e., $(|H|, |K|)=3$, a contradiction.

(d) $\bar{K} \cong Z_2$. Lemma 3.7 ensures $|K| \leq 5K_S^2$.

(e) $\bar{K}=1$. If $s_2(F_0) \geq 2$, then $|G| \leq 5K_{S/C}^2$. If $s_2(F_0)=1$, there is only one situation, i.e., the local equation of R' near F_0 is $(x^2-t)(x-a_1(t))(x-a_2(t))(x-a_3(t))(x-a_4(t))(x-a_5(t))$, $a_i(0) \neq 0$. Suppose that there is only one singular H-orbit. Then R' is a smooth sextuple cover of C. The contribution of each singular fiber to the ramification index equals 1. By Hurwitz's formula,

$$2g(R')-2=6(2g(C)-2)+|H| .$$

So $|H|$ is even, a contradiction. □

PROPOSITION 4.4. *If S is a surface of genral type which has a relatively minimal fibration $f: S \to C$ of genus 2 with $g(C)=0$. Then a cyclic automorphism group G of f satisfies*

$$|G| \leq 12.5K_S^2+90 .$$

PROOF. If R' is étale, we have $|G| \leq 5K_{S/C}^2$. If there is a singular fiber in a big H-orbit with $s_3(F)>0$, then $|G| \leq (50/7)K_{S/C}^2$. Now assume that R' has only negligible singularities or ramifications in big H-orbits. If $\bar{K} \cong Z_4$ or $\{1\}$, we have $|G| \leq 10K_{S/C}^2$ by

Lemma 2.8. When $\bar{K}\cong Z_3$ or Z_2, if f has only one H-orbit of singular fibers, then Lemmas 3.6 and 3.7 ensure $|G|\leq 6K_{S/C}^2$. Otherwise, by the proof of Proposition 4.2, f has at least two big H-orbits of singular fibers, hence $|G|\leq 10K_{S/C}^2$.

There remains the case of $\bar{K}\cong Z_5$. The proof of Proposition 4.3 tells us that if f has only one big H-orbit of singular fibers, then f has another singular fiber which is stabilized by H. By Lemma 2.4, we have

$$K_{S/C}^2\geq\frac{4}{5}(|H|+1),$$

so

$$|G|+10|H|\leq 12.5K_{S/C}^2-10=12.5K_S^2+90 .$$

□

When $g(C)\geq 2$, we need the following lemma on the order of some automorphisms of a curve. The proof of the lemma is just a slight modification of that of the theorem of Wiman [5]. For the convenience of the reader, we include its proof here which is a modified copy of the version given in [8, Lemma B].

LEMMA 4.3. *Let H be a cyclic group of automorphisms of a curve C of genus $g\geq 2$ such that the order of $|\mathrm{Stab}_H(p)|$ is odd for any $p\in C$. Then*

$$|H|\leq 3g+3 .$$

PROOF. Let x be a non-zero element in H with the maximal number of fixed points, H' the subgroup of H generated by elements fixing all fixed points of x, n the number of fixed elements of x, and k the order of H'. Then k must be odd. Let $C'=C/H'$, $g'=g(C')$, and let Σ be the image of the set of fixed points of H' on C'. We have

$$2g-2=2kg'-2k+n(k-1) \tag{1}$$

and the quotient group $H''=H/H'$ is a cyclic group of automorphisms of C' which satisfies the same condition imposed on H, i.e., $|\mathrm{Stab}_{H''}(p)|$ is odd for any $p\in C'$.

If $n=0$, then $g'\geq 2$ and $|H|\leq g-1$. If $n=2$, then because every non-zero element of H'' induces a non-trivial translation on Σ, we must have $|H''|\leq 2$, so $|H|\leq 2k$. Then $|H|\leq 2g$ by (1) (note that $g'\neq 0$ in this case). So we may assume $n\geq 3$.

Suppose $g'=1$ and H'' acts freely on C'. Considering the induced action H'' on Σ, we see that $|H''|\leq n$. So (1) gives $|H|\leq 2g+n-2$. On the other hand, since $k\geq 3$, (1) also gives $n\leq g-1$, therefore $|H|\leq 3g-3$ in this case.

Suppose $g'=1$ and H'' does not act freely on C'. Then H'' has a fixed point. By assumption, $|H''|$ must be odd. This implies $|H''|\leq 3$. So (1) gives $|H|\leq 2g+1$.

Now suppose that C' is a rational curve. Then the action of H'' has exactly two fixed points. So $|H''|$ must be odd. If one of these two points is in Σ, then $|H''|\leq n-1$ in view of the action of H'' on Σ. Since $|H''|$ is odd, we have $n\geq 4$. So $|H|\leq 3g+3$.

Suppose that Σ and the two fixed points ξ, η of H'' are disjoint. Let $H_1 \subset H$ be the stabilizer of a point in the inverse image of ξ. Then $[H:H_1]=k$. Since the stabilizer of a point in the inverse image of η is also of index k in H, we see that any non-zero element in H_1 fixes exactly $2k$ points, i.e., the inverse image of ξ and η. Now we can replace H' by H_1 and repeat the arguments above (note that the only conditions we used are that non-trivial elements in H' have the same fixed point set and that H/H' acts faithfully on Σ). But then Σ is composed of two orbits of H'', so $|H''| \le n/2$, whereby

$$|H| \le \frac{3}{2} g+3$$

by (1).

Finally, we use induction on g. Suppose that $g' \ge 2$ and $|H''| \le 3g'+3$. (1) gives

$$3g+3-(n-4)\frac{3(g-g')}{2g'-2+n} \ge |H|.$$

If $n \ge 4$, we are done. If $n=3$, by assumption, we must have $|H''| \le 3$. Therefore

$$|H| \le \frac{3(2g+1)}{2g'+1} \le \frac{3}{5}(2g+1) \le 3g+3\,.$$

□

Proposition 4.5. *If $f: S \to C$ is a relatively minimal fibration of genus* 2 *with $g(C) \ge 2$, then a cyclic automorphism group G of f satisfies*

$$|G| \le 5K_S^2+30$$

for $K_S^2 \ge 48$.

Proof. (1) Assume that $|H|=4g(C)+2$ and $|K|=10$. Let $g=g(C)$. By the theorem of Wiman (see the version given in [8, Lemma B]), C is a cyclic cover of $\boldsymbol{P}^1$ with ramification indices $r_1=2$, $r_2=2g+1$, $r_3=4g+2$ or $r_1=3$, $r_2=6$, $r_3=(4g+2)/3$. In fact, these r_i are the orders of $\mathrm{Stab}_H(p)$ for $p \in C$. Since Z_{10} is a maximal cyclic automorphism subgroup of a smooth curve of genus 2, by Lemma 4.2 we have $(|\mathrm{Stab}_H(p)|, |K|)=1$ if $f^{-1}(p)$ is a smooth fiber. But in Case 1, r_1 and r_3 are even, while in Case 2, r_2 and r_3 are even. So f has at least $(2g+10)/3$ singular fibers. By Lemma 2.4, we have $s_2(F) \ge 4$ for a singular fiber F. Hence

$$K_S^2-8(g-1)=K_{S/C}^2 \ge \frac{4}{5}\cdot\frac{2g+10}{3}=\frac{8(g+5)}{15},$$

$$|G|=10|H|=40g+20 \le \frac{75}{16}K_S^2+45 \le 5K_S^2+30$$

when $K_S^2 \ge 48$.

If $|K|\leq 8$ and $|K|$ is even, then by Lemma 4.3 there exist points $p\in C$ with $(|\operatorname{Stab}_H(p)|, 2)\neq 1$. Hence $K_S^2-8(g-1)=K_{S/C}^2\geq 1$ and

$$|G|\leq 8|H|=32g+16\leq 4K_S^2+44\leq 5K_S^2+30$$

when $K_S^2\geq 14$.

If $|K|$ is odd, then $|K|\leq 5$. The inequality is immediate.

(2) Assume that $|H|$ is odd. By Lemma 4.3, we have $|H|\leq 3g+3$. So

$$|G|\leq 10|H|\leq 30g+30\leq\frac{15}{4}K_S^2+60\leq 5K_S^2+30$$

when $K_S^2\geq 24$.

(3) Assume that $|H|$ is even and $|H|<4g+2$. If $|K|=10$, f must have more than one singular fibers by Lemma 2.4. So $K_S^2-8(g-1)=K_{S/C}^2\geq 2$. We get

$$|G|=10|H|\leq 40g\leq 5K_S^2+30\,.$$

If $|K|\leq 8$, it is not difficult to obtain this inequality. □

It seems that this bound is not the best possible. In Section 5 we will give an example to show that there are infinitely many fibrations which has an automorphism with order $3.75K_S^2+60$.

5. Examples.

EXAMPLE 5.1. Fibration with $|G|=50K_S^2$.

Let C be a Hurwitz curve, i.e., $|\operatorname{Aut}(C)|=84(g(C)-1)$, and let F be a curve of genus 2 with $|\operatorname{Aut}(F)|=48$. Let $S=C\times F$ with $f=\mathrm{pr}_1: S\to C$. Then $K_S^2=8(g(C)-1)$, $\operatorname{Aut}(f)\cong\operatorname{Aut}(C)\times\operatorname{Aut}(F)$,

$$|\operatorname{Aut}(f)|=|\operatorname{Aut}(C)|\cdot|\operatorname{Aut}(F)|=504K_S^2\,.$$

EXAMPLE 5.2. Fibrations with $|G|=126K_S^2$ which is not locally trivial.

Let $F=\boldsymbol{P}^1$. Let $p_1=0$, $p_2=\infty$, $p_3=1$, $p_4=\sqrt{-1}$, $p_5=-1$, $p_6=-\sqrt{-1}$ be six points on F. Let C be a Hurwitz curve. Then C has an H-orbit $\{q_1,\dots,q_m\}$ which contains $m=12(g(C)-1)$ points. Let $P=C\times F$. Taking $R=\mathrm{pr}_1^*(q_1+\cdots+q_m)+\mathrm{pr}_2^*(p_1+\cdots+p_6)$ as the branch locus, we construct a double cover of P. After desingularization, we get a smooth surface S with a genus 2 fibration $f: S\to C$. By computation, we obtain $K_S^2=32(g(C)-1)$, and $|G|=48\times 84(g(C)-1)=126K_S^2$.

EXAMPLE 5.3. Fibrations with $|G|=144K_S^2$ and $g(C)=1$.

Let F and $p_1,\dots,p_6$ be as in Example 5.2. Let C be an elliptic curve with the j-invariant $j(C)=0$. Fix a $q_1\in C$. Then the order of the group of automorphisms $\operatorname{Aut}(C, q_1)$ of C leaving q_1 fixed is equal to 6. Let $H_1\cong Z_m\oplus Z_m$ be a subgroup of translations of $\mathrm{Aut}(C)$. Take an extension subgroup $H_1 \subset H \subset \mathrm{Aut}(C)$ such that $H/H_1 \cong \mathrm{Aut}(C, q_1)$. Then $|H| = 6m^2$. Let $q_1, \ldots, q_{m^2}$ be the orbit of q_1 under H. Let $P = C \times F$. Using $R = \mathrm{pr}_1^*(q_1 + \cdots + q_{m^2}) + \mathrm{pr}_2^*(p_1 + \cdots + p_6)$ as the branch locus, we construct a double cover of P. After desingularization, we get a smooth surface S with a genus 2 fibration $f: S \to C$. By computation, we get $K_S^2 = 2m^2$. On the other hand, $|K| = 48$ gives $|G| = 288m^2 = 144K_S^2$.

EXAMPLE 5.4. Rational fibration with $|G| = 120(K_S^2 + 8)$.

Let F and $p_1, \ldots, p_6$ be as in Example 5.2. Let $C = \boldsymbol{P}^1$, $q_1, \ldots, q_{12}$ be the twelve vertices of an icosahedron. Let $P = C \times F$. Taking $R = \mathrm{pr}_1^*(q_1 + \cdots + q_{12}) + \mathrm{pr}_2^*(p_1 + \cdots + p_6)$ as the branch locus, we can construct a double cover of P. After desingularization, we obtain a genus 2 fibration $f: S \to C$ with $K_S^2 = 16$, $|H| = 60$, $|K| = 48$, $|G| = 2880 = 120(K_S^2 + 8)$.

EXAMPLE 5.5. Rational fibrations with $|G| = 48(K_S^2 + 8)$.

Let F and $p_1, \ldots, p_6$ be as in Example 5.2. Let $C = \boldsymbol{P}^1$ and let $q_1, \ldots, q_m$ be the m-th roots of unity. Then using the same construction as in Example 5.2, we obtain a genus 2 fibration with $K_S^2 = 2(m-4)$, $|K| = 48$, $|H| = 2m$, $|G| = 96m = 48(K_S^2 + 8)$.

EXAMPLE 5.6. Exceptional rational fibrations listed in the proof of Proposition 3.4.

Using the same construction as in Example 5.2, take $q_1, \ldots, q_{20}$ as the twenty vertices of a dodecahedron. We get a fibration with $K_S^2 = 32$ and $|G| = 2880 = 90K_S^2$. If we take $q_1, \ldots, q_6$ as the six vertices of an octahedron, we get a fibration with $K_S^2 = 4$ and $|G| = 1152 = 288K_S^2$. If we take $q_1, \ldots, q_8$ as the eight vertices of a cube, we get a fibration with $K_S^2 = 8$ and $|G| = 1152 = 144K_S^2$.

EXAMPLE 5.7. Fibrations the order of whose abelian automorphism group is $12.5(K_S^2 + 8)$.

Let $x_0, \ldots, x_{2m}, x_{2m+1}$ be the homogeneous coordinates in $\boldsymbol{P}^{2m+1}$, and let $\boldsymbol{P}^{2m}$ be the hyperplane defined by $x_{2m+1} = 0$. Let $\varphi: t \mapsto (1, t, \ldots, t^{2m}, 0)$ be a $2m$-ple embedding of $\boldsymbol{P}^1$ in $\boldsymbol{P}^{2m}$ and denote its image by Y. Then Y is a rational normal curve of degree $2m$. Let X be the cone over Y in $\boldsymbol{P}^{2m+1}$ with vertex $P_0 = (0, 0, \ldots, 0, 1)$. Denote $\eta = \exp(2\pi i/10m)$. Then the automorphism $\sigma: (x_0, \ldots, x_{2m+1}) \mapsto (x_0, x_1\eta, \ldots, x_{2m}\eta^{2m}, x_{2m+1})$ of $\boldsymbol{P}^{2m+1}$ is of order $10m$. The automorphism $\tau: (x_0, \ldots, x_{2m+1}) \mapsto (x_0, \ldots, x_{2m}, x_{2m+1}\eta^{2m})$ of $\boldsymbol{P}^{2m+1}$ is of order 5. The cone X is stabilized by these automorphisms σ and τ. Take a hypersurface H defined by $x_0^5 + x_{2m}^5 + x_{2m+1}^5$ which is also stabilized by σ and τ. Moreover, $P_0 \notin H$. How blowing up the cone X at the vertex P_0, we get the Hirzebruch surface $P = F_{2m}$ which has an automorphism $\tilde{\sigma}$ of order $10m$ induced by σ and an automorphism $\tilde{\tau}$ of order 5 induced by τ. The pull-back of the intersection $H \cap X$ is a smooth divisor R_1 on P which is linearly equivalent to $5C_0 + 10mF$. Taking $R = R_1 + C_0 \equiv 6C_0 + 10mF$, which is a smooth even divisor and stabilized under $\tilde{\sigma}$ and

$\tilde{\tau}$, as the branch locus, we can construct a double cover S of P which has a natural genus 2 fibration $f: S \to \boldsymbol{P}^1$. Since $K_P = -2C_0 - (2m+2)F$, we have $K_S^2 = 2(K_P + R/2)^2 = 8(m-1)$. The pull-back of $\tilde{\sigma}$ to S can generate a cyclic automorphism subgroup H of order $10m$. The pull-back of $\tilde{\tau}$ to S together with the hyperelliptic involution of the fibration f generates a cyclic automorphism subgroup $K \cong Z_{10}$. Since H and K commute, $G = KH \cong Z_{10} \oplus Z_{10m}$ is an abelian automorphism group of f with order $|G| = 100m = 12.5(K_S^2 + 8)$.

EXAMPLE 5.8. Rational fibrations which has an automorphism of order $12.5K_S^2 + 90$.

Let $x_0, \ldots, x_{2m}, x_{2m+1}$ be the homogeneous coordinates in $\boldsymbol{P}^{2m+1}$, and let $\boldsymbol{P}^{2m}$ be the hyperplane defined by $x_{2m+1} = 0$. Let $\varphi: t \mapsto (1, t, \ldots, t^{2m}, 0)$ be a $2m$-ple embedding of $\boldsymbol{P}^1$ in $\boldsymbol{P}^{2m}$ and denote its image by Y. Then Y is a rational normal curve of degree $2m$. Let X be the cone over Y in $\boldsymbol{P}^{2m+1}$ with vertex $P_0 = (0, 0, \ldots, 0, 1)$. Denote $\eta = \exp(2\pi i/(50m-5))$. Then the automorphism $\sigma: (x_0, \ldots, x_{2m+1}) \mapsto (x_0, x_1\eta^5, \ldots, x_{2m}\eta^{10m}, x_{2m+1}\eta)$ of $\boldsymbol{P}^{2m+1}$ is of order $50m-5$. The cone X is stabilized by this automorphism σ. Take a hypersurface H defined by $x_0^4 x_1 + x_{2m}^5 + x_{2m+1}^5$ which is also stabilized by σ and $P_0 \notin H$. Now blowing up the cone X at the vertex P_0, we get the Hirzebruch surface $P = F_{2m}$ which has an automorphism $\tilde{\sigma}$ of order $50m-5$ induced by σ. The pull-back of the intersection $H \cap X$ is a smooth divisor R_1 on P which is linearly equivalent to $5C_0 + 10mF$. Taking $R = R_1 + C_0 \equiv 6C_0 + 10mF$, which is a smooth even divisor and stabilized under $\tilde{\sigma}$, as the branch locus, we can construct a double cover S of P which has a natural genus 2 fibration $f: S \to \boldsymbol{P}^1$. Since $K_P \equiv -2C_0 - (2m+2)F$, we have $K_S^2 = 2(K_P + R/2)^2 = 8(m-1)$. The pull-back of $\tilde{\sigma}$ to S can generate a cyclic automorphism group G_1 of order $50m-5$. Since $|G_1|$ is odd, G_1 and the hyperelliptic involution of the fibration f generate a cyclic automorphism group G of S. Therefore $|G| = 100m - 10 = 12.5K_S^2 + 90$.

EXAMPLE 5.9. Fibrations which has an automorphism of order $5K_S^2$.

Let $F = \boldsymbol{P}^1$. Let $p_1 = 0$, $p_k = \exp(2k\pi i/5)$, $k = 1, \ldots, 5$, be six points in F. Let C be an elliptic curve, $\{q_1, \ldots, q_m\}$ an orbit of a cyclic translation group $H \subseteq \mathrm{Aut}(C)$ of order m, where m is an odd prime different from 5. Then using the same construction as in Example 5.2, we obtain a genus 2 fibration with $K_S^2 = 2m$, $K \cong Z_{10}$. Let $G = K \times H \cong Z_{10m}$. Then $|G| = 10m = 5K_S^2$.

EXAMPLE 5.10. Fibrations which has an automorphism of order $3.75K_S^2 + 60$.

Let $p_0 = 0$, $p_k = \exp(2k\pi i/3)$, $k = 1, 2, 3$ be four points in $C' = \boldsymbol{P}^1$. For any odd prime $m \neq 3, 5$, taking $D = p_0 + p_1 + (m-1)p_2 + (m-1)p_3$ as a branch locus, we can construct a cyclic cover $\sigma: C \to C'$ of degree m. Then $g(C) = m-1$. $H'' = \{x \mapsto x\exp(2k\pi i/3) \mid k = 1, 2, 3\} \cong Z_3$ is a cyclic automorphism group of C' which stabilizes the set $\{p_0, p_1, p_2, p_3\}$. On the other hand, the Galois group H' of the cyclic cover σ is isomorphic to Z_m. We obtain an extension

$$1 \to H' \to H \to H'' \to 1$$

such that $Z_{3m} \cong H \subseteq \mathrm{Aut}(C)$.

Let $q_0 = 0$, $q_k = \exp(2k\pi i/5)$, $k = 1, \dots, 5$, be six points in $F \cong \boldsymbol{P}^1$. Let $P = C \times F$. Taking $R = \mathrm{pr}_2^*(q_0 + q_1 + \cdots + q_5)$ as branch locus, we can construct a double cover $\theta: S \to P$ which is also a genus 2 fibration $f = p_1 \circ \theta: S \to C$. F has a cyclic automorphism group $K_1 = \{y \mapsto y \exp(2k\pi i/5) \mid k = 1, \dots, 5\} \cong Z_5$ which stabilizes the set $\{q_0, \dots, q_5\}$ and can be lift to P. It is not difficult to see that we can get $K \cong Z_{10}$ by adding the involution of the double cover. Then $G = K \times H \cong Z_{30m}$ is a cyclic automorphism group of f which satisfies

$$|G| = 30m = 30(g(C)+1) = \frac{15}{4} K_S^2 + 60 ,$$

because $K_S^2 = 8(g(C)-1)$.

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