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EXISTENCE OF ALMOST PERIODIC SOLUTIONS OF SOME FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY IN A BANACH SPACE

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Abstract. For functional differential equations with infinite delay in a Banach space, the existence of almost periodic solutions is studied under some stability assumptions.

1. Introduction. In this paper we are concerned with a system of functional differential equations with infinite delay

$$\text{(E)} \qquad \frac{du}{dt} = Au(t) + F(t, u_t)$$

on a phase space $\mathscr{B} = \mathscr{B}((-\infty, 0]; X)$ which possesses a fading memory property, where X is a Banach space and u_t is an element belonging to $\mathscr{B}((-\infty, 0]; X)$ defined by $u_t(s) = u(t+s)$ for $s \in (-\infty, 0]$. For functional differential equations on a uniform fading memory space $\mathscr{B}$ with $X = R^n$, Hino [5] obtained a result on the existence of almost periodic solutions by assuming the existence of a bounded solution which is $\mathscr{B}$-totally stable or $\mathscr{B}$-uniformly asymptotically stable. The $\mathscr{B}$-stability means that the solution remains small if the initial function is small with respect to the semi-norm $|\cdot|_{\mathscr{B}}$. However, as pointed out in [2], some integrodifferential equations can be set up as functional differential equations on a fading memory space (not uniform) and BC-stability is more practical, where BC-stability means that the solution remains small if the initial function is small with respect to the BC-norm, that is, $\sup_{-\infty < \theta \le 0} |\phi(\theta)|$. For these reasons, Murakami and Yoshizawa [10] have discussed the existence of an almost periodic solution for functional differential equations on a fading memory space with $X = R^n$ in the context of BC-stability.

Recently, Hino and Murakami [7] have established a result on the existence of an almost periodic solution when (E) is the nonhomogeneous linear system. The purpose of this paper is to treat a nonlinear equation (E) on a fading memory space $\mathscr{B}((-\infty, 0]; X)$ with a general Banach space X and to establish a result on the existence of almost periodic solutions by assuming the existence of a bounded solution which is

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BC-totally stable or BC-uniformly asymptotically stable. Hence our main results (Theorems 1 and 2) of this paper would be considered as some extensions of Hino [5, Theorem 4] and Murakami and Yoshizawa [10, Corollary 1] to the case where X is a general Banach space, and of Hino and Murakami [7, Theorem 1] to a nonlinear equation (E).

2. Fading memory spaces. Let X be a Banach space with norm $|\cdot|_X$. For any interval $J\subset R:=(-\infty,\infty)$, we denote by $BC(J;X)$ the space of all bounded and continuous functions mapping J into X. Clearly $BC(J;X)$ is a Banach space with the norm $|\cdot|_{BC(J;X)}$ defined by $|\phi|_{BC(J;X)}=\sup\{|\phi(t)|_X : t\in J\}$. If $J=R^-:=(-\infty,0]$, then we simply write $BC(J;X)$ and $|\cdot|_{BC(J;X)}$ as BC and $|\cdot|_{BC}$, respectively. For any function $u:(-\infty,a)\mapsto X$ and $t<a$, we define a function $u_t: R^-\mapsto X$ by $u_t(s)=u(t+s)$ for $s\in R^-$. Let $\mathscr{B}=\mathscr{B}(R^-;X)$ be a real linear space of functions mapping R^- into X with a complete seminorm $|\cdot|_{\mathscr{B}}$. The space $\mathscr{B}$ is assumed to have the following properties:

(A1) There exist a positive constant N and locally bounded functions $K(\cdot)$ and $M(\cdot)$ on $R^+:=[0,\infty)$ with the property that if $u:(-\infty,a)\mapsto X$ is continuous on $[\sigma,a)$ with $u_\sigma\in\mathscr{B}$ for some $\sigma<a$, then for all $t\in[\sigma,a)$,

(i) $u_t\in\mathscr{B}$,

(ii) u_t is continuous in t (w.r.t. $|\cdot|_{\mathscr{B}}$),

(iii) $N|u(t)|_X\le|u_t|_{\mathscr{B}}\le K(t-\sigma)\sup_{\sigma\le s\le t}|u(s)|_X+M(t-\sigma)|u_\sigma|_{\mathscr{B}}$.

(A2) If $\{\phi^k\}$ is a sequence in $\mathscr{B}\cap BC$ converging to a function ϕ uniformly on any compact interval in R^- and $\sup_k|\phi^k|_{BC}<\infty$, then $\phi\in\mathscr{B}$ and $|\phi^k-\phi|_{\mathscr{B}}\to 0$ as $k\to\infty$.

It is known (cf. [8, Proposition 7.1.1]) that the space $\mathscr{B}$ contains BC and that there is a constant $l>0$ such that

$$|\phi|_{\mathscr{B}}\le l|\phi|_{BC}, \qquad \phi\in BC. \tag{1}$$

Set $\mathscr{B}_0=\{\phi\in\mathscr{B}:\phi(0)=0\}$, and define an operator $S_0(t):\mathscr{B}_0\mapsto\mathscr{B}_0$ by

$$[S_0(t)\phi](s)=\begin{cases}\phi(t+s) & \text{if } t+s\le 0,\\ 0 & \text{if } t+s>0\end{cases}$$

for each $t\ge 0$. By virtue of (A1), one can see that the family $\{S_0(t)\}_{t\ge 0}$ is a strongly continuous semigroup of bounded linear operators on $\mathscr{B}_0$. The space $\mathscr{B}$ is called *a fading memory space*, if it satisfies the property

$$\text{(A3)} \qquad \lim_{t\to\infty}|S_0(t)\phi|_{\mathscr{B}}=0, \qquad \phi\in\mathscr{B}_0,$$

in addition to (A1) and (A2). It is known (cf. [8, Proposition 7.1.5]) that the functions $K(\cdot)$ and $M(\cdot)$ in (A1) can be chosen as $K(t)\equiv l$ and $M(t)\equiv(1+(l/N))\|S_0(t)\|$; here and hereafter, $\|\cdot\|$ denotes the operator norm of bounded linear operators. Note that (A3) implies $\sup_{t\ge 0}\|S_0(t)\|<\infty$ by the Banach-Steinhaus theorem. Therefore, whenever $\mathscr{B}$ is a fading memory space, we may assume that the functions $K(\cdot)$ and $M(\cdot)$ in

(A1) are constants $K(\cdot)\equiv K$ and $M(\cdot)\equiv M$.

We provide a typical example of fading memory spaces. Let $g: R^- \mapsto [1, \infty)$ be any continuous nonincreasing function such that $g(0)=1$ and $g(s)\to\infty$ as $s\to-\infty$. We set

$$C_g^0 := C_g^0(X) = \left\{\phi : R^- \mapsto X \text{ is continuous with } \lim_{s\to-\infty} |\phi(s)|_X / g(s) = 0\right\}.$$

Then the space C_g^0 equipped with the norm

$$|\phi|_g = \sup_{s\le 0} \frac{|\phi(s)|_X}{g(s)}, \qquad \phi\in C_g^0,$$

is a separable Banach space and satisfies (A1)–(A3).

Let $C(R^+; X)$ be the set of continuous functions defined on R^+ with values in X. A subset $\mathscr{F}$ of $C(R^+; X)$ is said to be uniformly equicontinuous on R^+, if $\sup\{|x(t+\delta)-x(t)|_X : t\in R^+, x\in\mathscr{F}\}\to 0$ as $\delta\to 0^+$. For any set $\mathscr{F}$ in $C(R^+; X)$ and any set S in $\mathscr{B}$, we set

$$R(\mathscr{F}) = \{x(t) \mid t\in R^+, x\in\mathscr{F}\}$$

$$W(S,\mathscr{F}) = \{x(\cdot): R\mapsto X \mid x_0\in S, x|_{R^+}\in\mathscr{F}\}$$

and

$$V(S,\mathscr{F}) = \{x_t \mid t\in R^+, x\in W(S,\mathscr{F})\}.$$

LEMMA 1. *Let $\mathscr{B}$ be a fading memory space. If S is a compact subset in $\mathscr{B}$ and if $\mathscr{F}$ is a uniformly equicontinuous set in $C(R^+, X)$ such that the set $R(\mathscr{F})$ is relatively compact in X, then the set $V(S,\mathscr{F})$ is relatively compact in $\mathscr{B}$.*

PROOF. We shall prove that any sequence $\{x_{t_k}^k\}$, $t_k\ge 0$, $x_{t_k}^k\in V(S,\mathscr{F})$, contains a convergent subsequence. Taking a subsequence if necessary, we may assume that $t_k\to t_0\le\infty$ and $x_0^k := \phi^k\to\phi$ in S as $k\to\infty$, because S is compact. Let

$$x_{t_k}^k = y_{t_k}^k + S_0(t_k)\psi^k,$$

where

$$y^k(s) = \begin{cases} x^k(s), & s\ge 0 \\ x^k(0), & s\le 0, \end{cases}$$

$$\psi^k = x_0^k - x^k(0)\chi$$

and

$$\chi(\theta) = 1, \qquad \theta\le 0.$$

Then $\psi^k\to\psi := \phi-\phi(0)\chi$ in $\mathscr{B}$ as $k\to\infty$. Clearly $\xi^k := y_{t_k}^k$ lies in BC, and the sequence $\{\xi^k\}$ is equicontinuous on $(-\infty, 0]$. Moreover, for each $\theta\le 0$ the set $\{\xi^k(\theta): k=1,2,\dots\}$

is relatively compact in X, because it is contained in the set $R(\mathscr{F})$ which is relatively compact in X. By applying the Ascoli-Arzéla theorem and (A2), we may assume that $\{\xi^k\}$ is a convergent sequence in $\mathscr{B}$. On the other hand, since $\sup_{t\geq 0}\|S_0(t)\|<\infty$, we have $|S_0(t_k)\psi^k-S_0(t_k)\psi|_{\mathscr{B}}\leq\sup_{t\geq 0}\|S_0(t)\||\psi^k-\psi|_{\mathscr{B}}\to 0$ as $k\to\infty$. If $t_0<\infty$, then $S_0(t_k)\psi\to S_0(t_0)\psi$ as $k\to\infty$, while if $t_0=\infty$, then $S_0(t_k)\psi\to 0$ as $k\to\infty$ by (A3). As a result, $\{S_0(t_k)\psi^k\}$ is a convergent sequence in $\mathscr{B}$. Therefore, the sequence $\{x^k_{t_k}\}$ has the desired property.

3. Asymptotically almost periodic functions and definitions of stabilities. Throughout the remainder of the paper, we assume that $\mathscr{B}$ is a fading memory space which is separable.

Now we shall consider the following functional differential equation

$$\frac{du}{dt}=Au(t)+F(t,u_t)\,, \tag{2}$$

where A is the infinitesimal generator of a compact semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators on X and $F(t,\phi)\in C(R\times\mathscr{B};X)$; here and hereafter, we denote by $C(R\times\mathscr{B};X)$ the set of continuous functions defined on $R\times\mathscr{B}$ with values in X.

We always impose the following conditions on (2):

(H1) $F(t,\phi)$ is almost periodic in t uniformly for $\phi\in\mathscr{B}$, where $F(t,\phi)$ is said to be almost periodic in t uniformly for $\phi\in\mathscr{B}$, if for any $\varepsilon>0$ and any compact set W in $\mathscr{B}$, there exists a positive number $l(\varepsilon,W)$ such that any interval of length $l(\varepsilon,W)$ contains a τ for which

$$|F(t+\tau,\phi)-F(t,\phi)|_X\leq\varepsilon$$

for all $t\in R$ and all $\phi\in W$;

(H2) For any $H>0$, there is an $L(H)>0$ such that $|F(t,\phi)|_X\leq L(H)$ for all $t\in R^+$ and $\phi\in\mathscr{B}$ such that $|\phi|_{\mathscr{B}}\leq H$;

(H3) Equation (2) has a (bounded) solution $u(t)$ defined on R^+ such that $u_0\in \mathrm{BC}$ and $|u_t|_{\mathscr{B}}\leq C_1$ for all $t\in R^+$.

By virtue of (H1) and (H2), it follows that for any $(\sigma,\phi)\in R\times\mathscr{B}$, there exists a function $v\in C((-\infty,t_1);X)$ such that $v_\sigma=\phi$ and the following relation holds:

$$v(t)=T(t-\sigma)\phi(0)+\int_\sigma^t T(t-s)F(s,v_s)ds\,,\qquad \sigma\leq t<t_1\,,$$

(cf. [3, Theorem 1]). The function v is called the (mild) solution of (2) defined on $[\sigma,t_1)$ through (σ,ϕ) and denoted by $x(\cdot,\sigma,\phi,F)$. In the above, t_1 can be taken as $t_1=\infty$ if $\sup_{t<t_1}|v(t)|_X<\infty$ (cf. [3, Corollary 2]).

For the solution $u(t)$ of (2) whose existence is assumed in (H3), we have the following lemma.

LEMMA 2. *$O_{u,R^+} := \overline{\{u(t) \mid 0 \le t\}}$ is compact in X, $u(t)$ is uniformly continuous on R^+ and $X_{u,R^+} := \overline{\{u_t \mid 0 \le t\}}$ is compact in $\mathscr{B}$.*

PROOF. First we prove that the set O_{u,R^+} is compact in X. To do so, we consider the sets $O_\eta = \{u(t) \mid t \ge \eta\}$ and $\tilde{O}_\eta = \{u(t) \mid 0 \le t \le \eta\}$ for each $\eta > 0$. Then $\alpha(O_{u,R^+}) = \max\{\alpha(O_\eta), \alpha(\tilde{O}_\eta)\}$, where $\alpha(\cdot)$ is Kuratowski's measure of noncompactness of sets in X. For the details of the properties of $\alpha(\cdot)$, see [9, Section 1.4]. Let $0 < \nu < \min\{1, \eta\}$. If $t \ge \eta$, then

$$\begin{aligned} u(t) &= T(t)u(0) + \int_0^t T(t-s)F(s, u_s)ds \\ &= T(\nu)\left[T(t-\nu)u(0) + \int_0^{t-\nu} T(t-\nu-s)F(s, u_s)ds\right] + \int_{t-\nu}^t T(t-s)F(s, u_s)ds \\ &= T(\nu)u(t-\nu) + \int_{t-\nu}^t T(t-s)F(s, u_s)ds\,. \end{aligned}$$

Note that $\sup_{t \ge 0} |F(t, u_t)|_X =: L(C_1) < \infty$. Since the set $T(\nu)\{u(t-\nu) \mid t \ge \eta\}$ is relatively compact in X because of the compactness of the semigroup $\{T(t)\}_{t \ge 0}$, it follows that

$$\alpha(O_\eta) \le C_2 L(C_1)\nu\,,$$

where $C_2 = \sup_{0 \le \tau \le 1} \|T(\tau)\|$. Letting $\nu \to 0$ in the above, we get $\alpha(O_\eta) = 0$ for all $\eta > 0$. Moreover, since the set $\tilde{O}_\eta$ is compact in X, we have $\alpha(\tilde{O}_\eta) = 0$. Consequently, $\alpha(O_{u,R^+}) = 0$, which shows that the set O_{u,R^+} is compact in X.

To establish the uniform continuity of u, let $0 \le s \le t \le s+1$. Then

$$\begin{aligned} |u(t) - u(s)|_X &\le |T(t-s)u(s) - u(s)|_X + \left|\int_s^t T(t-\tau)F(\tau, u_\tau)d\tau\right|_X \\ &\le \sup\{|T(t-s)z - z|_X : z \in O_{u,R^+}\} + C_2 L(C_1)|t-s|\,. \end{aligned}$$

Since the set O_{u,R^+} is compact in X, $T(\tau)z$ is uniformly continuous in $\tau \in [0, 1]$ uniformly for $z \in O_{u,R^+}$. This leads to $\sup\{|u(t) - u(s)|_X : 0 \le s \le t \le s+1\} \to 0$ as $|t-s| \to 0$, which proves the uniform continuity of u on R^+.

The compactness of X_{u,R^+} follows immediately from the above facts and Lemma 1.

A sequence $\{F^k\}$ in $C(R \times \mathscr{B}; X)$ is said to converge to G Bohr-uniformly on $R \times \mathscr{B}$ if F^k converges to G uniformly on $R \times W$ for any compact set W in $\mathscr{B}$ as $k \to \infty$. It is known (cf., e.g. [13, Theorems 2.2 and 2.3]) that $F(t, \phi)$ is almost periodic in t uniformly for $\phi \in \mathscr{B}$ if and only if for any sequence $\{t_k\}$ in R, the sequence $\{F(t+t_k, \phi)\}$ contains a Bohr-uniformly convergent subsequence.

We denote by $H(F)$ the set of all functions $G(t, \phi)$ such that $\{F(t+t_k, \phi)\}$ converges to $G(t, \phi)$ Bohr-uniformly for some sequence $\{t_k\}$. In particular, $\Omega(F)$ is the subset of $H(F)$ for $\{t_k\}$ which tends to ∞ as $k \to \infty$. Clearly $G(t, \phi)$ is almost periodic in t uniformly for $\phi \in \mathscr{B}$ if $G \in H(F)$. We shall denote by $\Omega(u, F)$ the set of all $(v, g) \in H(u, F)$ for which there exists a sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$, such that $F(t+t_k, \phi) \to G(t, \phi) \in H(F)$ Bohr-uniformly and $u(t+t_k) \to v(t)$ uniformly on any compact set in R. From the following lemma we see that $\Omega(u, F)$ is nonempty and that $v(t)$ is a solution of

$$\frac{dv}{dt} = Av(t) + G(t, v_t)\,, \tag{3}$$

whenever $(v, G) \in \Omega(u, F)$.

LEMMA 3. *For any sequence $\{t'_n\}$, $t'_n \to \infty$ as $n \to \infty$, there exists a subsequence $\{t_n\}$ of $\{t'_n\}$ and functions $v(t)$ and $G(t, \phi)$ such that*

$$F(t+t_n, \phi) \to G(t, \phi)$$

Bohr-uniformly on $R \times \mathscr{B}$ as $n \to \infty$, and that

$$u(t+t_n) \to v(t)$$

uniformly on any compact interval in R as $n \to \infty$. In this case, (v, G) is in $\Omega(u, F)$ and $v(t)$ is a bounded solution of (3) *defined on R.*

PROOF. By virtue of (H1) and (H2), it follows that there exists a subsequence $\{t_n\}$ of $\{t'_n\}$ and a continuous function $G(t, \phi)$ such that $F(t+t_n, \phi) \to G(t, \phi)$ as $n \to \infty$, uniformly on $R \times S$ for any compact set $S \subset \mathscr{B}$. Set $u^n(t) = u(t+t_n)$ for $t \in R$. Applying the Ascoli-Arzéla theorem and the diagonalization procedure one can choose a subsequence of $\{u^n(t)\}$ which is uniformly convergent on any compact interval in R, by Lemma 2. Without loss of generality, we may assume that $u^n(t) \to v(t)$ uniformly on any compact set in R as $n \to \infty$, for some function $v: R \mapsto X$. This completes the proof of the former part of the lemma.

By using (A2), we can see that $u_{t+t_n} \to v_t$ in $\mathscr{B}$ uniformly on any compact set in R as $n \to \infty$. Hence $F(t+t_n, u_{t+t_n}) \to G(t, v_t)$ uniformly on any compact set in R as $n \to \infty$. Letting $n \to \infty$ in the relation

$$u(t+t_n) = T(t)u(t_n) + \int_0^t T(t-s)F(s+t_n, u_{s+t_n})ds\,, \qquad t+t_n \geq 0\,,$$

we have

$$v(t) = T(t)v(0) + \int_0^t T(t-s)G(s, v_s)ds\,, \qquad t \in R\,,$$

which shows that $v(t)$ is a solution of (3) defined on R.

Let $g: [a, \infty) \mapsto X$ be a continuous function. $g(t)$ is said to be asymptotically almost periodic if it is a sum of a continuous almost periodic function $p(t)$ and a continuous function $q(t)$ defined on $a \leq t < \infty$ which tends to 0 as $t \to \infty$, that is,

$$g(t) = p(t) + q(t)\,.$$

It is known (cf., e.g. [13, pp. 20–30]) that when $X=R^n$, $g(t)$ is asymptotically almost periodic if and only if it satisfies the following property:

(L) for any sequence $\{t'_n\}$ such that $t'_n\to\infty$ as $n\to\infty$ there exists a subsequence $\{t_n\}$ of $\{t'_n\}$ for which $g(t+t_n)$ converges uniformly on $a\le t<\infty$.

Indeed, using Bochner's criterion (cf., e.g. [1, Section 1.2]) for almost periodic functions, we can see that the argument employed in [13, pp. 20–30] works even when X is any separable Banach space, under the additional assumption that the set $\{g(t): t\ge a\}$ is relatively compact in X. It is easy to see that the additional assumption is satisfied whenever $g(t)$ is asymptotically almost periodic or it satisfies Property (L). Therefore, the above equivalence holds true when X is a general separable Banach space, too.

PROPOSITION 1. *If $u(t)$ is an asymptotically almost periodic solution, then* (2) *has an almost periodic solution.*

PROOF. Let $u(t)=p(t)+q(t)$ be a decomposition, where $p(t)$ is almost periodic and $q(t)\to 0$ as $t\to\infty$. There exists a sequence $\{\tau_n\}$, $\tau_n\to\infty$ as $n\to\infty$, such that $F(t+\tau_n, \phi)$ converges to $F(t, \phi)$ Bohr-uniformly on $R\times\mathscr{B}$ as $n\to\infty$ and that $u(t+\tau_n)\to p(t)$ uniformly on R^+ and uniformly on any compact set in R as $n\to\infty$. By Lemma 3, $p(t)$ is a solution of (2). Hence $p(t)$ is an almost periodic solution of (2).

Now we shall give some definitions of stabilities.

DEFINITION 1. The bounded solution $u(t)$ of (2) is said to be BC-totally stable (BC-TS) if for any $\varepsilon>0$ there exists a $\delta(\varepsilon)>0$ with the property that $\sigma\in R^+$, $\phi\in \mathrm{BC}$ with $|u_\sigma-\phi|_{\mathrm{BC}}<\delta(\varepsilon)$ and $h\in \mathrm{BC}([\sigma,\infty); X)$ with $\sup_{t\in[\sigma,\infty)}|h(t)|_X<\delta(\varepsilon)$ imply $|u(t)-x(t,\sigma,\phi, F+h)|_X<\varepsilon$ for $t\ge\sigma$, where $x(\cdot,\sigma,\phi,F+h)$ denotes the solution of

$$\frac{dv}{dt}=Av(t)+F(t, v_t)+h(t)\,, \qquad t\ge\sigma\,,$$

through (σ, ϕ).

DEFINITION 2. The bounded solution $u(t)$ of (2) is said to be BC-uniformly asymptotically stable (BC-UAS) if for any $\varepsilon>0$ there exists a $\delta(\varepsilon)>0$ such that $\sigma\in R^+$ and $\phi\in\mathrm{BC}$ with $|u_\sigma-\phi|_{\mathrm{BC}}<\delta(\varepsilon)$ imply $|u(t)-x(t,\sigma,\phi,F)|_X<\varepsilon$ for $t\ge\sigma$; in addition, there exists a $\delta_0>0$ with the property that for any $\varepsilon>0$ there exists a $t_0(\varepsilon)>0$ such that $\sigma\in R^+$ and $\phi\in\mathrm{BC}$ with $|u_\sigma-\phi|_{\mathrm{BC}}<\delta_0$ imply $|u(t)-x(t,\sigma,\phi,F)|_X<\varepsilon$ for $t\ge\sigma+t_0(\varepsilon)$.

In the above, we can define *$\mathscr{B}$-total stability* ($\mathscr{B}$-TS) if we replace "$\phi\in\mathrm{BC}$ with $|u_\sigma-\phi|_{\mathrm{BC}}<\delta(\varepsilon)$" by "$\phi\in\mathscr{B}$ with $|u_\sigma-\phi|_{\mathscr{B}}<\delta(\varepsilon)$". Moreover, we can define *$\mathscr{B}$-uniform asymptotic stability* ($\mathscr{B}$-UAS) in a similar way. From the relation (1) it follows that BC-TS and BC-UAS respectively follow from $\mathscr{B}$-TS and $\mathscr{B}$-UAS.

4. Almost periodic solutions. In this section, we shall discuss the existence of an almost periodic solution of an almost periodic system (2).

THEOREM 1. *If the solution $u(t)$ of* (2) *is* BC-TS, *then it is asymptotically almost periodic in t. Consequently,* (2) *has an almost periodic solution.*

PROOF. For any sequence $\{\tau'_k\}$ such that $\tau'_k \to \infty$ as $k \to \infty$, there is a subsequence $\{\tau_k\}$ of $\{\tau'_k\}$ and a $(v, G) \in \Omega(u, F)$ such that $u(t+\tau_k)$ converges to $v(t)$ uniformly on any compact interval in R and $F(t+\tau_k, \phi)$ converges to $G(t, \phi)$ Bohr-uniformly on $R \times \mathscr{B}$. We shall show that $u(t+\tau_k)$ is convergent uniformly on R^+.

Suppose that $u(t+\tau_k)$ is not convergent uniformly on R^+. Then, for some $\varepsilon > 0$ there exist sequences $\{t_j\}$, $\{k_j\}$ and $\{m_j\}$ such that

$$k_j \to \infty\,, \quad m_j \to \infty \qquad \text{as} \quad j \to \infty\,,$$

$$|u(\tau_{k_j}+t_j) - u(\tau_{m_j}+t_j)|_X = \varepsilon \tag{4}$$

and

$$|u(\tau_{k_j}+t) - u(\tau_{m_j}+t)|_X < \varepsilon \qquad \text{on} \quad [0, t_j)\,. \tag{5}$$

Put $v^j(t) = u(\tau_{k_j}+t)$ and $w^j(t) = u(\tau_{m_j}+t)$. Since the sequences $\{v^j(t)\}$ and $\{w^j(t)\}$ converge to $v(t)$ uniformly on any compact interval in R, we may assume that

$$\rho(v^j_0, w^j_0) := \sum_{l=1}^{\infty} 2^{-l} |v^j_0 - w^j_0|_l / \{1 + |v^j_0 - w^j_0|_l\} < \frac{1}{j}\,, \qquad j = 1, 2, \dots, \tag{6}$$

where $|v^j_0 - w^j_0|_l = \sup_{-l \le \theta \le 0} |v^j(\theta) - w^j(\theta)|_X$. For each $j \in N$ and $r \in R^+$, we define a function $v^{j,r} : R \mapsto X$ by

$$v^{j,r}(t) = \begin{cases} v^j(t)\,, & -r \le t \\ v^j(-r) + w^j(t) - w^j(-r)\,, & t < -r\,, \end{cases}$$

where N denotes the set of all positive integers.

First, we shall show that

$$\sup\{|v^{j,r}_0 - v^j_0|_{\mathscr{B}} : j \in N\} \to 0 \qquad \text{as} \quad r \to \infty\,. \tag{7}$$

If this is not the case, then there exist an $\varepsilon_0 > 0$ and sequences $\{j_k\} \subset N$ and $\{r_k\}$, $r_k \to \infty$ as $k \to \infty$, such that $|v^{j_k, r_k}_0 - v^{j_k}_0|_{\mathscr{B}} \ge \varepsilon_0$ for $k = 1, 2, \dots$. Put $\psi^k = v^{j_k, r_k}_0 - v^{j_k}_0$. Clearly, $\{\psi^k\}$ is a sequence in BC which converges to the zero function uniformly on any compact set in R^- and $\sup_k |\psi^k|_{\mathrm{BC}} < \infty$. Then Axiom (A2) yields that $|\psi^k|_{\mathscr{B}} \to 0$ as $k \to \infty$, a contradiction.

Observe that $v^{j,r}(t) \to v(t)$ as $j \to \infty$, uniformly for $(t, r) \in J \times R^+$ for any compact interval J in R. Hence the set $\{v^j_0, v^{j,r}_0 : j \in N, r \in R^+\}$ is relatively compact in $\mathscr{B}$, because the set X_{u,R^+} is compact in $\mathscr{B}$ by Lemma 2 and $v^j_0 \in X_{u,R^+}$. Moreover, the set $\{v^j(t), v^{j,r}(t) : j \in N, r \in R^+, t \in R^+\}$ is contained in the compact set O_{u,R^+}. From these observations and Lemma 1 it follows that the set $W := \{v^j_t, v^{j,r}_t : j \in N, r \in R^+, t \in R^+\}$ is relatively compact in $\mathscr{B}$. Consequently,

(8) $$\sup\{|F(t+\tau_k, \phi)-G(t, \phi)|_X : t\in R, \phi\in W\}\to 0 \quad \text{as} \quad k\to\infty .$$

Define a continuous function $q^{j,r}$ on R^+ by

$$q^{j,r}(t)=\begin{cases} F(t+\tau_{k_j}, v_t^j)-F(t+\tau_{m_j}, (v^{j,r})_t) , & 0\le t\le t_j \\ q^{j,r}(t_j) , & t_j<t . \end{cases}$$

Since

$$|(v^{j,r})_t - v_t^j|_{\mathscr{B}} \le M|v_0^{j,r}-v_0^j|_{\mathscr{B}} \qquad (t\in R^+, j\in N)$$

by Axiom (A1), it follows from (7) that

(9) $$\sup\{|G(t, v_t^j)-G(t, (v^{j,r})_t)|_X : t\in R^+, j\in N\}\to 0 \quad \text{as} \quad r\to\infty .$$

Hence, by (8) and (9) we can choose $j_0=j_0(\varepsilon)\in N$ and $r=r(\varepsilon)\in N$ in such a way that

$$\sup\{|q^{j,r}(t)|_X : j\ge j_0, t\in R^+\}<\delta(\varepsilon/2)/2 ,$$

where $\delta(\cdot)$ is the one for BC-TS of the solution $u(t)$ of (2). Moreover, for this r, select an integer $j\ge j_0$ such that $j>2^r(1+\delta(\varepsilon/2))/\delta(\varepsilon/2)$. Then $2^{-r}|w_0^j-v_0^j|_r/[1+|w_0^j-v_0^j|_r]\le \rho(w_0^j, v_0^j)<2^{-r}\delta(\varepsilon/2)/[1+\delta(\varepsilon/2)]$ by (6), which implies that

$$|w_0^j-v_0^j|_r<\delta(\varepsilon/2) \quad \text{or} \quad |v_0^{j,r}-w_0^j|_{\mathrm{BC}}<\delta(\varepsilon/2) .$$

Since the function $v^{j,r}$ is a solution of

$$\frac{dx}{dt}=Ax(t)+F(t+\tau_{m_j}, x_t)+q^{j,r}(t)$$

for $t\in[0, t_j]$, and since $w^j(t)$ is a BC-TS solution of

$$\frac{dx}{dt}=Ax(t)+F(t+\tau_{m_j}, x_t)$$

with the same $\delta(\cdot)$ as the one for $u(t)$, from the fact that $\sup_{t\ge 0}|q^{j,r}(t)|<\delta(\varepsilon/2)$ it follows that $|(v^{j,r})(t)-w^j(t)|_X<\varepsilon/2$ on $[0, t_j]$. In particular, we have $|(v^{j,r})(t_j)-w^j(t_j)|_X<\varepsilon$ or $|v^j(t_j)-w^j(t_j)|_X<\varepsilon$, which contradicts (4).

The existence of an almost periodic solution follows immediately from Proposition 1.

We say that the solution $u(t)$ is unique for the initial value problem if $u_\sigma=\phi$ implies $u(t)=x(t, \sigma, \phi, F)$. Furthermore, we say that (2) is regular if for any $G\in\Omega(F)$, each solution of (3) is unique for the initial value problem.

THEOREM 2. *Assume that system* (2) *is regular. If the solution* $u(t)$ *is* BC-UAS, *then it is asymptotically almost periodic in* t. *Consequently*, (2) *has an almost periodic solution.*

PROOF. We shall show that $u(t)$ is BC-TS. Then the conclusion follows immediately from Theorem 1.

Let $(\delta(\cdot), \delta_0, t_0(\cdot))$ be the triple for BC-UAS of $u(t)$, where we may assume $\delta_0 < \delta(1)$. We first establish that

$$\begin{aligned} &\sigma \in R^+, \quad (v, G) \in \Omega(u, F) \quad \text{and} \quad |\phi - v_\sigma|_{BC} < \delta(\eta/2) \\ &\text{imply } |x(t, \sigma, \phi, G) - v(t)|_X < \eta \quad \text{for } t \geq \sigma. \end{aligned} \tag{10}$$

Select a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $u(t+t_n) \to v(t)$ uniformly on any compact set in R and $F(t+t_n, \phi) \to G(t, \phi)$ Bohr-uniformly on $R \times \mathscr{B}$, and consider any solution $x(\cdot, \sigma+t_n, \phi - v_\sigma + u_{\sigma+t_n}, F)$. For any $n \in N$, set $x^n(t) = x(t+t_n, \sigma+t_n, \phi - v_\sigma + u_{\sigma+t_n}, F)$, $t \in R$. Since the solution $u(t)$ of (2) is BC-UAS, from the fact that $|x^n_\sigma - u_{\sigma+t_n}|_{BC} = |\phi - v_\sigma|_{BC} < \delta(\eta/2)$ it follows that

$$|x^n(t) - u(t+t_n)|_X < \eta/2 \qquad \text{for all} \quad t \geq \sigma \text{ and } n \in N. \tag{11}$$

Observe that the set $\{x^n(\sigma) : n \in N\}$ is relatively compact in X. By virtue of this fact and (11), repeating almost the same argument as in the proof of Lemma 2 we can see that the set $\{x^n(t) : t \geq \sigma, n \in N\}$ is relatively compact in X and that the sequence $\{x^n\}$ is uniformly equicontinuous on $[\sigma, \infty)$. Thus we may assume that $x^n(t) \to y(t)$ as $n \to \infty$, uniformly on any compact set in $[\sigma, \infty)$ for some function $y(t) : [\sigma, \infty) \mapsto X$. Since $x^n(\sigma) = \phi(0) - v(\sigma) + u(\sigma + t_n)$, we obtain $y(\sigma) = \phi(0)$. Hence, if we extend the function y by setting $y_\sigma = \phi$, then $y \in C(R, X)$ and $|x^n_t - y_t|_{\mathscr{B}} \to 0$ uniformly on any compact set in $[\sigma, \infty)$ as $n \to \infty$. Letting $n \to \infty$ in the relation

$$x^n(t) = T(t-\sigma)\{\phi(0) - v(\sigma) + u(\sigma + t_n)\} + \int_\sigma^t T(t-s)F(s+t_n, x^n_s)ds, \qquad t \geq \sigma,$$

we obtain

$$y(t) = T(t-\sigma)\phi(0) + \int_\sigma^t T(t-s)G(s, y_s)ds, \qquad t \geq \sigma,$$

which means that $y(t) \equiv x(t, \sigma, \phi, G)$ for $t \geq \sigma$ by the regularity assumption. Then (10) follows from (11) by letting $n \to \infty$.

Repeating the same argument as in the proof of (11), we see that $\sup\{|x^n(t) - u(t+t_n)|_X : t \geq \sigma, n \in N\} \leq 1$ and $\sup\{|x^n(t) - u(t+t_n)|_X : t \geq \sigma + t_0(\varepsilon/2), n \in N\} \leq \varepsilon/2$ whenever $\sigma \in R^+$, $(v, G) \in \Omega(u, F)$ and $|\phi - v_\sigma|_{BC} < \delta_0$ $(< \delta(1))$. Therefore, by the same reason as that for (10), we obtain that

$$\begin{aligned} &\sigma \in R^+, \quad (v, G) \in \Omega(u, F) \quad \text{and} \quad |\phi - v_\sigma|_{BC} < \delta_0 \\ &\text{imply } |x(t, \sigma, \phi, G) - v(t)|_X < \varepsilon \quad \text{for } t \geq \sigma + t_0(\varepsilon/2). \end{aligned} \tag{12}$$

Now, we suppose that the solution $u(t)$ is not BC-TS. Then there exist an ε, $0 < \varepsilon < \delta_0$, sequences $\{\tau_n\} \subset R^+$, $\{r_n\}$, $r_n > 0$, $\{\phi_n\} \subset BC$, $\{h_n\}$, $h_n \in BC([\tau_n, \infty); X)$, and solutions

$\{x(\cdot, \tau_n, \phi_n, F+h_n)\}$ such that, for all $n \in N$,

$$(13) \qquad |\phi_n - u_{\tau_n}|_{\mathrm{BC}} < \frac{1}{n} \quad \text{and} \quad \sup_{t \geq \tau_n} |h_n(t)|_X < \frac{1}{n}$$

and

$$(14) \quad |z^n(\tau_n + r_n) - u(\tau_n + r_n)|_X = \varepsilon \quad \text{and} \quad |z^n(t) - u(t)|_X < \varepsilon \qquad \text{for } t \in (-\infty, \tau_n + r_n),$$

where $z^n(t) := x(t, \tau_n, \phi_n, F+h_n)$. We first consider the case where $\{r_n\}$ is unbounded. Without loss of generality, we may assume that $u(t+\tau_n+r_n-t_0) \to v(t)$ uniformly on any compact set in R and $F(t+\tau_n+r_n-t_0, \phi) \to G(t, \phi)$ Bohr-uniformly on $R \times \mathscr{B}$ for some $(v, G) \in \Omega(u, F)$ and that $z^n(t+\tau_n+r_n-t_0) \to z(t)$ uniformly on any compact set in $(-\infty, t_0]$ for some function z, where $t_0 = t_0(\varepsilon/2)$. Repeating almost the same argument as in the proof of the claim (10), we see by (13) that z satisfies (3) on $[0, t_0]$. Let $n \to \infty$ in (14) to obtain $|z(t) - v(t)|_X \leq \varepsilon$ on $(-\infty, t_0]$ and $|z(t_0) - v(t_0)|_X = \varepsilon$. This is a contradiction, because $|z_0 - v_0|_{\mathrm{BC}} \leq \varepsilon < \delta_0$ implies $|z(t_0) - v(t_0)|_X < \varepsilon$ by (12). Therefore the sequence $\{r_n\}$ must be bounded. Thus we may assume that $\{r_n\}$ converges to some r, $0 \leq r < \infty$. Moreover, we may assume that $\{z^n(\tau_n + t)\}$ converges to a function ξ uniformly on any compact set in $(-\infty, r]$ as $n \to \infty$. Consider the case where the sequence $\{\tau_n\}$ is unbounded; hence we may assume that $\{u(t+\tau_n)\}$ converges to a function w uniformly on any compact set in R, and $F(t+\tau_n, \phi) \to H(t, \phi)$ Bohr-uniformly on $R \times \mathscr{B}$ for some $(w, H) \in \Omega(u, F)$. Then $\xi(t)$ satisfies

$$\xi(t) = T(t)\xi(0) + \int_0^t T(t-s)H(s, \xi_s)ds$$

on $[0, r]$, and moreover we have $|\xi_0 - w_0|_{\mathrm{BC}} = 0$ and $|\xi(r) - w(r)|_X = \varepsilon$ by letting $n \to \infty$ in (13) and (14). This is a contradiction, because we must have $\xi \equiv w$ on $[0, r]$ by the regularity assumption. Thus the sequence $\{\tau_n\}$ must be bounded, too. Hence we may assume that $\lim_{n \to \infty} \tau_n = \tau$ for some $\tau < \infty$. Then $\xi(t-\tau)$ satisfies (2) on $[\tau, \tau+r]$, and moreover we have $|\xi_0 - u_\tau|_{\mathrm{BC}} = 0$ and $|\xi(r) - u(\tau+r)|_X = \varepsilon$ by (13) and (14). This again contradicts the fact that the solution $u(t)$ of (1) is BC-UAS.

The $\mathscr{B}$-stability implies the BC-stability. Therefore, the following results are direct consequences of Theorems 1 and 2.

COROLLARY 1. *If the solution $u(t)$ of* (2) *is $\mathscr{B}$*-TS, *then it is asymptotically almost periodic in t. Consequently,* (2) *has an almost periodic solution.*

COROLLARY 2. *Assume that system* (2) *is regular. If the solution $u(t)$ of* (2) *is $\mathscr{B}$*-UAS, *then it is asymptotically almost periodic in t. Consequently,* (2) *has an almost periodic solution.*

As an example, we consider the following integrodifferential equation with diffusion

$$(15)\qquad \frac{\partial u}{\partial t}(t,x)=\frac{\partial^2 u}{\partial x^2}(t,x)-u^3(t,x)+\int_{-\infty}^{t}k(t,s,x)u(s,x)ds+h(t,x)\,,$$
$$t>0\,,\quad 0<x<\pi\,.$$

In [4, p. 85], the equation (15) with $k\equiv 0$ and $h\equiv 0$ was treated under the Dirichlet boundary condition, and the uniform asymptotic stability of the zero solution was derived. In what follows, we shall treat the equation (15) under the Neumann boundary condition

$$(16)\qquad \frac{\partial u}{\partial x}(t,0)=\frac{\partial u}{\partial x}(t,\pi)=0\,,\qquad t>0\,,$$

and give a sufficient condition under which (15) possesses a bounded solution which is BC-TS.

We assume that the functions $h(t,x)$ and $k(t,s,x)$ are continuous functions satisfying $2<h(t,x)<7$ and $0\le k(t,s,x)\le K(t-s)$ for some continuous function $K(\tau)$ with $\int_0^\infty K(\tau)d\tau<1/4$. Note that one can choose a continuous nonincreasing function $g: R^-\mapsto[1,\infty)$ so that $g(0)=1$, $\lim_{s\to-\infty}g(s)=\infty$ and $\int_0^\infty K(\tau)g(-\tau)d\tau<\infty$ (cf. [2]). We now consider the Banach space $X=C([0,\pi];R)$, and define a linear operator A in X by

$$(A\xi)(x)=\frac{d^2\xi}{dx^2}(x)\,,\qquad 0<x<\pi\,,$$

for

$$\xi\in D(A):=\{\xi\in C^2[0,\pi]:\xi'(0)=\xi'(\pi)=0\}\,.$$

Then the operator A generates a compact semigroup $T(t)$ on X, and (15)–(16) is represented as the functional differential equation (2) on X with

$$F(t,\phi)(x)=h(t,x)-\phi^3(0,x)+\int_{-\infty}^{0}k(t,t+s,x)\phi(s,x)ds$$

for $\phi\in C_g^0(X)$. We refer to a (mild) solution of (2) as a (mild) solution of (15)–(16).

LEMMA 4. *Let $\phi(\theta,x)\equiv 3/2$ for all $(\theta,x)\in R^-\times[0,\pi]$. Then the solution $u(t,x)$ of* (15)–(16) *through* $(0,\phi)$ *satisfies the inequality*

$$1<u(t,x)<2\qquad \text{on}\quad [0,\infty)\times[0,\pi]\,.$$

PROOF. It is clear that there exists a (unique) local solution $u(t,x)\in C([0,a)\times[0,\pi])$ of (15)–(16) through $(0,\phi)$ for some $a>0$. We shall prove that

(17) $$1 < u(t, x) < 2 \quad \text{on} \quad [0, a) \times [0, \pi]\,.$$

Then $a = \infty$ and the conclusion of the lemma must hold. We will prove (17) by contradiction. Assume that (17) is false. Then there exists a $t_1 \in (0, a)$ such that $1 \leq u(t, x) \leq 2$ on $[0, t_1] \times [0, \pi]$ and $u(t_1, x_1) = 2$ or $u(t_1, x_1) = 1$ for some $x_1 \in [0, \pi]$. Consider a function $p(t, x) \in C([0, t_1] \times [0, \pi])$ defined by $p(t, x) = \int_{-\infty}^{t} k(t, s, x)u(s, x)ds + h(t, x)$, and choose a sequence $\{p_n(t, x)\} \in C^1([0, t_1] \times [0, \pi])$ such that $2 \leq p_n(t, x) \leq 7.5$ on $[0, t_1] \times [0, \pi]$ and that $p_n(t, x) \to p(t, x)$ uniformly on $[0, t_1] \times [0, \pi]$ as $n \to \infty$. There exists a (classical) solution $v_n(t, x)$ of the initial-boundary value problem

$$\frac{\partial v}{\partial t}(t, x) = \frac{\partial^2 v}{\partial x^2}(t, x) - v^3(t, x) + p_n(t, x)\,, \qquad 0 < t \leq t_1\,, \quad 0 < x < \pi\,,$$

$$\frac{\partial v}{\partial x}(t, 0) = \frac{\partial v}{\partial x}(t, \pi) = 0\,, \qquad 0 < t \leq t_1\,,$$

$$v(0, x) = 3/2\,, \qquad 0 < x < \pi\,.$$

Clearly, $v_n(t, x) \to u(t, x)$ uniformly on $[0, t_1] \times [0, \pi]$. We now assert that

(18) $$\sqrt[3]{1.9} < v_n(t, x) < \sqrt[3]{7.6} \quad \text{on} \quad [0, t_1] \times [0, \pi]\,.$$

If this assertion holds true, letting $n \to \infty$ in (18) we obtain $\sqrt[3]{1.9} \leq u(t, x) \leq \sqrt[3]{7.6}$ on $[0, t_1] \times [0, \pi]$, which contradicts $u(t_1, x_1) = 1$ or $u(t_1, x_1) = 2$. Consider the case where (18) does not hold true. Then there exists a $(t_2, x_2) \in [0, t_1] \times [0, \pi]$ such that $v_n(t_2, x_2) = \sqrt[3]{7.6}$ (or $v_n(t_2, x_2) = \sqrt[3]{1.9}$) and that $\sqrt[3]{1.9} < v_n(t, x) < \sqrt[3]{7.6}$ on $[0, t_2) \times [0, \pi]$. If $x_2 \in (0, \pi)$, then $\partial^2 v_n/\partial x^2 \leq 0$ and $\partial v_n/\partial t \geq 0$ (or $\partial^2 v_n/\partial x^2 \geq 0$ and $\partial v_n/\partial t \leq 0$) at (t_2, x_2); consequently $7.5 \geq p_n = -\partial^2 v_n/\partial x^2 + \partial v_n/\partial t + v_n^3 \geq v_n^3 = 7.6$ (or $2 \leq p_n = -\partial^2 v_n/\partial x^2 + \partial v_n/\partial t + v_n^3 \leq v_n^3 = 1.9$) at (t_2, x_2), a contradiction. Thus we must get $\sqrt[3]{1.9} < v_n(t, x) < \sqrt[3]{7.6}$ for all $(t, x) \in [0, t_2] \times (0, \pi)$ and $x_2 = 0$ or $x_2 = \pi$; say $x_2 = \pi$. Hence, by the strong maximum principle (cf., e.g. [12, Theorem 3.7]) we get $(\partial v_n/\partial x)(t_2, \pi) > 0$ (or $(\partial v_n/\partial x)(t_2, \pi) < 0$), a contradiction, because of $(\partial v_n/\partial x)(t_2, \pi) = 0$. Therefore, we must have the assertion (18).

PROPOSITION 2. *Let $u(t, x)$ be the solution of* (15)–(16) *ensured in Lemma* 4. *Then $u(t, x)$ is* BC-*totally stable. Hence, if $h(t, x)$ and $k(t, t+s, x)$ are almost periodic in t uniformly for $(s, x) \in R^- \times [0, \pi]$, then $u(t, x)$ is asymptotically almost periodic in t uniformly for $x \in [0, \pi]$.*

PROOF. In order to show that $u(t, x)$ is BC-TS, it is sufficient to show that

(19) $$|u(t, x) - v(t, x)| < \varepsilon\,, \qquad t \geq \sigma\,, \quad x \in [0, \pi]\,,$$

whenever $v(t, x)$ is a solution of

$$\frac{\partial v}{\partial t}(t,x)=\frac{\partial^2 v}{\partial x^2}(t,x)-v^3(t,x)+\int_{-\infty}^{t}k(t,s,x)v(s,x)ds+h(t,x)+r(t,x)\,,$$
$$t>\sigma\,,\quad 0<x<\pi\,,$$
$$\frac{\partial v}{\partial x}(t,0)=\frac{\partial v}{\partial x}(t,\pi)=0\,,\qquad t>\sigma\,,$$

where $r(t,x)\in C([\sigma,\infty)\times[0,\pi])$ with $\sup_{t\geq\sigma,0\leq x\leq\pi}|r(t,x)|<\varepsilon/2$ and $\sup_{\theta\leq\sigma,0\leq x\leq\pi}|u(\theta,x)-v(\theta,x)|<\varepsilon$. Set $w(t,x)=u(t,x)-v(t,x)$. Then $w(t,x)$ is a (mild) solution of

$$\frac{\partial w}{\partial t}(t,x)=\frac{\partial^2 w}{\partial x^2}(t,x)-w(u^2(t,x)+u(t,x)v(t,x)+v^2(t,x))$$
$$+\int_{-\infty}^{t}k(t,s,x)w(s,x)ds-r(t,x)\,,\qquad t>\sigma\,,\quad 0<x<\pi\,,$$
$$\frac{\partial w}{\partial x}(t,0)=\frac{\partial w}{\partial x}(t,\pi)=0\,,\qquad t>\sigma\,.$$

Assume that (19) is not true. Then there exists a $(t_3,x_3)\in(\sigma,\infty)\times[0,\pi]$ such that $|w(t,x)|<\varepsilon$ on $[\sigma,t_3)\times[0,\pi]$ and $|w(t_3,x_3)|=\varepsilon$, say $w(t_3,x_3)=\varepsilon$. Consider a function $V(t,x)$ defined by $V(t,x)=w(t,x)-\varepsilon$ for $(t,x)\in[\sigma,t_3]\times[0,\pi]$. Clearly, $V(t,x)$ is a (mild) solution of

$$\frac{\partial V}{\partial t}(t,x)=\frac{\partial^2 V}{\partial x^2}(t,x)-(V+\varepsilon)(u^2(t,x)+u(t,x)v(t,x)+v^2(t,x))$$
$$+\int_{-\infty}^{t}k(t,s,x)w(s,x)ds-r(t,x)\,,\qquad \sigma<t\leq t_3\,,\quad 0<x<\pi\,,$$
$$\frac{\partial V}{\partial x}(t,0)=\frac{\partial V}{\partial x}(t,\pi)=0\,,\qquad \sigma<t\leq t_3\,.$$

If $V(t,x)$ is smooth, then

$$\frac{\partial^2 V}{\partial x^2}(t,x)-\frac{\partial V}{\partial t}(t,x)-(u^2(t,x)+u(t,x)v(t,x)+v^2(t,x))V(t,x)$$
$$=\varepsilon(u^2(t,x)+u(t,x)v(t,x)+v^2(t,x))-\int_{-\infty}^{t}k(t,s,x)w(s,x)ds+r(t,x)$$
$$\geq\varepsilon\{(v(t,x)+(1/2)u(t,x))^2+(3/4)u^2(t,x)\}-\varepsilon\int_{-\infty}^{t}k(t,s,x)ds-\varepsilon/2$$
$$>3\varepsilon/4-\varepsilon/4-\varepsilon/2=0$$

on $(\sigma,t_3]\times[0,\pi]$. Then, repeating the same argument as in the proof of Lemma 4, we get a contradiction by applying the strong maximum principle. When $V(t, x)$ is not smooth, we get a contradiction again by approximating $V(t, x)$ by some smooth functions as in the proof of Lemma 4. Thus we must have (19).

REFERENCES

[1] L. AMERIO AND G. PROUSE, Almost-Periodic Functions and Functional Equations, Van Nostrand Reinhold, New York, 1971.

[2] F. V. ATKINSON AND J. R. HADDOCK, On determining phase spaces for functional differential equations, Funkcial. Ekvac. 31 (1988), 331–347.

[3] H. R. HENRÍQUEZ, Periodic solutions of quasi-linear partial functional differential equations with unbounded delay, Funkcial. Ekvac. 37 (1994), 329–343.

[4] D. HENRY, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math. 840, Springer-Verlag, Berlin-Heidelberg, 1981.

[5] Y. HINO, Stability and existence of almost periodic solutions of some functional differential equations, Tôhoku Math. J. 28 (1976), 389–409.

[6] Y. HINO AND S. MURAKAMI, Stability properties in abstract linear functional differential equations with infinite delay, WSSIAA (Dynamical Systems and Applications), (1995), 329–341.

[7] Y. HINO AND S. MURAKAMI, Existence of an almost periodic solutions of abstract linear functional differential equations with infinite delay, Dynamic Systems and Applications, 5 (1996), 385–398.

[8] Y. HINO, S. MURAKAMI AND T. NAITO, Functional Differential Equations with Infinite Delay, Lecture Notes in Math. 1473, Springer-Verlag, Berlin-Heidelberg, 1991.

[9] V. LAKSHMIKANTHAM AND S. LEELA, Nonlinear Differential Equations in Abstract Spaces, Pergamon Press, Oxford-New York, 1981.

[10] S. MURAKAMI AND T. YOSHIZAWA, Relationships between BC-stabilities and ρ-stabilities in functional differential equations with infinite delay, Tôhoku Math. J. 44 (1992), 45–57.

[11] A. PAZY, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Math. Sciences 44, Springer-Verlag, New York, 1983.

[12] M. H. PROTTER AND H. F. WEINBERGER, Maximum Principles in Differential Equations, Springer-Verlag, New York, 1984.

[13] T. YOSHIZAWA, Stability Theory and the Existence of Peroidic Solutions and Almost Periodic Solutions, Applied Math. Sciences 14, Springer-Verlag, New York, 1975.

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