Tôhoku Math. J.
49 (1997), 115–131

LOCAL AND GLOBAL PROPERTIES OF FUNCTIONS AND THEIR FOURIER TRANSFORMS

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(Received October 24, 1995, revised May 30, 1996)

Abstract. We show that an integrable function on the real line with a nonnegative Fourier transform is square-integrable near the origin if and only if the transform belongs to the amalgam space comprised of functions that are locally integrable and globally square-integrable. We use this to give another proof that there are integrable functions on the real line that have nonnegative Fourier transforms and are square-integrable near the origin, but are not square-integrable on the whole real line. Our methods work on all locally compact abelian groups that are not compact. They also apply to other questions related to the one discussed above.

1. Introduction. It is known that if a function on the interval $[-\pi, \pi)$ belongs to L^1, has nonnegative Fourier coefficients, and is square-integrable in some neighbourhood of 0, then the function is square-integrable on all of $[-\pi, \pi)$. In his book on entire functions, Boas [6] proved a special case of this, and credited it to N. Wiener, but it is not clear (cf. [21]) that the proof given by Boas is the same as Wiener's. A proof different from the one in Boas was presented by Shapiro [21] and Rains [19]; that method extended to all compact abelian groups. Further extensions to other norms of other classes of groups are in [1], [17], [5], and [15].

On the real line, a simple analogue of Wiener's theorem would be the statement that if a function in $L^1(R)$ has a nonnegative Fourier transform and is square-integrable near 0, then the function must be square-integrable on the whole line. It was recently shown by Kawazoe, Onoe, and Tachizawa [16] that this simple analogue is false; this is also the case (cf. [14]) on some noncompact Lie groups. Our goal here is to use basic facts about amalgams (cf. [12]) of L^p and l^q norms to show exactly what does follow when a function on the real line is square-integrable near 0 and has a nonnegative Fourier transform; our methods extend easily to all locally compact abelian groups. In particular, these methods show that the simple analogue of Wiener's theorem is true if and only if the group is compact, and they provide a sharp substitute when the group is not compact.

We state our main results for the real line in this section, prove them in that context in the next section, and discuss their extensions to all locally compact abelian groups in the final section. For each integer n, denote the interval $[n-1/2, n+1/2)$ by I_n, and

1991 *Mathematics Subject Classification.* Primary 42A38; Secondary 43A15, 43A25, 43A35.
* Research partially supported by NSERC grant 4822.

denote the indicator function of I_n by 1_{I_n}. Given a measurable function f on the real line, and two indices p and q in the extended real interval $[1, \infty]$, form the norms $\|f \cdot 1_{I_n}\|_p$ and let $\|f\|_{p,q}$ be the l^q-norm of the sequence $(\|f \cdot 1_{I_n}\|_p)_{n=-\infty}^{\infty}$. The *amalgam* (L^p, l^q) is the space of (equivalence classes of) measurable functions f for which $\|f\|_{p,q} < \infty$. See the surveys [12] and [8] for more information about these spaces.

THEOREM 1.1. *The following conditions are equivalent for any function in $L^1(R)$ with a nonnegative transform.*

(a) *The function is square-integrable in some neighbourhood of* 0.

(b) *The transform of the function belongs to (L^1, l^2).*

(c) *The function belongs to (L^2, l^∞).*

The condition (c) asserts that there is a constant C for which the function, f say, satisfies

$$\int_{n-1/2}^{n+1/2} |f|^2 \leq C \qquad \text{for all } n .$$

Of course, this implies the condition (a) without any other assumptions about f. It will be shown in Section 2 that (a) implies (b) when the transform is nonnegative; that argument is essentially the one used in [19] and [21] to prove Wiener's theorem.

The space (L^1, l^2) stands out among amalgams. It is the largest solid space of functions on the real line with the property that all its members have (distributional) Fourier transforms that are functions (cf. [22]). It is the Köthe dual of the space of Fourier transforms of continuous functions with compact support (cf. [7]). It is an endpoint for the Hausdorff-Young theorem for amalgams (cf. [13]), which in particular states that if the Fourier transform of a function belongs to (L^1, l^2), then the function belongs to (L^2, l^∞), that is, the condition (b) implies the condition (c) without any other assumptions about the function.

We use the symbols $\hat{R}$ and $\hat{I}_0$ to denote dual copies of the real line R and the interval $[-1/2, 1/2)$. In comparing Theorem 1.1 with the simple, but false, analogue of Wiener's theorem, we note that the space $L^1(\hat{I}_0)$ is strictly larger than $L^2(\hat{I}_0)$, so that the amalgam $(L^1, l^2)(\hat{R})$ that occurs in Theorem 1.1 is strictly larger than $(L^2, l^2)(\hat{R})$. The latter amalgam coincides with $L^2(\hat{R})$, which is also the space of transforms of functions that are square-integrable on R. This suggests a strategy for disproving the simple analogue of Wiener's theorem.

THEOREM 1.2. *The amalgam $(L^1, l^2)(\hat{R})$ contains nonnegative functions that are transforms of integrable functions, but that do not belong to $L^2(\hat{R})$.*

COROLLARY 1.3. *There are functions in $L^1(R)$ that have nonnegative transforms and that belong to $(L^2, l^\infty)(R)$, but that do not belong to $L^2(R)$.*

We will give explicit examples of functions with these properties. We will not need the theory of amalgams to verify that these functions have the properties specified in Corollary 1.3, but our constructions are motivated by that theory. We now outline how it can be used to prove the two statements above.

The Hausdorff-Young theorem for amalgams (cf. [13]) states that if $1 \leq p, q \leq 2$ and if a function belongs to (L^p, l^q), then the Fourier transform of the function belongs to $(L^{q'}, l^{p'})$, where p' and q' are the indices conjugate to p and q; note that these conjugate indices occur in the reverse order in the specification of the amalgam $(L^{q'}, l^{p'})$. Moreover, there is a constant C so that $\|\hat{f}\|_{q',p'} \leq C\|f\|_{q,p}$ in these cases. The theorem also applies to the inverse Fourier transform.

The counterpart of Young's inequality for convolution states that if $f_1 \in (L^{p_1}, l^{q_1})$ and if $f_2 \in (L^{p_2}, l^{q_2})$, then $\int_R |f_1(x-y)f_2(y)|\,dy$ is finite for almost all x provided that $1/p_1 + 1/p_2 \geq 1$ and $1/q_1 + 1/q_2 \geq 1$, and in that case the convolution $f_1 * f_2$ defined by letting

$$f_1 * f_2(x) = \int_R f_1(x-y)f_2(y)\,dy \qquad \text{for almost all } x$$

belongs to (L^p, l^q), where $1/p = 1/p_1 + 1/p_2 - 1$ and $1/q = 1/q_1 + 1/q_2 - 1$; moreover, there is a constant C so that $\|f_1 * f_2\|_{p,q} \leq C\|f_1\|_{p_1,q_1}\|f_2\|_{p_2,q_2}$ for all such functions f_1 and f_2.

It is known (cf. [11]) that the indices in these theorems are best possible. The following statement addresses the sharpness of these conclusions in another context.

THEOREM 1.4. *There is a function in $L^2(R)$ that does not belong to $L^4(R)$, but whose Fourier transform belongs to $(L^1, l^{4/3})(\hat{R})$.*

The condition that the transform belongs to $(L^1, l^{4/3})(\hat{R})$ and the Hausdorff-Young theorem for the inverse transform guarantee that the function above must belong to the amalgam $(L^4, l^\infty)(R)$, which is a slightly larger space than $L^4(R) = (L^4, l^4)(R)$. The point of the theorem is that adding the requirement that the (transform of the) function be square-integrable still does not force the function to belong to $L^4(R)$.

To deduce Theorem 1.2 from Theorem 1.4, split the Fourier transform of the function above into real and imaginary parts, and write these parts as differences of minimal nonnegative functions. These four nonnegative functions will all belong to $(L^1, l^{4/3})(\hat{R})$ and to $L^2(\hat{R})$. So they all have inverse transforms in $L^2(R)$, but at least one of them does not have an inverse transform in $L^4(R)$, since the function in the statement of the theorem does not belong to $L^4(R)$.

Hence the conclusion of Theorem 1.4 holds for some function, f say, that has a nonnegative Fourier transform. Then f^2 belongs to $L^1(R)$ but not to $L^2(R)$. The transform of f^2 is the convolution square $\hat{f} * \hat{f}$ of the nonnegative function $\hat{f}$. Hence $\widehat{f^2}$ is also nonnegative, and being the convolution of two functions in $(L^1, l^{4/3})(\hat{R})$, the function $\widehat{f^2}$ must belong to $(L^1, l^2)(\hat{R})$, that is, f^2 has the properties specified in Theorem 1.2.

The first example in [16] also came from a suitable convolution, but that operation occurred on R rather than on $\hat{R}$. The key properties of that example can be summarized as follows. Given a function g on R, let $\tilde{g}(x)=g(-x)^-$ for all x in R.

THEOREM 1.5. *There is an integrable function g in $(L^{4/3}, l^2)$ such that $\tilde{g}*g \notin L^2(R)$.*

The function $\tilde{g}*g$ is also integrable and has the nonnegative transform $|\hat{g}|^2$. Young's inequality for convolution and the assumption that g belongs to $(L^{4/3}, l^2)(R)$ guarantee that $\tilde{g}*g$ belongs to $(L^2, l^\infty)(R)$; so $\tilde{g}*g$ has all the properties specified in Corollary 1.3. The conclusion that $\tilde{g}*g \in (L^2, l^\infty)(R)$ follows without the assumption that g is integrable. Part of the point of the theorem is that adding that assumption still does not force $\tilde{g}*g$ to belong to $L^2(R)$, which is a smaller space than $(L^2, l^\infty)(R)$. This also bears on the Hausdorff-Young theorem, which guarantees that $\hat{g} \in (L^2, l^4)(\hat{R})$ if $g \in (L^{4/3}, l^2)(R)$; by the theorem above, adding the assumption that $g \in L^1(R)$ still does not force $\hat{g}$ to belong to $L^4(\hat{R})$, which is a smaller space than $(L^2, l^4)(\hat{R})$.

Finally, we consider variants of parts of Theorem 1.1 where the index 2 is replaced by other numbers in the interval $(1, \infty]$. We first state the counterpart of the theorem in [1].

THEOREM 1.6. *Let $1<p<2$. If f is an integrable function on R with the property that $\hat{f} \geq 0$ and if $|f|^p$ is integrable in some neighbourhood of 0, then $\hat{f} \in (L^1, l^{p'})(\hat{R})$.*

In contrast to the situation in Theorem 1.1, the converse is false here. In fact, it does not even help to replace $(L^1, l^{p'})(\hat{R})$ by the smaller amalgam $(L^\infty, l^{p'})(\hat{R})$.

THEOREM 1.7. *There is an integrable function f on R with the properties that $\hat{f} \geq 0$ and $\hat{f} \in (L^\infty, l^q)(\hat{R})$ for all indices $q>2$, but for which there is no index $p>1$ such that $|f|^p$ is integrable in some neighbourhood of 0.*

The main part of Theorem 1.1 is that the condition (a) implies the condition (b) when the transform is nonnegative. The fact (a) implies (c) in this case is also of interest, and a version of that fact survives for some other indices.

THEOREM 1.8. *Let p be an even positive integer, or let $p=\infty$. If f is an integrable function on R such that $\hat{f} \geq 0$ and if there is a neighbourhood U of 0 for which the restriction of f to U belongs to $L^p(U)$, then $f \in (L^p, l^\infty)(R)$.*

The requirement that p be even or infinite is essential here.

THEOREM 1.9. *Let $p \in (1, \infty)$, and suppose that p is not even. Then there is a function f in $L^1(R)$ with the properties that $\hat{f} \geq 0$ and $|f|^p$ is integrable in some neighbourhood of 0, but such that every nonempty open set V in R has a translate $t+V$ for which $\int_{t+V} |f|^p = \infty$.*

In Theorem 1.1, the implication (c)$\Rightarrow$(a) was obvious and did not require any further assumptions about the function. The corresponding rephrasing of Theorem 1.8 is that if p is even or infinite, then the following statements are equivalent for an integrable function f with a nonnegative transform.

(a) The restriction of f to some neighbourhood of 0 belongs to L^p of that neighbourhood.

(c) $f \in (L^p, l^\infty)(R)$.

In this context, the counterpart of the statement (b) in Theorem 1.1 is that

(b) $\hat{f} \in (L^1, l^{p'})(\hat{R})$.

Our Theorem 1.1 says that this statement is equivalent to the other two when $p=2$ and $\hat{f} \geq 0$. It was pointed out in [16, Theorem 3.1] that this is also the case when $p=\infty$ and $\hat{f} \geq 0$; then (b) asserts that $\hat{f} \in L^1$. For the other values of p arising in Theorem 1.8 the condition (b) implies the condition (c), and hence (a), by the Hausdorff-Young theorem for amalgams, without assuming that $\hat{f} \geq 0$. Adding the latter assumption and more does not make the converse true.

THEOREM 1.10. *There is a function that belongs to $(L^p, l^1)(R)$ for all $p < \infty$ and that has a nonnegative transform that belongs to $(L^1, l^q)(\hat{R})$ only when $q \geq 2$.*

Since $l^2 \subset l^q$ when $q \geq 2$, Theorem 1.1 guarantees that the transform of the function above must belong to $(L^1, l^q)(\hat{R})$ for all such values of q. None of these values have the form p' for any $p > 2$. So the conditions (a) and (c) above and the positivity of $\hat{f}$ do not imply the condition (b) when $2 < p < \infty$.

2. Proofs on the real line. Dilating or translating the variables leads to the same amalgam spaces with equivalent norms. Except for constants in estimates, our results are not affected by rescalings of the variables used in computing the Fourier transform and the amalgam norms. For definiteness, we choose a specific version of the transform, namely the one given by letting

$$\hat{f}(y) = \int_{x \in R} f(x) e^{-2\pi i x y}\, dx \qquad \text{when} \quad f \in L^1(R)\,. \tag{2.1}$$

Then the inverse transform follows the same pattern except that the integration runs over the dual copy $\hat{R}$ of the real line, and there is no minus-sign in the exponential.

We will repeatedly use properties of the functions ϕ_δ defined for positive values of the parameter δ by letting

$$\phi_\delta(x) = \begin{cases} 1 - |x|/\delta\,, & \text{if} \quad |x| \leq \delta\,; \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{2.2}$$

Then

$$\widehat{\phi_\delta}(y) = \frac{1}{\delta}\left[\frac{\sin(\pi y \delta)}{\pi y}\right]^2 \tag{2.3}$$

for all $y \neq 0$. This is also true for the inverse transform of ϕ_δ. We note that $\widehat{\phi_\delta} \in L^1(\hat{R})$.

So the Fourier inversion theorem applies and allows us to write ϕ_δ as the inverse transform of $\widehat{\phi_\delta}$.

To prove that (a)⇒(b) in Theorem 1.1, let f be an integrable function that is square-integrable near 0 and has a nonnegative transform. Choose δ small enough to make the restriction of f to $(-\delta, \delta)$ square-integrable, and small enough that $\sin(\pi y\delta)/(\pi y) \geq \delta/2$ for all values of y in $[-1, 1)$, that is,

$$\widehat{\phi_\delta}(y) \geq \frac{\delta}{4} \quad \text{if} \quad y \in [-1, 1)\,. \tag{2.4}$$

Then the product $f \cdot \phi_\delta$ is both integrable and square-integrable. By the Plancherel theorem, the L^2-norm of this product is equal to the L^2-norm of its transform. We have

$$(f \cdot \phi_\delta)^\wedge(y) = \int_{\hat{R}} \hat{f}(y-t)\widehat{\phi_\delta}(t)\,dt = \hat{f} * \widehat{\phi_\delta}(y)\,. \tag{2.5}$$

Since $\hat{f}$ and $\widehat{\phi_\delta}$ are both nonnegative, there is no cancellation in the integral giving the convolution in the formula above. So the condition (2.4) gives that

$$(f \cdot \phi_\delta)^\wedge(y) \geq \frac{\delta}{4}\int_{z=y-1}^{z=y+1} \hat{f}(z)\,dz\,. \tag{2.6}$$

Note that if $y \in I_n$, then $I_n \subset [y-1, y+1)$. Therefore,

$$\int_{I_n} (f \cdot \phi_\delta)^\wedge(y)^2\,dy \geq \frac{\delta^2}{16}\left[\int_{I_n} \hat{f}(t)\,dt\right]^2 \qquad \text{for all } n\,. \tag{2.7}$$

Summing with respect to n then gives that

$$\|\hat{f}\|_{1,2} \leq \frac{4}{\delta}\|f \cdot \phi_\delta\|_2\,. \tag{2.8}$$

This completes the proof that (a)⇒(b) in Theorem 1.1. As noted earlier, the fact that (b)⇒(c) was already known, and it is obvious that (c)⇒(a); moreover, these steps do not require $\hat{f}$ to be nonnegative.

The assumption that the function belongs to L^1 is not essential here. Replacing ϕ_δ with a suitable nonnegative test function with a nonnegative transform show that if f is a tempered distribution that coincides with a square-integrable function near 0 and if the (distributional) transform of f is a nonnegative function, then again $\hat{f} \in (L^1, l^2)(\hat{R})$, with an estimate for $\|\hat{f}\|_{1,2}$ in terms of the L^2-norm of the restriction of f to any fixed neighbourhood of 0. Assuming square-integrability of f near 0 and that $\hat{f}$ is a nonnegative distribution forces that transform to be a nonnegative Borel measure, μ say, and then the conclusion is that

(2.9)
$$\sum_{n=-\infty}^{\infty} |\mu|(\hat{I}_n)^2 < \infty ;$$

again there is an estimate for the square root of the left side above. Conversely, (cf. [4]) if a distribution on R has a transform that is a Borel measure satisfying the condition (2.9), then that distribution is given by a function in $(L^2, l^\infty)(R)$, and its norm in that space is bounded above by a constant times the square-root of the left side of (2.9).

We will give an independent proof of Theorem 1.2, but we note first that if follows immediately from Theorem 1.1 and Corollary 1.3, which is the main result in [16]. Theorem 1.2 also follows easily from the Hausdorff-Young theorem for amalgams and Theorem 1.5, which specifies some properties of the first example in [16]. Indeed, since the function g in the statement of Theorem 1.5 belongs to $(L^{4/3}, l^2)(R)$, its transform must belong to $(L^2, l^4)(\hat{R})$, and hence $|\hat{g}|^2$ belongs to $(L^1, l^2)(\hat{R})$. Also, $|\hat{g}|^2$ is nonnegative, and it is the transform of the integrable function $\tilde{g}*g$. Since that function does not belong to L^2, neither does $|\hat{g}|^2$. So $|\hat{g}|^2$ has all the properties specified in Theorem 1.2.

To prove that theorem directly, we consider linear combinations of translates of functions ϕ_δ on $\hat{R}$. Given any function F on $\hat{R}$, and a number t, we use the symbol $f(\cdot - t)$ to denote the translate of F that maps y to $F(y-t)$ for all y in $\hat{R}$. Given two positive integers j and k, we let

(2.10)
$$F_{j,k} = \sum_{-k<m<k} \left(1 - \frac{|m|}{k}\right) \phi_{1/(2j)}(\cdot - m) .$$

In contrast to our use of the functions ϕ_δ in Theorem 1.1, we now take their domains to be $\hat{R}$ and the domains of their (inverse) transforms to be R.

Clearly, these functions $F_{j,k}$ are nonnegative. Because the translates $\phi_{1/(2j)}(\cdot - m)$ are supported by the distinct intervals I_m, the amalgam norm $\|F_{j,k}\|_{p,q}$ is simply the product of the L^p-norm of $\phi_{1/(2j)}$ and the l^q norm of the coefficients multiplying the translates of $\phi_{1/(2j)}$ in formula (2.10). In particular,

(2.11)
$$\|F_{2^n,4^n}\|_{1,2} \le \sqrt{2}, \quad \text{but} \quad \|F_{2^n,4^n}\|_{2,2} \ge \frac{1}{2} 2^{n/2}$$

for all n. So the series

(2.12)
$$\sum_{n=1}^{\infty} 2^{-n/4} F_{2^n,4^n}$$

converges in the norm $\|\cdot\|_{1,2}$ to some function, F say. Then $F \ge 0$, and $F \in (L^1, l^2)(\hat{R})$. On the other hand, because the terms in (2.12) are all nonnegative,

$$\|F\|_2 = \|F\|_{2,2} \ge 2^{-n/4} \|F_n\|_{2,2} \ge \frac{1}{2} 2^{n/4} \qquad \text{for all } n .$$

Hence $F \notin L^2(\hat{R})$.

To verify that F is the transform of some integrable function, consider the inverse transforms of the terms in the series (2.12). As noted earlier, the inverse transform of ϕ_δ coincides with the transform of ϕ_δ. Translating ϕ_δ by m on $\hat{R}$ corresponds to multiplying $\widehat{\phi_\delta}(x)$ by $e^{2\pi i m x}$ on R. So the (inverse) transform $f_{j,k}$ say, of $F_{j,k}$ is the product of the transform of $\phi_{1/(2j)}$ and the function

$$h_k : x \mapsto \sum_{-k<m<k} \left(1 - \frac{|m|}{k}\right) e^{2\pi i m x}.$$

Now h_k is a periodic Féjer kernel with period 1. So its L^1-norm over every interval of length 1 is equal to 1, and hence $\|h_k\|_{1,\infty} = 1$. On the other hand, it is easy to verify from formula (2.3) that

$$\|\widehat{\phi_\delta}\|_{\infty,1} \leq 4 \qquad \text{when} \quad 0 < \delta \leq \frac{1}{2}. \tag{2.13}$$

Then

$$\|f_{j,k}\|_1 \leq \|[\phi_{1/(2j)}]^\wedge\|_{\infty,1} \|h_k\|_{1,\infty} \leq 4$$

for all positive integers j and k. Hence the series

$$\sum_{n=-\infty}^{\infty} 2^{-n/4} f_{2^n, 4^n} \tag{2.14}$$

converges in norm in $L^1(R)$, to a function f say. Then $\hat{f} = F$.

This completes our proof of Theorem 1.2. Corollary 1.3 then follows from Theorems 1.1 and 1.2. A reader who wants to avoid the use of the less-obvious theorems about amalgams can verify Corollary 1.3 as follows. The argument immediately above shows in an elementary way that the series (2.14) converges in L^1-norm and that its sum f has the nonnegative transform F. Since F does not belong to L^2, neither does f. One can show that f belongs to (L^2, l^∞) by estimating $\|h_k\|_{2,\infty}$ and $\|(\phi_{1/(2j)})^\wedge\|_\infty$ in terms of the l^2-norm of the Fourier coefficients of the restriction of h_k to $[-1/2, 1/2)$ and the L^1 norm of $\phi_{1/(2j)}$. This provides estimates for $\|f_{j,k}\|_{2,\infty}$ that force the series (2.14) to converge in the space $(L^2, l^\infty)(R)$.

Proving Theorem 1.4 is easier because of the lack of a positivity requirement. If the theorem were false, then the inverse transform would map the intersection of the spaces $L^2(\hat{R})$ and $(L^1, l^{4/3})(\hat{R})$ into the space $L^4(R)$. That intersection space is complete with respect to the norm $\|\cdot\|_2 + \|\cdot\|_{1,4/3}$, and the inverse transform mapping into $L^4(R)$ would then have a closed graph. To verify the latter claim, suppose that $\widehat{g_n} \to G$ in the norm on the intersection space on $\hat{R}$, and that $g_n \to g$ in the L^4-norm on R. Then the sequence (g_n) converges to the inverse transform $\check{G}$ of G in $L^2(R)$, and hence in measure, but that sequence also converges in measure to g; so $g = \check{G}$.

By the closed graph theorem, there would be a constant C so that

$$\|g\|_4 \leq C(\|\hat{g}\|_2 + \|\hat{g}\|_{1,4/3}) \qquad \text{whenever} \quad \hat{g} \in L^2(\hat{R}) \cap (L^1, l^{4/3})(\hat{R}) . \tag{2.15}$$

To see that there is in fact no such constant C, consider the functions

$$g_j = \frac{1}{\sqrt{j}} f_{j,j^2} \quad \text{and} \quad \widehat{g_j} = \frac{1}{\sqrt{j}} F_{j,j^2} . \tag{2.16}$$

As in our proof of Theorem 1.2, the transforms $\widehat{g_j}$ form bounded sequences in both of the spaces $L^2(\hat{R})$ and $(L^1, l^{4/3})(\hat{R})$; moreover, the norms $\|\widehat{g_j}\|_2$ are also bounded away from 0, so that the same is true for the norms $\|g_j\|_2$. On the other hand, $\|g_j\|_1 \to 0$ as $j \to \infty$, and by Hölder's inequality,

$$\|g_j\|_2 \leq (\|g_j\|_1)^{1/3} (\|g_j\|_4)^{2/3} .$$

The numbers on the left stay bounded away from 0, while the first factors on the right tend to 0; so the second factors on the right tend to ∞. This contradicts the inequality in line (2.15).

To prove Theorem 1.5, simply inspect the first example in [16]. Theorem 1.6 follows by the same method that was used to prove that (a)$\Rightarrow$(b) in Theorem 1.1, with the Hausdorff-Young inequality for the indices p and p' replacing the equality between L^2-norms on R and $\hat{R}$. Theorem 1.7 follows easily from the corresponding fact for Fourier series; lacking a reference for the latter, we prove it first.

We claim that it suffices in that setting to construct a function, g say, in $L^1[-1/2, 1/2)$ so that its Fourier coefficients

$$\hat{g}(n) = \int_{-1/2}^{1/2} g(x) e^{-2\pi i n x} dx$$

are nonnegative and belong to l^q for all $q > 2$, but for which

$$\sum_{j=1}^{\infty} |\hat{g}(3^j)|^2 = \infty . \tag{2.17}$$

To verify this claim, note first (cf. [23, Chapter XII, Theorem 7.6]) that the coefficients of any function in any of the spaces $L^p[-1/2, 1/2)$ with $p > 1$ have the property that their restriction to the set $\{3^j\}_{j=1}^{\infty}$ belongs to l^2. So the condition (2.17) guarantees that g does not belong to any of these spaces.

If $|g|^p$ is integrable near 0 for some $p > 1$, and if δ is small enough, then the restriction to $\{3^j\}_{j=1}^{\infty}$ of the coefficients of the product $g \cdot \phi_\delta$ must belong to l^2. But, as in the proof that (a)$\Rightarrow$(b) in Theorem 1.1, the nonnegativity of the coefficients of g and ϕ_δ and the fact that $\widehat{\phi_\delta}(0) > 0$ would then force the restriction of the coefficients of g to $\{3^j\}_{j=1}^{\infty}$ to belong to l^2. So the condition (2.17) and the condition that $\hat{g} \geq 0$ guarantee that there is no index $p > 1$ for which $|g|^p$ is integrable near 0.

To get an integrable function g with coefficients satisfying the conditions above, use Riesz products. Start with any nonnegative sequence $(d_j)_{j=1}^{\infty}$ that belongs to l^q for all $q>2$ but does not belong to l^2, and rescale the sequence, if necessary so that $d_j \le 1/2$ for all j. Define functions g_m by letting

$$(2.18)\qquad g_m(x)=\prod_{j=1}^{m}[1+2d_j\cos(3^j 2\pi x)]=\prod_{j=1}^{m}[1+d_je^{3^j2\pi ix}+d_je^{-3^j2\pi ix}]\qquad \text{for all } x.$$

Then (cf. [23, Chapter V, §7]) these functions have the following properties. The norm of g_m in $L^1[-1/2, 1/2)$ is equal to 1. Moreover, $\widehat{g_m}(n)=0$ unless

$$(2.19)\qquad n=\sum_{j=1}^{m}\varepsilon_j 3^j$$

with each number ε_j taking one of the values -1, 0, and 1. There is at most one such representation of any particular integer n, and then

$$(2.20)\qquad \widehat{g_m}(n)=\prod_{j=1}^{m}(d_j)^{|\varepsilon_j|},$$

with the convention that $0^0=1$. Since the sequence (g_m) is bounded in $L^1[-1/2, 1/2)$ it has a weak-star limit point, μ say, in the space of bounded Borel measures on the set $[-1/2, 1/2)$. For each integer n, the Fourier-Stieltjes coefficient $\hat{\mu}(n)$ must be a limit point of the sequence $(\widehat{g_m}(n))_{m=1}^{\infty}$. In fact, that sequence is ultimately constant for each n, and it follows that $\hat{\mu}(n)=0$ unless n has a representation in the form (2.19) for some m, in which case $\hat{\mu}(n)$ is equal to the right side of the equation (2.20).

One consequence of this is that when $2<q<\infty$,

$$(2.21)\qquad \sum_{n=-\infty}^{\infty}|\hat{\mu}(n)|^q\le\prod_{m=1}^{\infty}[1+2|d_m|^q]\le\prod_{m=1}^{\infty}[e^{2|d_m|^q}]\le e^{2(\|d\|_q)^q}<\infty.$$

Another is that $\sum_{m=1}^{\infty}|\hat{\mu}(3^m)|^2=\infty$. There are functions in $L^1[-1/2, 1/2)$ whose Fourier coefficients are nonnegative and tend to 0 as slowly as one likes. Choose such a function, h say, so that

$$\sum_{m=1}^{\infty}|\hat{h}(3^m)\hat{\mu}(3^m)|^2=\infty.$$

Then the convolution $g=h*\mu$ has the desired properties.

To lift this example from $L^1[-1/2, 1/2)$ to $L^1(R)$, extend g to have period 1 on R, thereby getting a function in $(L^1, l^\infty)(R)$, and multiply that extension by the inverse transform of $\phi_{1/2}$. It follows from the line (2.13) that this product, f say, belongs to $L^1(R)$. From the line (2.3), the (inverse) transform of $\phi_{1/2}$ is bounded away from 0 in all small enough neighbourhoods of 0. When $1<p<\infty$, the factor g is not p-th power integrable in any such neighbourhood, and so neither is f.

The following statements should be clear in a formal sense. First, the distributional transform of the periodic extension of g is a discrete measure on $\hat{R}$ supported by the set of integers n with representations in the form (2.19) for sufficiently large values of m. Second, the mass assigned by that measure to such a point n is equal to the right side of the formula (2.20). Third, $\hat{f}$ is the convolution of the compactly supported function ϕ_δ with that measure on $\hat{R}$. This makes it plausible that $\hat{f}\geq 0$ and that $\hat{f}\in(L^\infty, l^q)$ for all $q>2$.

This argument can be made rigorous using a theory (cf. [9]) of distributions and Fourier transforms where the function $\phi_{1/2}$ and its transform are test functions. An alternative is to use the conventional theory of tempered distributions, replacing $\phi_{1/2}$ in the construction of f by any nontrivial test function with a nonnegative transform. Both approaches produce a function f with all the properties specificed in Theorem 1.7.

One can also bypass theories of distributional transforms and construct such a function as a sum of a series in the style of our proof of Theorem 1.2. Each term in the series would be the product of a periodic trigonometric polynomial and the (inverse) transform of ϕ_δ for some fixed $\delta>0$. The n-th trigonometric polynomial would be required to have a relatively small norm in $L^1[-1/2, 1/2)$ and to have nonnegative coefficients with a relatively small l^{p_n}-norm, where (p_n) is a strictly decreasing sequence converging to 2, but to have the l^2-norm of the restriction of those coefficients to the set $\{3^j\}_{j=1}^\infty$ be relatively large.

To prove Theorem 1.8, let U be a neighbourhood of 0 for which $f\,|\,U\in L^p(U)$. Choose a smaller neighbourhood V of 0 so that $V+V\subset U$. Then choose a sequence of functions $(\psi_k)_{k=1}^\infty$ so that each function ψ_k vanishes outside V and has a nonnegative transform belonging to $L^1(\hat{R})$, and so that $\|f-f*\psi_k\|_1\to 0$ as $k\to\infty$. Rescale, if necessary, so that $\|\psi_k\|_1=1$ for all k.

Since $\|\psi_k\|_1=1$, Young's inequality for convolution yields that

$$\|(f*\psi_k)\,|\,V\|_p\leq\|f\,|\,U\|_p \qquad \text{for all } k. \tag{2.22}$$

Choose a positive number δ so that the interval $(-\delta, \delta)$ is contained in the set V. Then

$$\|(f*\psi_k)\cdot\phi_\delta\|_p\leq\|f\,|\,U\|_p \qquad \text{for all } k. \tag{2.23}$$

Both factors $(f*\psi_k)$ and ϕ_δ above have nonnegative, integrable transforms. So the transform of the product $(f*\psi_k)\cdot\phi_\delta$ is the convolution of these nonnegative transforms. The same reasoning applies to the transform of the product of $(f*\psi_k)$ with any translate of ϕ_δ, except that the transforms of those translates are unlikely to be nonnegative. Translating ϕ_δ does not change the absolute value of its transform, however, so that absolute value of the transform of the product of $(f*\psi_k)$ with any translate of ϕ_δ is majorized by the transform of $(f*\psi_k)\cdot\phi_\delta$.

The values of the index p in Theorem 1.8 are precisely those for which $L^p(R)$ has the *upper majorant property* (cf. [18]) with constant 1, that is, if g and h are integrable and have integrable transforms, and if $|\hat{g}|\leq\hat{h}$, then $\|g\|_p\leq\|h\|_p$. Applying this with

$$g=(f*\psi_k)\cdot[\phi_\delta(\cdot-t)] \quad\text{and}\quad h=(f*\psi_k)\cdot\phi_\delta$$

yields that

$$\|(f*\psi_k)\cdot[\phi_\delta(\cdot-t)]\|_p \le \|(f*\psi_k)\cdot\phi_\delta\|_p \le \|f|U\|_p \quad \text{for all real } t. \tag{2.24}$$

Letting $k\to\infty$ through a subsequence for which $f*\psi_k$ converges to f almost everywhere, and applying Fatou's theorem then gives the conclusion that

$$\|f\cdot[\phi_\delta(\cdot-t)]\|_p \le \|f|U\|_p \quad \text{for all real } t. \tag{2.25}$$

Since $\phi_\delta \ge 1/2$ on the interval $[-\delta/2, \delta/2)$, the L^p-norms of the restrictions of f to the translates of that interval are all majorized by $2\|f|U\|_p$. This completes the proof of Theorem 1.8.

The counterpart of Theorem 1.9 for Fourier series is already known (cf. [21]). As in the case of Theorem 1.7, it can be lifted to R. Given a noneven index p in the interval $(1, \infty)$, choose a function g in $L^1(-1/2, 1/2]$ with nonnegative Fourier coefficients so that $|g|^p$ is integrable in some neighbourhood of 0 but so that g does not belong to $L^p[-1/2, 1/2)$. Extend g to have period 1 on all of R, and then multiply it by ϕ_1 to get f.

The function f belongs to $L^1(R)$ and has a nonnegative transform. Moreover, $|f|^p$ is integrable in a neighbourhood of 0, but the fact that $\phi_1 \ge 1/2$ on $[-1/2, 1/2)$ implies that the restriction of f to $[-1/2, 1/2)$ does not belong to L^p, because $g \notin L^p[-1/2, 1/2)$. Theorem 1.9 follows since the interval $[-1/2, 1/2]$ can be covered by finitely-many translates of any nonempty open subset of R.

The same lifting procedure that made Theorem 1.7 follow from its periodic counterpart also works for Theorem 1.10. The periodic version of the latter theorem asserts that there is a function that belongs to $L^p[-1/2, 1/2)$ for all finite values of p and has nonnegative Fourier coefficients that belong to l^q only when $q \ge 2$. To get such a function, start with any nonnegative sequence $(d_j)_{j=1}^{\infty}$ that belongs to l^2 but not to l^q for any $q<2$. Then (cf. [23, Chapter V, §8]) the L^2-function represented by the series $\sum_{j=1}^{\infty} d_j e^{3^j 2\pi i x}$ belongs to $L^p[-1/2, 1/2)$ for all finite values of p.

3. Extensions to locally compact abelian groups. Each of the theorems in Section 1 is a special case of a more general statement on some class of locally compact abelian groups. We now specify those statements and outline proofs of them. Our methods do *not* cover the counterparts [15], [5], [17] and [14] of some of the results in Section 1 on many nonabelian groups.

We use the same system for numbering theorems and corollaries as in Section 1. The symbols G and $\hat{G}$ always denote a locally compact abelian group and its dual group, with the Haar measures on these groups normalized for the inversion theorem. Various equivalent definitions of amalgam of L^p and l^q in this setting can be found in [3]. We write the groups operation on G as addition and denote the identity element in G by 0.

THEOREM 3.1. *The following conditions are equivalent for a function in $L^1(G)$ with a nonnegative transform.*

(a) *The function is square-integrable in some neighbourhood of* 0.

(b) *The transform of the function belongs to* $(L^1, l^2)(\hat{G})$.

(c) *The function belongs to* $(L^2, l^\infty)(G)$.

Again (b)⇒(c)⇒(a) without the assumption that the transform of the function is nonnegative; the first of these implications is part of the Hausdorff-Young theorem for amalgams, and the second is obvious. The fact that (a)⇒(b) when the transform is nonnegative has a proof along the lines of [19]. Given a function f that has a nonnegative transform and is integrable near 0, choose a neighbourhood U of 0 so that $\|f|U\|_2<\infty$. Then choose a symmetric neighbourhood V of 0 so that the closure of V is compact and so that $V+V\subset U$. Let ϕ be the convolution of 1_V, the indicator function of V, with itself.

The ϕ is continuous and has compact support. The symmetry of V implies that ϕ can also be written in the form 1_V*1_{-V}, and this means that $\hat{\phi}$ is equal to the square of the modulus of the transform of 1_V, and hence that $\hat{\phi}\geq 0$. It also follows here that $\|\hat{\phi}\|_1=(\|\widehat{1_V}\|_2)^2=(\|1_V\|_2)^2$, and therefore that $\hat{\phi}\in L^1(\hat{G})$. The product $\phi\cdot f$ belongs to $L^2(G)$, so that its transform belongs to $L^2(\hat{G})$. Writing ϕ as the inverse transform of the integrable function $\hat{\phi}$ confirms that the transform of $\phi\cdot f$ is indeed equal to the convolution of the two functions $\hat{\phi}$ and $\hat{f}$. The same reasoning as in Section 2 then shows that this convolution of nonnegative functions can only belong to $L^2(\hat{G})$ if $\hat{f}\in(L^1, l^2)(\hat{G})$.

It is again not essential that the function be integrable. If a distribution on G has a transform that is a nonnegative function, then the distribution must coincide with a measurable function, and the three conditions in Theorem 3.1 are still equivalent. If the transform is a nonnegative measure, μ say, then the conditions (a) and (c) are both equivalent to the requirement that

$$\int_{\hat{G}} \mu(y+\widehat{I_0})^2\,dy<\infty\,. \tag{3.1}$$

If G is compact, then $\hat{G}$ is discrete, and the amalgam $(L^1, l^2)(\hat{G})$ is just $l^2(\hat{G})$ with an equivalent norm. In this case, (L^2, l^∞) also coincides with $L^2(G)$. The conditions (b) and (c) become the requirements that $\hat{f}\in L^2(\hat{G})$ and $f\in L^2(G)$ respectively, and Theorem 3.1 becomes the version of Wiener's theorem proved for compact abelian groups in [19].

If G is discrete, then the singleton set $\{0\}$ is a neighbourhood of 0, and all functions are square-integrable in that neighbourhood. In this case, $\hat{G}$ is compact, and $(L^1, l^2)(\hat{G})$ is just $L^1(G)$ with an equivalent norm. The main assertion of the theorem, that (a)⇒(b) when $\hat{f}\geq 0$, reduces in this case to the statement that if a function on the discrete group G has a nonnegative transform, then that transform must belong to $L^1(\hat{G})$ if it is a function, and the transform must assign only finite mass to $\hat{G}$ if it is a measure.

While these two cases of the theorem seem to contain nothing new, the method of proof also provides estimates for amalgam norms. Combinations of such estimates at various scales can be useful (cf. [2]) even when the groups are compact or discrete.

THEOREM 3.2. *If $\hat{G}$ is neither discrete nor compact, then the amalgam $(L^1, l^2)(\hat{G})$ contains nonnegative functions that are transforms of integrable functions but that do not belong to $L^2(\hat{G})$.*

COROLLARY 3.3. *If G is neither compact nor discrete, then there are functions in $L^1(G)$ that have nonnegative transforms and that belong to $(L^2, l^\infty)(G)$ but that do not belong to $L^2(G)$.*

The assumptions about the groups $\hat{G}$ and G in the two statements above are essential.

The corollary follows from Theorem 3.2 and the Hausdorff-Young theorem for amalgams. We opt to deduce the theorem above from the two statements below. It follows from them in the same way that Theorem 1.2 followed from Theorems 1.4 and 1.5.

THEOREM 3.4. *If G is neither compact nor discrete, then there is a function in $L^2(G)$ that does not belong to $L^4(G)$ but whose transform belongs to $(L^1, l^{4/3})(\hat{G})$.*

THEOREM 3.5. *If G is neither compact nor discrete, then there is an integrable function that belongs to $(L^{4/3}, l^2)(G)$ but whose transform does not belong to $L^4(\hat{G})$.*

The function, g say, in Theorem 3.5 must then also have the property that $\tilde{g}*g \notin L^2(G)$, because the transform of $\tilde{g}*g$ is equal to $|\tilde{g}|^2$. The same method proves both theorems. We used it to prove Theorem 1.4, which is a special case of Theorem 3.4. Here, we will concentrate instead on Theorem 3.5. The results in [14] cover that theorem when $G=R^d$ for some positive integer d. Our method is slightly different.

We use the fact (cf. [20, §2.4]) that G must have an open subgroup of form $H=R^d \times K$, where K is compact. If $d>0$, then we can construct a function with the properties specified in Theorem 3.5 and vanishing off H. To this end, let Φ_δ be the function on R^d that is the tensor product of d copies of the function ϕ_δ that was used in Section 2. Regard Φ_δ as a function on G by composing it with the projection of H onto the factor R^d to lift it to H, and then extending it to be equal to 0 off H.

Amalgams on G can be defined by covering G with disjoint translates, $(I_\beta)_{\beta\in B}$ say, of the set $I=[-1/2, 1/2)^d \times K$, computing L^p-norms on each such translate of I, and then forming the l^q-norm of the resulting function on the index set B. In any case, $\widehat{\Phi_\delta}$ has norm 1 in $L^1(\hat{G})$. For values of $\delta \le 1/2$, the functions Φ_δ are all supported by the set I; it follows (cf. [4]) from this and the fact that the L^1-norms of the transforms of these functions are all equal that

$$\|\widehat{\Phi_\delta}\|_{\infty,1} \le C$$

for all such values of δ, with a constant C that depends only on the dimension d.

Take the functions h_k of Section 2, and redefine them on the dual group $\hat{R}^d$ by applying them to the first coordinate; then extend them to all of $\hat{G}$ as was done for Φ_δ on G. Then $\| h_k \|_{1,\infty} = 1$ for all k. So the product of h_k with the transform of $\Phi_{1/(2j^{1/d})}$ has L^1-norm at most C for all positive integers j and k. In this setting, denote the inverse transform of this product by $F_{j,k}$.

This function is a sum of translates of $\Phi_{1/(2j^{1/d})}$ multiplied by the same scalars as in the formula (2.10). That makes it easy to control the sizes of various amalgam norms of $F_{j,k}$. This time, let $g_j = (1/j) F_{j,j^2}$; then there is a positive constant c so that the norms $\| g_j \|_2$ and $\| g_j \|_{4/3,2}$ all lie between $1/c$ and c. On the other hand,

$$\| \widehat{g_j} \|_1 \to 0 \qquad \text{as} \quad j \to \infty \,. \tag{3.2}$$

Since $\| \widehat{g_j} \|_2$ stays between $1/c$ and c, Hölder's inequality implies as in the proof of Theorem 1.4 that

$$\| \widehat{g_j} \|_4 \to \infty \qquad \text{as} \quad j \to \infty \,. \tag{3.3}$$

This case of Theorem 3.5 then follows from the closed graph theorem. We can also form a series that converges in norm in $L^1(G)$ and $(L^{4/3}, l^2)(G)$ to a function with the properties specified in the theorem.

Suppose now that $d=0$. Then the compact group K must be infinite because G is not discrete. The quotient group G/K must also be infinite because G is not compact. The idea again is to form functions on K with the essential properties of the functions Φ_δ as $\delta \to 0$, and then to take suitable linear combinations of suitable translates of these functions.

The first part of this, on the compact group K, was carried out by the referee of [10] in Lemma 0 of that paper. The second part requires suitable functions on the infinite compact abelian group that is dual to the discrete group G/K. These functions should have the following properties of the functions h_k that were used when $d>0$. First, $\| h_k \|_1 = 1$, next the support of $\widehat{h_k}$ is finite and of cardinality comparable to k, and finally $\widehat{h_k} \geq 1/2$ on a large part of its support.

If the group G/K includes a copy of the integers, simply transfer the functions $\widehat{h_k}$, as defined in Section 2, to G/K. Also do this if G/K has subgroups of arbitrarily large finite order. All that remains is the case where there is a finite upper bound on the orders of the members of G/K. In that case, there is a prime number P for which G/K has subgroups that are products of arbitrarily many copies of the cyclic group Z_P with P elements. Then the inverse transforms of the indicator functions of these subgroups have the desired properties, with the same subgroup being used and yielding the same function h_k for many different values of k. The rest of the proof is essentially the same as in the case where $d>0$.

THEOREM 3.6. *Let $1<p<2$, and let f be an integrable function on G with the properties that $\hat{f} \geq 0$ and $|f|^p$ is integrable in some neighbourhood of 0. Then $\hat{f} \in (L^1, l^{p'})(\hat{G})$.*

The statement above is again proved in the same way as the implication (a)$\Rightarrow$(b) in Theorem 3.1.

THEOREM 3.7. *If G is not discrete, then there is an integrable function f on G with the properties that $\hat{f}\geq 0$ and $f\in(L^\infty, l^q)(\hat{G})$ for all $q>2$ but such that there is no index p for which $|f|^p$ is integrable in some neighbourhood of* 0.

The proof of this splits into cases as in the proof of Theorem 3.5. If the dimension d is positive, then the function used in the proof of Theorem 1.7 on R can be lifted to have the specified properties on the group G. If $d=0$, then the compact open subgroup K must be infinite. One can again use suitable Riesz products and convolution to obtain a suitable function on this subgroup, and then extend that function to all of G by setting it equal to 0 off K.

THEOREM 3.8. *Let p be an even integer or let $p=\infty$. If f is an integrable function on G with the properties that $\hat{f}\geq 0$ and $\|f|U\|_p<\infty$ for some neighbourhood U of* 0, *then* $f\in(L^p, l^\infty)(G)$.

The proof is essentially the same as in the case where $G=R$. If G is discrete, then $L^1(G)$ is included in all the amalgams (L^p,l^q) for which the indices satisfy the condition that $1\leq p, q\leq\infty$. So in this case the conclusion of Theorem 3.8 holds without any special conditions on p or $\|f|U\|_p$.

THEOREM 3.9. *Suppose that G is not discrete, that $p\in(1,\infty)$, and that p is not even. Then there is a function f in $L^1(g)$ with the properties that $\hat{f}\geq 0$ and $\|f|U\|_p<\infty$ for some neighbourhood U of* 0, *but such that every neighbourhood V of* 0 *has a translate $x+V$ for which $\int_{x+V}|f|^p=\infty$.*

As noted in Section 2, the case of this where G is the unit circle was proved in [21]. The case where G is a general infinite compact abelian group was proved in [19]. When G is not compact, consider the open subgroup that was used in the proof of Theorem 3.5. If $d>1$, use the special case of the theorem that was proved in Section 2. Apply the function from that case to the first coordinate in R^d, and then multiply that function on R^d by a suitable compactly supported function to prove the theorem on R^d. Extend that product to G as before. If $d=0$, use the result in [19] to get a suitable function on K, and then extend it to all of G by setting it equal to 0 off K.

THEOREM 3.10. *If G is not discrete, then there is a function that belongs to $(L^p, l^1)(G)$ for all $p<\infty$ and that has a nonnegative transform that belongs to $(L^1, l^q)(\hat{G})$ only when $q\geq 2$.*

Again, this can be lifted from the case where G is compact and infinite. In that case, proceed as in Section 2 with the set $\{3^j\}_{j=1}^\infty$ replaced by any infinite Sidon subset of $\hat{G}$.

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