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CERTAIN ALGEBRAIC SURFACES OF GENERAL TYPE WITH IRREGULARITY ONE AND THEIR CANONICAL MAPPINGS

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Abstract. In this paper, we show the existence of certain algebraic surfaces of general type with irregularity one, and investigate the canonical mappings of these surfaces. Such a surface has a pencil of non-hyperelliptic curves of genus 3 over an elliptic curve, and is obtained as the minimal resolution of a relative quartic hypersurface with at most rational double points as singularities, of the projective plane bundle over an elliptic curve. We use some results on locally free sheaves over elliptic curves by Atiyah and Oda to prove the existence.

1. Introduction. Let S be a minimal nonsingular projective surface defined over $\boldsymbol{C}$. S is said to be canonical if the rational mapping $\Phi_{|K_S|}$ defined by the canonical linear system $|K_S|$ is birational.

In this paper, we show for all values of $p_g(S)\geq 2$ the existence of minimal algebraic surfaces of general type with $K_S^2=3p_g(S)$ and $q(S)=1$, and study their canonical mappings. Note that the case $p_g(S)=1$ was studied by Catanese and Ciliberto [7].

(I) (Castelnuovo-Horikawa's inequality, cf. [5, Théorème 5.5], [12, Lemma 1.1]). If S is a canonical surface, then

$$K_S^2\geq 3p_g(S)-7\,.$$

(II) Castelnuovo obtained canonical surfaces with $K_S^2=3p_g(S)-7$ (cf. [6]). Such a surface S satisfies $q(S)=0$, and with a few exceptions S is birational to a relative quartic hypersurface of a $\boldsymbol{P}^2$-bundle over $\boldsymbol{P}^1$ which has at most rational double points as singularities.

In general, a nonsingular relative quartic hypersurface in a $\boldsymbol{P}^2$-bundle over a nonsingular curve C of genus b satisfies

$$K_S^2=3p_g(S)+7(b-1)\,,\quad q(S)=b\,.$$

We may ask whether a canonical surface S satisfying these equalities is obtained as the minimal resolution of a relative quartic hypersurface with at most rational double points, of a $\boldsymbol{P}^2$-bundle over a nonsingular curve C of genus b. Konno [15, Lemma 3.1, Theorem 3.2] proved that it is the case if $b=1$. Namely, if S is a canonical surface satisfying $K_S^2=3p_g(S)$ and $q(S)=1$, then S is the minimal resolution of a relative quartic

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hypersurface in a P^2-boundle over an elliptic curve.

More precisely, S has a pencil $f: S \to C = \mathrm{Alb}(S)$ whose general fiber is a non-hyperelliptic curve of genus 3. Hence, the direct image $f_*\omega_{S/C}$ of the relative dualizing sheaf $\omega_{S/C} := \omega_S \otimes f^*\omega_C^{-1}$ is a locally free sheaf of rank 3 over C. If we let π: $W := P(f_*\omega_{S/C}) \to C$ to be the P^2-bundle associated to $f_*\omega_{S/C}$, $T \in \mathrm{Pic}(W)$ a tautological divisor with $\pi_*\mathcal{O}_W(T) \cong f_*\omega_{S/C}$, and $D \in \mathrm{Pic}(C)$ a divisor with $\mathcal{O}_C(D) \cong \det f_*\omega_{S/C}$, then there exists a member $S' \in |4T - \pi^*D|$ which has at most rational double points as singularities, and S is the minimal resolution of S' (cf. [15]).

Not all the irreducible relative quartic hypersurfaces in the P^2-bundles over elliptic curves which have at most rational double points as singularities have canonical surfaces as the minimal resolutions of singularities. For example, we have the possibilities $p_g(S) = 1$, 2, 3, and S is not canonical in these cases.

In this paper, we study whether a complete linear system of $\mathcal{O}_W(4T) \otimes \pi^* \det E^\vee$ has members which have at most rational double points as singularities for every locally free sheaf E of rank three over an elliptic curve C, where π: $W := P(E) \to C$ is the P^2-bundle associated to E and T is a tautological divisor with $\pi_*\mathcal{O}_W(T) \cong E$. In particular, we check for all values of $p_g(S) \geq 2$ the existence of minimal algebraic surfaces of general type satisfying $K_S^2 = 3p_g(S)$ and $q(S) = 1$. We then study their canonical mappings $\Phi_{|K_S|}$ including the cases $p_g(S) \leq 3$.

We obtain the following results on the existence of minimal algebraic surfaces with $K_S^2 = 3p_g(S)$ and $q(S) = 1$, using the results about vector bundles over an elliptic curve C by Atiyah [4] and Oda [19].

(1) The case where $f_*\omega_{S/C}$ is isomorphic to the direct sum of three invertible sheaves over C (§3.1): $p_g(S) \geq 3$ is necessary, and conversely, for every integer $N \geq 3$, there exists minimal algebraic surfaces of general type with $p_g(S) = N$, $K_S^2 = 3p_g(S)$ and $q(S) = 1$. (See Theorem 3.1.)

(2) The case where $f_*\omega_{S/C}$ is isomorphic to the direct sum of an invertible sheaf and an indecomposable locally free sheaf of rank 2 over C (§3.2): $p_g(S) \geq 2$ is necessary, and conversely, for every integer $N \geq 2$, there exist minimal algebraic surfaces of general type with $p_g(S) = N$, $K_S^2 = 3p_g(S)$ and $q(S) = 1$. (See Theorems 3.9 and 3.10.)

(3) The case where $f_*\omega_{S/C}$ is indecomposable (§3.3): $p_g(S) \geq 2$ is necessary, and conversely, for every integer $N \geq 2$, there exist minimal algebraic surfaces of general type with $p_g(S) = N$, $K_S^2 = 3p_g(S)$ and $q(S) = 1$. (See Theorem 3.19.)

As for the canonical mappings of the above surfaces, we obtain the following results:

(1) In the case where $f_*\omega_{S/C}$ is the direct sum of three invertible sheaves, if $p_g(S) \geq 6$ holds, then $\Phi_{|K_S|}$ is always birational onto its image with the exception of only one case $f_*\omega_{S/C} \cong L_0^{\oplus 3}$ where L_0 is an invertible sheaf of degree 2 over C.

If $p_g(S) = 5$ and if $f_*\omega_{S/C}$ is not some special locally free sheaf, then $\Phi_{|K_S|}$ is always birational onto its image, too.

If $p_g(S) = 5$ and $f_*\omega_{S/C}$ is some special locally free sheaf, or if $p_g(S) = 4$, then $\Phi_{|K_S|}$ is birational onto its image in most cases. Although there is a possibility of the existence of a surface whose canonical mapping is not birational onto its image, we have not obtained an example of such a surface.

If $p_g(S)=3$, then $\Phi_{|K_S|}$ is a generically finite mapping onto the projective plane whose degree varies according to the isomorphism class of $f_*\omega_{S/C}$. In most cases, the degree of the canonical mapping is 6, 8 or 9.

(2) In the case where $f_*\omega_{S/C}$ is the direct sum of an invertible sheaf and an indecomposable locally free sheaf of rank 2, if $p_g(S)\geq 5$ holds, then $\Phi_{|K_S|}$ is always birational onto its image.

If $p_g(S)=4$, then $\Phi_{|K_S|}$ is birational onto its image in most cases. Although there is a possibility of the existence of a surface whose canonical mapping is not birational onto its image, we have not obtained an example of such a surface.

If $p_g(S)=3$, then $\Phi_{|K_S|}$ is a generically finite mapping onto the projective plane whose degree varies according to the isomorphism class of $f_*\omega_{S/C}$. In most cases, the degree of the canonical mapping is 4, 8 or 9.

If $p_g(S)=2$, then $|K_S|$ is a linear pencil and the genus of a general member is 7.

(3) In the case where $f_*\omega_{S/C}$ is indecomposable, if $p_g(S)\geq 5$ holds, then $\Phi_{|K_S|}$ is always holomorphic and birational onto its image.

If $p_g(S)=4$, then $\Phi_{|K_S|}$ is birational onto its image in most cases. Although there is a possibility of the existence of a surface whose canonical mapping is not birational onto its image, we have not obtained an example of such a surface.

If $p_g(S)=3$, then $\Phi_{|K_S|}$ is a generically finite mapping of degree 8 onto the projective plane in most cases.

If $p_g(S)=2$, then $|K_S|$ is a linear pencil and the genus of a general member is 7.

We obtain some examples of canonical surfaces whose canonical mappings are not holomorphic. Such surfaces do not appear in the cases treated by Ashikaga [2] and Konno [16].

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2. Preliminaries.

Let us mention some results which we need later.

THEOREM 2.1 (cf. Konno [15, Corollary 6.4]). *If S is a canonical surface with $q(S)=1$ and $K_S^2\leq(10/3)\chi(\mathcal{O}_S)$, then a general fiber of the Albanese mapping f: $S\to C:=\mathrm{Alb}(S)$ is a nonsingular curve of genus* 3.

THEOREM 2.2 (cf. Konno [15, Lemma 3.1, and Theorem 3.2]). *Let $f: S\to C$ be a relatively minimal non-hyperelliptic fibration of genus* 3, *where S is a nonsingular surface,* *and C is a nonsingular curve of genus b. Then*

$$(*) \qquad K_S^2 \geq 3\chi(\mathcal{O}_S) + 10(b-1)\,.$$

Let $\pi\colon W := \boldsymbol{P}(f_*\omega_{S/C}) \to C$ *be the* $\boldsymbol{P}^2$*-bundle over* C *defined by the locally free sheaf* $f_*\omega_{S/C}$ *of rank* 3, T *a tautological divisor with* $\pi_*\mathcal{O}_W(T) \cong f_*\omega_{S/C}$, *and* $\psi\colon S \cdots \to W$ *the rational mapping induced by the natural sheaf homomorphism* $f^*f_*\omega_{S/C} \to \omega_{S/C}$. *If the equality holds in* $(*)$, *then* $S' = \psi(S)$ *has at most rational double points as singularities, and we have*

$$\mathcal{O}_W(S') \cong \mathcal{O}_W(4T) \otimes \pi^* \det(f_*\omega_{S/C})^\vee\,,$$

where $(f_*\omega_{S/C})^\vee$ *is the* $\mathcal{O}_C$*-module dual to* $f_*\omega_{S/C}$.

REMARK. The inequality stated in the first half of Theorem 2.2 was proved by Horikawa [13], [14, Proposition 2.1] and Reid [20] in a different way. Konno [16, Theorem 2.1] himself also gave another proof.

PROPOSITION 2.3. *Let* C *be a nonsingular curve of genus* b, *and* E *a locally free sheaf of rank* 3 *over* C. *Let* $\pi\colon W := \boldsymbol{P}(E) \to C$ *be the* $\boldsymbol{P}^2$*-bundle over* C *associated to* E, T *a tautological divisor with* $\pi_*\mathcal{O}_W(T) \cong E$, *and* $D \in \mathrm{Div}(C)$ *a divisor on* C *such that* $\mathcal{O}_C(D) \cong \det E$. *If* $|4T - \pi^*D|$ *has an irreducible member* S' *with at most rational double points as singularities, then the following equalities hold for the minimal resolution* $\nu\colon S \to S'$ *of singularities.*

$$K_S^2 = 3\deg E + 16(b-1)\,,$$

$$p_g(S) = \deg E + 3(b-1) + \dim H^0(C, E^\vee)\,,$$

$$q(S) = b + \dim H^0(C, E^\vee)\,.$$

Furthermore, if we denote $f := \pi \circ \nu$, *then we have* $f_*\omega_{S/C} \cong E$.

PROOF. We have $\omega_{S'}^2 = \omega_S^2$, $p_g(S') = p_g(S)$ and $q(S') = q(S)$ by the hypothesis that S' has at most rational double points as singularities. Since $\omega_{S'} \cong \mathcal{O}_{S'} \otimes_{\mathcal{O}_W} \mathcal{O}_W(T + \pi^*K_C)$ by the adjunction formula, and since $T^3 - (\deg E)T^2F = 0$, we have $\omega_{S'}^2 = 3\deg E + 16(b-1)$.

By considering the cohomology long exact sequence induced by the exact sequence

$$0 \to \omega_W \to \mathcal{O}_W(T + \pi^*K_C) \to \omega_{S'} \to 0\,,$$

we obtain the equality for $p_g(S)$ and $q(S)$.

Since S' has at most rational double points as singularities, $\nu^*\omega_{S'/C} \cong \omega_{S/C}$ and $\nu_*\mathcal{O}_S \cong \mathcal{O}_{S'}$ hold. Since $\omega_{S'/C} \cong \mathcal{O}_W(T) \otimes_{\mathcal{O}_W} \mathcal{O}_{S'}$ by the adjunction formula, we have

$$f_*\omega_{S/C} \cong \pi_*\nu_*\nu^*\omega_{S'/C} \cong \pi_*\omega_{S'/C} \cong \pi_*(\mathcal{O}_W(T) \otimes_{\mathcal{O}_W} \mathcal{O}_{S'})\,.$$

Since we have the long exact sequence

$$\begin{aligned} 0 \to \pi_*(\mathcal{O}_W(-3T) \otimes_{\mathcal{O}_W} \pi^* \det E) &\to \pi_*\mathcal{O}_W(T) \\ &\to \pi_*(\mathcal{O}_W(T) \otimes_{\mathcal{O}_W} \mathcal{O}_{S'}) \to R^1\pi_*(\mathcal{O}_W(-3T) \otimes_{\mathcal{O}_W} \pi^* \det E)\,, \end{aligned}$$

and since $R^j\pi_*(\mathcal{O}_W(-3T)\otimes_{\mathcal{O}_W}\pi^*\det E)\cong(R^j\pi_*\mathcal{O}_W(-3T))\otimes_{\mathcal{O}_C}\det E=0$ for $j=0$, 1, we obtain $E=\pi_*\mathcal{O}_W(T)\cong\pi_*(\mathcal{O}_W(T)\otimes_{\mathcal{O}_W}\mathcal{O}_{S'})$, and hence $f_*\omega_{S/C}\cong E$. q.e.d.

REMARK. By the last assertion of Proposition 2.3, we see that two different $\boldsymbol{P}^2$-bundles do not contain the same surfaces.

THEOREM 2.4 (cf. Atiyah [4, Theorem 5, Theorem 7 and Corollary, Theorem 9], Oda [19, Theorem 1.2]). *Let C be an elliptic curve and $\mathcal{E}_C(r, d)$ $(r, d\in\boldsymbol{Z})$ the set of isomorphism classes of indecomposable locally free sheaves of rank r and degree d over C.*

(1) *If $(r, d)=1$, and if we fix any isogeny $\varphi: \tilde{C}\to C$ of degree r, we have a bijection*

$$\{L_0\in\operatorname{Pic}(\tilde{C})\mid\deg L_0=d\}\ni L_0\mapsto\varphi_*L_0\in\mathcal{E}_C(r, d)\,.$$

Denote $G=\ker\varphi$, and let T_σ be the translation by $\sigma\in G$ on $\tilde{C}$. Then we get

$$\varphi^*\varphi_*L_0\cong\bigoplus_{\sigma\in G}T_\sigma^*L_0\,.$$

(2) *For any $r\in\boldsymbol{N}$, there exists a unique $F_r\in\mathcal{E}_C(r, 0)$ such that $H^0(C, F_r)\neq 0$. F_r is a successive extension of $\mathcal{O}_C$, and $F_r\cong S^{r-1}(F_2)$ holds. Furthermore,* $\dim H^i(C, F_r)=1$ $(i=0, 1)$. *We have the following bijective mapping for $m\in\boldsymbol{Z}$:*

$$\{L_0\in\operatorname{Pic}(C)\mid\deg L_0=m\}\ni L_0\mapsto F_r\otimes_{\mathcal{O}_C}L_0\in\mathcal{E}_C(r, rm)\,.$$

REMARK. Although not necessary in this paper, we have the following in general: If $(r, d)=h$, then $\mathcal{E}_C(r/h, d/h)\ni F'\mapsto F'\otimes F_h\in\mathcal{E}_C(r, d)$ is a bijective mapping.

We use the following lemma in §3.2 and §3.3:

LEMMA 2.5. *Let C be an elliptic curve, $\mu: Y=\boldsymbol{P}(F_2)\to C$ the ruled surface associated to F_2, and $C'\subset Y$ the unique section of μ with $\mu_*\mathcal{O}_Y(C')\cong F_2$. For any point $p\in C$ and for any positive integer i, we have* $\operatorname{Bs}|iC'+\Gamma_p|=\{y_0\}$, *where $\Gamma_p:=\mu^{-1}(p)$ and $y_0:=C'\cap\Gamma_p$. Furthermore, general members of $|iC'+\Gamma_p|$ are nonsingular at y_0, and all the members which are nonsingular have the same tangent at y_0. If i and j are positive integers with $i\neq j$, then a nonsingular member of $|iC'+\Gamma_p|$ and a nonsingular member of $|jC'+\Gamma_p|$ have different tangents at y_0.*

PROOF. We have $\operatorname{Bs}|iC'+\Gamma_p|\subset C'\cup\Gamma_p$. Since $\dim\operatorname{Im}\{H^0(Y, \mathcal{O}_Y(iC'+\Gamma_p))\to H^0(\Gamma_p, \mathcal{O}_{\Gamma_p}(iC'))\}=i$, and since $\dim H^0(\Gamma_p, \mathcal{O}_{\Gamma_p}(iC'))=i+1$, there exists at most one base point on Γ_p. On the other hand, since $\dim H^0(\Gamma_p, \mathcal{O}_Y(iC'+\Gamma_p))=i+1\neq i=\dim H^0(Y, \mathcal{O}_Y((i-1)C'+\Gamma_p))$, C' is not a fixed component of $|iC'+\Gamma_p|$. Furthermore, since $(iC'+\Gamma_p)C'=1$ and $N_{C'/Y}\cong\mathcal{O}_{C'}$, only $y_0=C'\cap\Gamma_p$ is the base point of $|iC'+\Gamma_p|$ lying on C'. Hence, we obtain $\operatorname{Bs}|iC'+\Gamma_p|=\{y_0\}$. Since $(C'+\Gamma_p)C'=1$, general members of $|iC'+\Gamma_p|$ are nonsingular at y_0.

Let $M\in|iC'+\Gamma_p|$ be a nonsingular member. If we consider the cohomology long exact sequence induced by the exact sequence of sheaves

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(M) \to \mathcal{O}_M(M) \to 0\,,$$

we get $\dim H^0(M, \mathcal{O}_M(M)) = i+1$, and $\dim \mathrm{Im}\{H^0(Y, \mathcal{O}_Y(M)) \to H^0(M, \mathcal{O}_M(M))\} = i$. The subsystem of the complete linear system of $M|_M$ corresponding to the image of $H^0(Y, \mathcal{O}_Y(M)) \to H^0(M, \mathcal{O}_M(M))$ may be regarded as the complete linear system of $M|_M - y_0$, and its dimension is $i-1$. On the other hand, since $\deg \omega_M = (iC' + \Gamma_p)((i-2)C' + \Gamma_p) = 2i-2$, the genus of M is equal to i. Since $M|_M - 2y_0 \sim K_M$ by the adjunction formula, the complete linear system of $M|_M - 2y_0$ is also $(i-1)$-dimensional. Hence, y_0 is the base point of the complete linear system of $M|_M - y_0$, and the intersection multiplicity of any nonsingular member $M' \in |iC' + \Gamma_p|$ with M at y_0 is at least two, i.e., M and M' have the same tangent.

The last assertion can be proved in the same way as above. q.e.d.

The following lemma is trivial:

LEMMA 2.6. *Let X be a complete variety, and D an effective divisor on X. Assume* $\dim X \geq 2$, *and* $\dim |D| \geq 2$. *If $D_1 \cap D_2 \neq \mathrm{Bs}\,|D|$ for any distinct members $D_1, D_2 \in |D|$, then $|D|$ is not composite with a pencil. In particular, $|D|$ is not composite with a pencil if one of the following holds*:

(i) $\mathrm{Bs}\,|D| = \emptyset$ *and* $D^n > 0$,

(ii) $\mathrm{Bs}\,|D| = \emptyset$ *and the dimension of any component of* $\mathrm{Bs}\,|D|$ *is less than* $\dim X - 2$.

3. Existence and birationality. By Theorems 2.1 and 2.2, to classify canonical surfaces with $K_S^2 = 3p_g(S)$ and $q(S) = 1$, we need to have a necessary and sufficient condition for the complete linear system $|4T - \pi^* D|$, on the $\boldsymbol{P}^2$-bundle $W = \boldsymbol{P}(E)$ associated to a locally free sheaf E of rank 3 over an elliptic curve C, to have irreducible members with at most rational double points as singularities, where T is a tautological divisor on W with $\pi_* \mathcal{O}_W(T) \cong E$, and $D \in \mathrm{Div}(C)$ satisfies $\mathcal{O}_C(D) \cong \det E$. We should then choose those members whose minimal resolutions are canonical.

Locally free sheaves of rank 3 over an elliptic curve C are expressed uniquely up to order as direct sums of indecomposable locally free sheaves (cf. [4]). Hence we should consider the following three cases:

(1) E is the direct sum of three invertible sheaves.

(2) E is the direct sum of an invertible sheaf and an indecomposable locally free sheaf of rank 2.

(3) E is indecomposable.

DEFINITION. Let $\pi: W \to C$ be the $\boldsymbol{P}^2$-bundle over an elliptic curve C associated to a locally free sheaf E of rank 3, T a tautological divisor with $\pi_* \mathcal{O}_W(T) \cong E$, and $D \in \mathrm{Div}(C)$ a divisor with $\mathcal{O}_C(D) \cong \det E$. We say that E satisfies the *condition* (A) if $|4T - \pi^* D|$ has a member S' satisfying the following conditions:

(i) S' has at most rational double points as singularities,

(ii) The minimal resolution S of S' is of general type,

(iii) S satisfies $K_S^2 = 3p_g(S)$ and $q(S) = 1$.

REMARK. We only have to consider the locally free sheaves E with $H^0(C, E^\vee) = 0$ by Proposition 2.3. If E satisfies the condition (A), then $\chi(\mathcal{O}_S) = \deg E > 0$. Furthermore, by Fujita [8, (1.2) Proposition], we only have to consider locally free sheaves such that any quotient locally free sheaf has nonnegative degree.

3.1. The case where E is a direct sum of three invertible sheaves. Let L_0, L_1, L_2 be invertible sheaves over an elliptic curve C such that $E \cong L_0 \oplus L_1 \oplus L_2$, and denote $d_i := \deg L_i$ $(i = 0, 1, 2)$. Furthermore, let $\pi\colon W \to C$ be the $\boldsymbol{P}^2$-bundle associated to E, and T a tautological divisor with $\pi_* \mathcal{O}_W(T) \cong E$. In §3.1, we prove the existence of a surface S of general type with $K_S^2 = 3p_g(S)$, $q(S) = 1$ and $p_g(S) = N$ for any integer $N \geq 3$ by obtaining such a locally free sheaf E of rank three satisfying the condition (A) (Theorem 3.1). We then study the canonical mapping of the surfaces thus obtained. The results about the canonical mappings are stated in Corollaries 3.3 and 3.4, and Propositions 3.5, 3.7 and 3.8.

3.1.1. Existences. We may assume $d_0 \leq d_1 \leq d_2$. We only have to consider the case $d_0 \geq 0$, $d_1 \geq 0$ and $d_2 > 0$ by the remark immediately before §3.1.

THEOREM 3.1. *Let $\pi\colon W = \boldsymbol{P}(E) \to C$ be the $\boldsymbol{P}^2$-bundle over an elliptic curve C associated to $E \cong L_0 \oplus L_1 \oplus L_2$, T a tautological divisor with $\pi_* \mathcal{O}_W(T) \cong E$, and $D \in \mathrm{Div}(C)$ satisfies $\mathcal{O}_C(D) \cong \det E$. Denote $d_i := \deg L_i$ $(i = 0, 1, 2)$, and suppose $0 \leq d_0 \leq d_1 \leq d_2$ and $d_2 > 0$. Then the locally free sheaf E satisfies the condition* (A) *if and only if the following* (1), (2) *and* (3) *hold.*

(1) *One of the following* (i), (ii) *and* (iii) *holds*:
  (i) $d_0 + d_2 < 3d_1$,
  (ii) $L_0 \otimes L_2 \cong L_1^{\otimes 3}$,
  (iii) $d_0 = d_1$ *and at least one of* $L_1^{\otimes 2}$, $L_0 \otimes L_1$, $L_0^{\otimes 2}$ *and* $L_0^{\otimes 3} \otimes L_1^{-1}$ *is isomorphic to* L_2.
(2) *One of the following* (i), (ii) *and* (iii) *holds*:
  (i) $d_1 < 2d_0$,
  (ii) $L_1 \cong L_0^{\otimes 2}$,
  (iii) $2d_0 = d_1 = d_2$ *and* $L_2 \cong L_0^{\otimes 2}$.
(3) *If* $d_0 = d_1 = d_2 = 1$ *holds, then one of* L_0, L_1, L_2 *is not isomorphic to the others.*

PROOF. We can choose $X_i \in H^0(W, \mathcal{O}_W(T) \otimes \pi^* L_i^{-1})$ $(i = 0, 1, 2)$ which give homogeneous coordinates on each fiber of π. Then any $\Psi \in H^0(W, \mathcal{O}_W(4T) \otimes \pi_* \det E^\vee) \cong H^0(C, S^4 E \otimes \det E^\vee)$ can be written as

$$\Psi = \sum_{\substack{i, j \geq 0 \\ i + j \leq 4}} \psi_{ij} X_0^{4-i-j} X_1^i X_2^j, \quad \psi_{ij} \in H^0(C, L_0^{\otimes(3-i-j)} \otimes L_1^{\otimes(i-1)} \otimes K_2^{\otimes(j-1)}).$$

In the same way as in the proof of Claim III in [3], we can show that E does not satisfy the condition (A) when one of (1) and (2) does not hold. If (3) does not hold, then $\mathrm{Bs}|4T-\pi^*D|$ consists of a fiber of π.

From now on, we assume that (1), (2) and (3) hold.

(I) Let us look at the case where $3d_0>d_1+d_2$ or $L_0^{\otimes 3}\cong L_1\otimes L_2$. (If, moreover, $d_0=d_1$ and $L_0^{\otimes 3}\cong L_1\otimes L_2$ hold, we may assume $L_1^{\otimes 3}\cong L_0\otimes L_2$.)

Clearly, $|4T-\pi^*D|$ has no base point if and only if $3d_0-d_1-d_2\neq 1$. If $3d_0-d_1-d_2=1$ and $-d_0+3d_1-d_2\geq 2$, then $\mathrm{Bs}|4T-\pi^*D|$ consists of one point. Assume $3d_0-d_1-d_2=-d_0+3d_1-d_2=1$ and $-d_0-d_1+3d_2\geq 2$. If $L_0^{\otimes 3}\otimes L_1^{-1}\otimes L_2^{-1}\not\cong L_0^{-1}\otimes L_1^{\otimes 3}\otimes L_2^{-1}$, then $\mathrm{Bs}|4T-\pi^*D|$ consists of two points. If $L_0^{\otimes 3}\otimes L_1^{-1}\otimes L_2^{-1}\cong L_0^{-1}\otimes L_1^{\otimes 3}\otimes L_2^{-1}$, then $\mathrm{Bs}|4T-\pi^*D|$ is a line contained in some fiber of π. Next assume $3d_0-d_1-d_2=-d_0+3d_1-d_2=-d_0-d_1+3d_2=1$, i.e., $d_0=d_1=d_2=1$. If L_0, L_1 and L_2 are pairwise different, $\mathrm{Bs}|4T-\pi^*D|$ consists of three points. If two of L_0, L_1 and L_2 are isomorphic, then $\mathrm{Bs}|4T-\pi^*D|$ consists of a point and a line contained in some fiber of π. We can show that a general member of $|4T-\pi^*D|$ is nonsingular at any point of $\mathrm{Bs}|4T-\pi^*D|$ in any case above by considering the local equation. Clearly, $|4T-\pi^*D|$ is not composite with a pencil in any case above by Lemma 2.6. Hence a general member of $|4T-\pi^*D|$ is irreducible and nonsingular by Bertini's theorem.

(II) Let us look at the case where $3d_0<d_1+d_2$ or ($3d_0=d_1+d_2$ and $L_0^{\otimes 3}\not\cong L_1\otimes L_2$). We have $Z_0\subset\mathrm{Bs}|4T-\pi^*D|$, where Z_0 is a curve defined by $X_1=X_2=0$.

If $-d_0+3d_1-d_2\geq 2$, or $L_1^{\otimes 3}\cong L_0\otimes L_2$, then we have $\mathrm{Bs}|4T-\pi^*D|=Z_0$. If $-d_0+3d_1-d_2=1$ and $-d_0-d_1+3d_2\geq 2$, then $\mathrm{Bs}|4T-\pi^*D|$ consists of Z_0 and a point. If $-d_0+3d_1-d_2=-d_0-d_1+3d_2=1$ and $L_0^{-1}\otimes L_1^{\otimes 3}\otimes L_2^{-1}\not\cong L_0^{-1}\otimes L_1^{-1}\otimes L_2^{\otimes 3}$, then $\mathrm{Bs}|4T-\pi^*D|$ consists of Z_0 and two points. If $-d_0+3d_1-d_2=-d_0-d_1+3d_2=1$ and $L_0^{-1}\otimes L_1^{\otimes 3}\otimes L_2^{-1}\cong L_0^{-1}\otimes L_1^{\otimes 3}\otimes L_2^{-1}$, then $\mathrm{Bs}|4T-\pi^*D|$ consists of Z_0 and a line contained in some fiber of π. If $-d_0+3d_1-d_2=0$ and $L_1^{\otimes 3}\not\cong L_0\otimes L_2$, then we must have $2d_0=2d_1=d_2$, and $\mathrm{Bs}|4T-\pi^*D|=Z_0\cup Z_1$, where Z_1 is the curve defined by $X_0=X_2=0$.

We can show that a general member of $|4T-\pi^*D|$ is nonsingular at the base points which are not contained in Z_0 when $Z_1\not\subset\mathrm{Bs}|4T-\pi^*D|$ holds by considering the local equations. Hence it is sufficient to look at the multiplicity of a general member of $|4T-\pi^*D|$ at Z_0 when $Z_1\not\subset\mathrm{Bs}|4T-\pi^*D|$, or at $Z_0\cup Z_1$ when $Z_1\subset|4T-\pi^*D|$.

Let us look at the case where $2d_0>d_2$ or ($L_0^{\otimes 2}\cong L_2$ and $d_0<d_1$). (When $L_0^{\otimes 2}\cong L_2$, if we assume $d_1=d_2$ and $L_0^{\otimes 2}\not\cong L_1$, further, interchange L_1 and L_2 and regard this case as the case $L_0^{\otimes 2}\not\cong L_2$. Hence, we may assume $L_0^{\otimes 2}\cong L_1$ when $L_0^{\otimes 2}\cong L_2$ and $d_1=d_2$.) In this case, we have $Z_1\not\subset\mathrm{Bs}|4T-\pi^*D|$. Since we have $H^0(C, L_0^{\otimes 2}\otimes L_2^{-1})\neq 0$ and $H^0(C, L_0^{\otimes 2}\otimes L_1^{-1})\neq 0$, a general member of $|4T-\pi^*D|$ is nonsingular at Z_0 except in the case where $2d_0-d_1=2d_0-d_2=1$ and $L_1\cong L_2$ hold. In this case, we can show that any general member of $|4T-\pi^*D|$ has a rational double point of type A_1 on Z_0 by considering the local equation.

Let us look at the case where $2d_0<d_2$ or ($2d_0=d_2$, $d_0<d_1$ and $L_0^{\otimes 2}\not\cong L_2$). In this case, the coefficients of X_0^4 and $X_0^3X_1$ are 0, and $Z_1\not\subset\mathrm{Bs}|4T-\pi^*D|$ holds. If $2d_0=d_1$,

then we have $L_0^{\otimes 2} \cong L_1$ by the assumption of the theorem, and the coefficient of $X_0^3 X_2$ is a constant. Hence a general member is nonsingular at Z_0. Assume $d_1 < 2d_0$ holds, and that $\Psi \in H^0(W, \mathcal{O}_W(4T) \otimes \pi^* \det E^\vee)$ is general. Let $p \in C$ be one of the points with $\psi_{01}(p)=0$ for $0 \neq \psi_{01} \in H^0(C, L_0^{\otimes 2} \otimes L_1^{-1})$, and t a local coordinate around p. Furthermore, let $q \in W$ be a point with $t = X_1 = X_2 = 0$, and denote $x_1 := X_1/X_0$ and $x_2 := X_2/X_0$. Then Ψ can be written as

$$\begin{aligned}\Psi &= tx_2 + \psi_{20}x_1^2 + \psi_{11}x_1x_2 + \psi_{02}x_2^2 + \cdots \\ &= x_2(t + \psi_{11}x_1 + \psi_{02}x_2 + \cdots) + \psi_{20}x_1^2 + \psi_{30}x_1^3 + \psi_{40}x_1^4 ,\end{aligned}$$

around q. The equation $\Psi = 0$ gives a rational double point of type A_1 at q except in the case

$$L_0 \otimes L_1 \otimes L_2^{-1} \cong L_0^{\otimes 2} \otimes L_1^{-1}, \quad \text{and} \quad d_0 + d_1 - d_2 = 2d_0 - d_1 = 1 .$$

In this case, $\psi_{20} = c't$ holds around q for some constant $c' \in \boldsymbol{C}$, and the equation $\Psi = 0$ gives a rational double point of type A_2 at q. If $2d_0 = 2d_1 = d_2$, we can show that a general member of $|4T - \pi^*D|$ has at most rational double points of tyep A_1 on $Z_0 \cup Z_1$ in the same way as above.

Let S_1 and S_2 be general members of $|4T - \pi^*D|$, and F a general fiber of π. Furthermore, denote $q_i := Z_i \cap F$ for $i = 0, 1$. We can show that the intersection multiplicity of $S_1|_F$ and $S_2|_F$ at q_0 in F is at most two. When $Z_1 \subset \mathrm{Bs}\,|4T - \pi^*D|$, we can also show that the intersection multiplicity of $S_1|_F$ and $S_2|_F$ at q_1 in F is at most two. Since $S_1|_F$ and $S_2|_F$ are quartic curves, they have other intersection points. Therefore, we see that $|4T - \pi^*D|$ is not composite with a pencil by Lemma 2.6. Hence a general member of $|4T - \pi^*D|$ is irreducible and nonsingular by Bertini's theorem. q.e.d.

3.1.2. The canonical mappings. In this section, we consider the canonical mappings of those surfaces whose existences were shown in §3.1.1.

LEMMA 3.2. *Let L_0, L_1, L_2 be invertible sheaves over an elliptic curve C, and denote $d_i := \deg L_i$, $(i = 0, 1, 2)$. Assume that L_0, L_1, L_2 satisfy the conditions of Theorem 3.1. If $\pi\colon W := \boldsymbol{P}(E) \to C$ is the $\boldsymbol{P}^2$-bundle over C associated to $E := L_0 \oplus L_1 \oplus L_2$, and T is a tautological divisor such that $\pi_*\mathcal{O}_W(T) \cong E$, then $\Phi_{|T|}$ is birational onto its image when one of the following holds.*

(i) $d_0 + d_1 + d_2 \geq 7$.

(ii) $(d_0, d_1, d_2) = (1, 2, 3)$.

(iii) $(d_0, d_1, d_2) = (2, 2, 2)$ *and one of L_0, L_1, L_2 is not isomorphic to the others.*

(iv) $(d_0, d_1, d_2) = (1, 2, 2)$ *and $L_1 \not\cong L_2$.*

PROOF. If F is a general fiber of π, we have $H^1(W, \mathcal{O}_W(T - F)) = 0$. Hence the restriction mapping $H^0(W, \mathcal{O}_W(T)) \to H^0(F, \mathcal{O}_F(T))$ is surjective, and the restriction of $\Phi_{|T|}$ to F gives an isomorphism of F onto its image.

Let $X_i \in H^0(W, \mathcal{O}_W(T) \otimes \pi^* L_i^{-1})$ $(i = 0, 1, 2)$ be as in the proof of Theorem 3.1. Any $\Psi \in H^0(W, \mathcal{O}_W(T))$ can be written as

$$\Psi=\psi_0X_0+\psi_1X_1+\psi_2X_2\,, \qquad \psi_i\in H^0(C, L_i) \quad (i=0, 1, 2)\,.$$

We can easily prove that there exists a Zariski open subset of W such that the restriction of $\Phi_{|T|}$ on it gives an isomorphism onto the image under the assumption of the lemma. q.e.d.

COROLLARY 3.3. *The canonical mapping of any surface S, whose existence is guaranteed by Theorem* 3.1 *and the condition* (A), *is a birational morphism if one of the following holds.*

(i) $d_0+d_1+d_2\geq 7$ *and* $d_0\geq 2$,

(ii) $(d_0, d_1, d_2)=(2, 2, 2)$, *and one of* L_0, L_1, L_2 *is not isomorphic to the others.*

PROOF. Let the notation be as in Proposition 2.3. Since $\omega_{S'}\cong\mathcal{O}_{S'}(T)$, and since $H^i(W, \omega_W)=0$, for $i=0$, 1, we have $H^0(W, \mathcal{O}_W(T))\cong H^0(S', \omega_{S'})$. Since S' has at most rational double points as singularities, we have $\Phi_{|K_S|}=\psi\circ\Phi_{|T|}$, where $\psi\colon S\to S'$ is a minimal resolution. Clearly, S' has nonempty intersection with the Zariski open subset of W appearing in the proof of Lemma 3.2, and hence the birationality follows from Lemma 3.2. Since $d_0\geq 2$, we have $\mathrm{Bs}\,|T|=\emptyset$, and hence $\mathrm{Bs}\,|K_S|=\emptyset$ holds. q.e.d.

COROLLARY 3.4. *The canonical mapping of any surface S, whose existence is guaranteed by Theorem* 3.1 *and the condition* (A), *is birational onto its image but is not a morphism, and its image is non-normal, if one of the following holds*:

(i) $(d_0, d_1, d_2)=(1, 2, 5)$,

(ii) $(d_0, d_1, d_2)=(1, 2, 4)$,

(iii) $(d_0, d_1, d_2)=(1, 2, 3)$,

(iv) $(d_0, d_1, d_2)=(1, 2, 2)$ *and* $L_1\not\cong L_2$.

PROOF. We can show the birationality of $\Phi_{|K_S|}$ as in the proof of Corollary 3.3.

In the rest of the proof, we use our notation in Theorem 3.1. By considering $\mathrm{Bs}\,|T|$ and $\mathrm{Bs}\,|4T-\pi^*D|$, we see that $|K_S|$ has only one base point q_0. The restriction of $|T|$ to the fiber F_0 containing q_0 may be regarded as a subsystem of the complete linear system of $\mathcal{O}_{\boldsymbol{P}^2}(1)$ consisting of all lines going through q_0. Each line of this system intersects the fiber $\mathscr{F}$ of S at four points, one of which is q_0. Hence we have $\deg(\Phi_{|K_S|}|_{\mathscr{F}})=3$, and the canonical image of S is non-normal by Zariski's main theorem. q.e.d.

PROPOSITION 3.5. *In the notation of Lemma* 3.2, *assume* $d_i=2$ $(i=0, 1, 2)$ *and* $L_0\cong L_1\cong L_2$. *Then the canonical mapping of a general member of* $|4T-\pi^*D|$ *is a morphism of degree* 2 *onto the image, where* $D\in\mathrm{Div}(C)$ *satisfies* $\mathcal{O}_C(D)\cong\det E$.

PROOF. If we denote $\nu:=\Phi_{|L_0|}\colon C\to\boldsymbol{P}^1$, we have $L_0\cong\nu^*\mathcal{O}_{\boldsymbol{P}^1}(1)$, and hence $E\cong\nu^*(\mathcal{O}_{\boldsymbol{P}^1}(1)^{\oplus 3})$. Therefore, if we denote $\pi_0\colon W_0:=\boldsymbol{P}(\mathcal{O}_{\boldsymbol{P}^1}(1)^{\oplus 3})\to\boldsymbol{P}^1$, we have the following commutative diagram:

$$\begin{tikzcd} W \arrow[r, "\tilde{\nu}"] \arrow[d, "\pi"'] & W_0 \arrow[d, "\pi_0"] \\ C \arrow[r, "\nu"'] & \boldsymbol{P}^1 \end{tikzcd}$$

Let T_0 be a tautological divisor with $\pi_{0*}\mathcal{O}_{W_0}(T_0)\cong\mathcal{O}_{\boldsymbol{P}^1}(1)^{\oplus 3}$. We have $\tilde{\nu}^*T_0\sim T$, and both $H^0(W, \mathcal{O}_W(T))$ and $H^0(W_0, \mathcal{O}_{W_0}(T_0))$ are 6-dimensional. Hence, we get $\Phi_{|T|}=\Phi_{|T_0|}\circ\tilde{\nu}$. Since $\Phi_{|T_0|}: W_0 \hookrightarrow \boldsymbol{P}^5$ is an embedding we have $\deg\Phi_{|T|}=2$.

Since $\dim|4T-\pi^*D|=\dim|4T_0-\pi_0^*D_0|$, and since $\mathrm{Bs}|T|=\varnothing$, the canonical mapping of a general member of $|4T-\pi^*D|$ is a morphism of degree 2 onto the image. q.e.d.

REMARK. W_0 in the proof of Proposition 3.5 is isomorphic to $\boldsymbol{P}^2\times\boldsymbol{P}^1$. Let $S\in|4T-\pi^*D|$ be a nonsingular member. If S_0 is the image of S in W_0, and if $r: W_0\to\boldsymbol{P}^2$ is a natural projection, we see that $K_{S_0}^2=-7$ holds, and that $r|_{S_0}: S_0\to\boldsymbol{P}^2$ is a birational morphism by easy caluculations. Hence, $r|_{S_0}$ is the blowing-up at sixteen points of $\boldsymbol{P}^2$, and maps each fiber of $S_0\to\boldsymbol{P}^1$ onto a plane quartic curve birationally. Therefore, the surfaces in Proposition 3.5 are obtained in another way as follows:

Let B_1, B_2, B_3, $B_4\subset\boldsymbol{P}^2$ be nonsingular quartic curves intersecting each other at sixteen points $A_1,\dots,A_{16}$ transversally. Let $\xi: X\to\boldsymbol{P}^2$ be the blowing-up at $A_1,\dots,A_{16}$, and $\tilde{B}_j$ the proper transform of B_j ($j=1, 2, 3, 4$), and denote $\mathcal{E}_i:=\xi^{-1}(A_i)$. We have $\sum_{j=1}^4\tilde{B}_j\sim 16\xi^*H-4\sum_{i=1}^{16}\mathcal{E}_i$, where $H\subset\boldsymbol{P}^2$ is a line. Let $h: S\to X$ be a double covering branched along $\sum_{j=1}^4\tilde{B}_j$. Then $K_S\sim h^*(5\xi^*H-\sum_{i=1}^{16}\mathcal{E}_i)$ holds, and we have $K_S^2=18$. On the other hand, we have $p_g(S)=\dim H^0(X, \mathcal{O}_X(5\xi^*H-\sum_{i=1}^{16}\mathcal{E}_i))=\dim|5\xi^*H-\sum_{i=1}^{16}\mathcal{E}_i|+1$. Since $\dim|5\xi^*H-\sum_{i=1}^{16}\mathcal{E}_i|$ is equal to the dimension of the subsystem of $|5H|$ which consists of all the quintic curves going through $A_1,\dots,A_{16}$, we obtain $\dim|5\xi^*H-\sum_{i=1}^{16}\mathcal{E}_i|=\dim|5H|-15=5$ by using the Caylay-Bacharach theorem (cf., e.g., [10]), and hence $p_g(S)=6$. Since

$$\chi(\mathcal{O}_S)=\frac{1}{2}\left(8\xi^*H-2\sum_{i=1}^{16}\mathcal{E}_i\right)\left(5\xi^*H-\sum_{i=1}^{16}\mathcal{E}_i\right)+2\chi(\mathcal{O}_{\boldsymbol{P}^2})=6\,,$$

we have $q(S)=1$.

LEMMA 3.6. *Let L_0, L_1, L_2 be invertible sheaves over an elliptic curve C, $\pi: W:=\boldsymbol{P}(E)\to C$ be $\boldsymbol{P}^2$-bundle associated to the locally free sheaf $E:=L_0\oplus L_1\oplus L_2$, and T a tautological divisor with $\pi_*\mathcal{O}_W(T)\cong E$, and denote $d_i:=\deg L_i$ ($i=0, 1, 2$). If L_0, L_1 and L_2 satisfy one of the following* (i) *and* (ii), *then* $\deg\Phi_{|T|}=2$ *holds*:

(i) $(d_0, d_1, d_2)=(1, 2, 2)$ *and* $L_0^{\otimes 2}\cong L_1\cong L_2$.

(ii) $(d_0, d_1, d_2)=(1, 1, 2)$.

PROOF. In the case (i), let $p\in C$ be a point with $L_0\cong\mathcal{O}_C(p)$. There exists a point $q\in\pi^{-1}(p)$ with $\mathrm{Bs}|T|=\{q\}$. Denote $E':=\mathcal{O}_C\oplus L_1\oplus L_1$ and $F':=(L_1\oplus L_1)\otimes\mathcal{O}_p$. We have the following commutative diagram as a special case of Maruyama [17, Chapter 1]:

$$\begin{tikzcd} \bar{W} \arrow[r, "\phi'"] \arrow[d, "\phi"'] & W' \arrow[r, "\Phi"] \arrow[d, "\pi'"] & W_0 \arrow[d, "\pi_0"] \\ W \arrow[r, "\pi"'] & C \arrow[r, "\Phi_{|L_1|}"'] & \boldsymbol{P}^1 \end{tikzcd}$$

where π': $W' := \boldsymbol{P}(E') \to C$ is the $\boldsymbol{P}^2$-boundle associated to E', ϕ: $\bar{W} \to W$ is the blowing-up at q $(=\boldsymbol{P}(\mathcal{O}_P(p)))$, ϕ': $\bar{W} \to W'$ is the blowing-up along $\boldsymbol{P}(F')$, and π_0: $W_0 \to \boldsymbol{P}^1$ is the $\boldsymbol{P}^2$-bundle associated to $E_0 := \mathcal{O}_{\boldsymbol{P}^1} \oplus \mathcal{O}_{\boldsymbol{P}^1}(1) \oplus \mathcal{O}_{\boldsymbol{P}^1}(1)$. Let T' be a tautological divisor of W' with $\pi_* \mathcal{O}_{W'}(T') \cong E'$, and $\bar{T}$ the proper transform of T by ϕ. We have $\phi'(\bar{T}) \sim T'$. If T_0 is a tautological divisor of W_0 satisfying $\pi_{0*} \mathcal{O}_{W_0}(T_0) \cong E_0$, then we have $\Phi^* T_0 \sim T'$, and $\dim |T'| = \dim |T_0| = 4$. Hence we have $\Phi_{|T'|} = \Phi_{|T_0|} \circ \Phi$. We can show that $\Phi_{|T_0|}$ is a birational morphism onto the image in a way similar to Lemma 3.2. Therefore we have $\deg \Phi_{|T|} = \deg \Phi_{|T'|} = \deg \Phi = 2$.

In the case (ii), if we assume $L_0 \not\cong L_1$, then the statement can be proved in a way similar to that in the case (i).

Assume $L_0 \cong L_1$ in the case (ii). If $p \in C$ is the point with $\mathcal{O}_C(p) \cong L_0$, then there exists a line $Z \subset \pi^{-1}(p)$ with $\mathrm{Bs}\,|T| = Z$. We obtain the same commutative diagram as above, and in this case, ϕ: $\bar{W} \to W$ is the blowing-up along Z. We can show that $\deg \Phi_{|T|} = 2$ by the same argument as in the case (i). q.e.d

PROPOSITION 3.7. *Let the notation and the assumption be as in Lemma* 3.6. *Then the minimal resolution of a general member* $S \in |4T - \pi^* D|$ *is canonical.*

PROOF. First, we consider the case $E \cong L_0 \oplus L_1 \oplus L_1$, $(L_0 \in \mathscr{E}_C(1, 1),\ L_1 \cong L_0^{\otimes 2})$. There is nothing to prove if $\Phi_{|K_S|}$ is birational onto its image. Thus suppose $\Phi_{|K_S|}$ is not birational. Hence $\Phi_{|K_S|}$ gives an unramified two-to-one covering

$$h: S \setminus \bigcup_{i=0}^{3} F_i \to \tilde{S}_0$$

where $F_i := \pi^{-1}(p_i)$ with $p_i \in C$ $(i = 0, 1, 2, 3)$ the ramification points of $\Phi_{|L_1|}$: $C \to \boldsymbol{P}^1$, and $\tilde{S}_0$ is the image. Let $C_0 \subset W$ be a curve which is the base locus of $\mathcal{O}_W(T) \otimes \pi^* L_1^{-1}$. Fix a point $q \in S \setminus (C_0 \cup \bigcup_{i=0}^{3} F_i)$ and let $q' \in W$ be the other point which is mapped to $\Phi_{|T|}(q)$ by the two-to-one map $\Phi_{|T|}$.

Since $\dim H^0(W, \mathcal{O}_W(T) \otimes \pi^* L_0^{-1}) = 3$, we obtain $X_0 \in H^0(W, \mathcal{O}_W(T) \otimes \pi^* L_0^{-1})$ such that X_0 vanishes at q and q' and that the divisor (X_0) is irreducible. Similarly, since $\dim H^0(W, \mathcal{O}_W(T) \otimes \pi^* L_1^{-1}) = 2$, we obtain X_1, $X_2 \in H^0(W, \mathcal{O}_W(T) \otimes \pi^* L_1^{-1})$ such that X_1 vanishes at q and q', and that X_2 does not vanish at q and q'. Furthermore, the divisors (X_1) and (X_2) are irreducible. A global section $\Psi \in H^0(W, \mathcal{O}_W(4T) \otimes \pi^* \det E^\vee)$ defining S satisfies $\Psi(q') = \psi_{04}(q') X_2(q')^4$ by our choice of X_0, X_1, X_2, where

ψ_{04} is as in the previous section. Hence q' is not contained in S if and only if $\psi_{04}(q')\neq 0$ holds. Since S is general, we are done.

Next, we consider the case $E\cong L_0\oplus L_1\oplus L_2$, ($L_0$, $L_1\in\mathscr{E}_C(1, 1)$, $L_2\in\mathscr{E}_C(1, 2)$).

Let $S\in|4T-\pi^*D|$ be a general member. Except when the following (i) and (ii) are satisfied, if we assume $\deg\Phi_{|K_S|}=2$, then a fiber of $S'\to C$ which has a multiple component and another fiber which has no multiple component are mapped onto the same curve isomorphically, which is absurd.

(i) $L_0^{\otimes 2}\cong L_1^{\otimes 2}\cong L_2$.

(ii) $L_0^{\otimes 2}$, $L_1^{\otimes 2}\ncong L_2$, $L_0\otimes L_1\cong L_2$ and $L_0\ncong L_1$.

In the cases (i) and (ii), we can show that S is a canonical surface in the same way as in the case $E\cong L_0\oplus L_1\oplus L_2$, ($L_0\in\mathscr{E}_C(1, 1)$, $L_1\cong L_0^{\otimes 2}$). q.e.d

REMARK. In the situation of Proposition 3.7, we have a possibility that there exist special members, with at most rational double points as singularities, of $|4T-\pi^*D|$ whose canonical mapping is of degree 2.

PROPOSITION 3.8. *Let L_0, L_1 and L_2 be invertible sheaves over an elliptic curve C satisfying* $\deg L_i=1$ ($i=0, 1, 2$) *and the condition* (3) *of Theorem* 3.1, π: $W:=\boldsymbol{P}(E)\to C$ *the $\boldsymbol{P}^2$-bundle associated to $E:=L_0\oplus L_1\oplus L_2$, T a tautological divisor with $\pi_*\mathcal{O}_W(T)\cong E$, $D\in\mathrm{Div}(C)$ a divisor with $\mathcal{O}_C(D)\cong\det E$, and $S\in|4T-\pi^*D|$ a general member. We have the following about $\Phi_{|K_S|}$:*

(i) *If $L_0^{\otimes 2}\ncong L_1\otimes L_2$, $L_1^{\otimes 2}\ncong L_2\otimes L_0$ and $L_2^{\otimes 2}\ncong L_0\otimes L_1$, then $\Phi_{|K_S|}$ gives a covering of degree* 9 *onto $\boldsymbol{P}^2$.*

(ii) *If only one of $L_0^{\otimes 2}\cong L_1\otimes L_2$, $L_1^{\otimes 2}\cong L_2\otimes L_0$ and $L_2^{\otimes 2}\cong L_0\otimes L_1$ holds, then $|K_S|$ has one isolated base point, and $\Phi_{|K_S|}$ gives a covering of degree* 8 *over $\boldsymbol{P}^2$.*

(iii) *If all of $L_0^{\otimes 2}\cong L_1\otimes L_2$, $L_1^{\otimes 2}\cong L_2\otimes L_0$ and $L_2^{\otimes 2}\cong L_0\oplus L_1$ hold, then $|K_S|$ has three isolated base points, and $\Phi_{|K_S|}$ gives a covering of degree* 6 *over $\boldsymbol{P}^2$.*

PROOF. First we assume that L_0, L_1, L_2 are pairwise non-isomorphic. Let the notation be as in Theorem 3.1. If $q_i\in W$ ($i=0, 1, 2$) is the point defined by $\psi_i=X_{\tau(i)}=X_{\tau^2(i)}=0$, where $\psi_i\in H^0(C, L_i)\setminus\{0\}$, and τ is the cyclic permutation (012), then we have $\mathrm{Bs}\,|T|=\{q_0, q_1, q_2\}$.

In the case (i), we have $\mathrm{Bs}\,|T|\cap\mathrm{Bs}\,|4T-\pi^*D|=\emptyset$. Hence $\Phi_{|K_S|}$ is a surjective morphism onto $\boldsymbol{P}^2$. Since $K_S^2=9$ and the degree of $\boldsymbol{P}^2$ is 1, we are done in the case (i).

Next, we consider the case (ii). We only have to consider the case $L_0^{\otimes 2}\cong L_1\otimes L_2$ by renumbering L_0, L_1 and L_2 if necessary. In this case, all the members of $|4T-\pi^*D|$ go through q_0. Since $S\in|4T-\pi^*D|$ is general, it does not contain q_1 and q_2. Hence we obtain $\mathrm{Bs}\,|K_S|=\{q_0\}$. Let ϕ: $\tilde{W}\to W$ be the blowing-up at q_0, and $\tilde{T}$ a proper transform of T by ϕ. It is easy to see that $\mathrm{Bs}\,|\tilde{T}|=\emptyset$ holds, and hence q_0 is the simple base point of $|K_S|$. Denote $\xi:=\phi|_{\tilde{S}}$, where $\tilde{S}$ is a proper transform of S by ϕ, and $E:=\xi^{-1}(q_0)\subset\tilde{S}$. If $|V|$ is the variable part of $|\xi^*K_S|$, then we have $|\xi^*K_S|=|V|+E$. Therefore, we obtain

$$\deg \Phi_{|K_S|} = V^2 = (\xi^* K_S - E)^2 = 8 .$$

Similarly, since q_i $(i=0, 1, 2)$ is the simple base point of $|K_S|$ in the case (iii), we have $\deg \Phi_{|K_S|} = K_S^2 - 3 = 6$.

The proof is essentially the same when $L_0 \not\cong L_1 \cong L_2$, $L_1 \not\cong L_2 \cong L_0$, or $L_2 \not\cong L_0 \cong L_1$. q.e.d.

3.2. E is the direct sum of an invertible sheaf and an indecomposable locally free sheaf of rank 2. We denote $E = E_0 \oplus L$, where E_0 is an indecomposable locally free sheaf of rank 2 with $\deg E_0 =: e$, and L is an invertible sheaf over an elliptic curve C with $\deg L =: d$. We only have to consider the case $e \geq 0$, $d \geq 0$ and $(e, d) \neq (0, 0)$ by the remark immediately before §3.1.

We prove the existence of a surface S with $K_S^2 = 3p_g(S)$, $q(S) = 1$ and $p_g(S) = N$ for any integer $N \geq 2$ in §3.2.1 (Theorem 3.9) when e is even, and in §3.2.2 (Theorem 3.10) when e is odd. (When e is even, however, the case $p_g(S) = 2$ does not occur.) In §3.2.3, we study the canonical mapping of the surfaces obtained in §3.2.1 and §3.2.2. The results about the canonical mappings are stated in Corollary 3.13, and Propositions 3.14, 3.17, 3.18 and 3.30.

Let π: $W := \boldsymbol{P}^2(E) \to C$ be the $\boldsymbol{P}^2$-bundle associated to E, and T a tautological divisor with $\pi_* \mathcal{O}_W(T) \cong E$. If $\rho: X \to W$ is the blowing-up along $C_1 := \boldsymbol{P}(E/E_0) \subset W$, then X is a $\boldsymbol{P}^1$-bundle $\sigma: X \to Y := \boldsymbol{P}(E_0)$. Let $\mu: Y \to C$ be the ruling, and denote $Y_1 := \rho^* T$ and $Y_\infty := \rho^{-1}(C_1)$. If $C_0 \in \mathrm{Div}(Y)$ is a tautological divisor with $\mu_* \mathcal{O}_Y(C_0) \cong E_0$, then we have $Y_1 \sim Y_\infty + \sigma^* C_0$, and $\sigma_* \mathcal{O}_X(Y_1) \cong \mathcal{O}_Y(C_0) \oplus \mu^* L$. Let $Y_0 \in \mathrm{Div}(X)$ be a divisor with $\mathcal{O}_X(Y_0) \cong \mathcal{O}_X(Y_1) \otimes \sigma^* \mu^* L^{-1}$, and let $Z_0 \in H^0(X, \mathcal{O}_X(Y_0))$, $Z_\infty \in H^0(X, \mathcal{O}_X(Y_\infty))$ be global sections with $(Z_0) = Y_0$ and $(Z_\infty) = Y_\infty$. Then Z_0 and Z_∞ give homogeneous coordinates of each fiber of the $\boldsymbol{P}^1$-bundle σ.

We study the complete linear system of the invertible sheaf $\mathcal{O}_X(4Y_1) \otimes \sigma^* \mu^* \det E^\vee \cong \rho^*(\mathcal{O}_W(4T) \otimes \pi^* \det E^\vee)$ over X.

Any $\Psi \in H^0(X, \mathcal{O}_X(4Y_1) \otimes \sigma^* \mu^* \det E^\vee) \cong H^0(Y, S^4(\mathcal{O}_Y(C_0) \oplus \mu^* L) \otimes \mu^* \det E^\vee)$ can be written as

$$\Psi = \sum_{j=0}^{4} \psi_j Z_0^{4-j} Z_\infty^j , \quad \psi_j \in H^0(Y, \mathcal{O}_Y(jC_0) \otimes \mu^*(L^{\otimes(4-j)} \otimes \det E^\vee)) , \quad (j=0, \ldots, 4) .$$

3.2.1. Existence in the case where e is even. Denote $e = 2e_0$. There exist $L_0 \in \mathscr{E}_C(1, e_0)$, and $L_1 \in \mathscr{E}_C(1, d - e_0)$, with $E_0 \cong L_0 \otimes F_2$, $L \cong L_0 \otimes L_1$, hence we have $E \cong L_0 \otimes (F_2 \oplus L_1)$.

THEOREM 3.9. *Let the conditions and notation be as above. Then the locally free sheaf E satisfies the condition* (A) *if and only if one of the following* (1), (2) *and* (3) *holds*:

(1) $e = d > 0$ *and* $L_0 \cong L_1$,

(2) $d < e < 4d$,

(3) $e = 4d > 0$ *and* $L_0 \otimes L_1^{\otimes 2} \cong \mathcal{O}_C$.

PROOF. We deal with different cases. Let $D \in \mathrm{Div}(C)$ satisfy $\mathcal{O}_C(D) \cong \det E$.

(i) The case where ($e=d$ and $L_0 \not\cong L_1$), or ($e<d$). Since we have

$$H^0(Y, \mathcal{O}_Y(4C_0) \otimes \mu^* \det E^\vee) \cong H^0(C, F_5 \otimes L_0 \otimes L_1^{-1}) = 0,$$

by Theorem 2.4, Ψ is divisible by Z_0. Hence, the image of (Ψ) in W is reducible.

(ii) The case where ($L_0 \cong L_1$), ($d<e<3d$), or ($L_0 \otimes L_1^{\otimes 3} \cong \mathcal{O}_C$).

Let us look at the complete linear system of $L^{\otimes 4} \otimes \det E^\vee$. Since we have $\deg(L^{\otimes 4} \otimes \det E^\vee) = 3d-e \geq 0$, it does not have base points when $3d-e \neq 1$. If $3d-e = 1$ holds, we have $\Gamma := Y_\infty \cap (\mu \circ \sigma)^{-1}(q) \subset \mathrm{Bs}\,|4Y_1 - \sigma^*\mu^*D|$, where $q \in C$ satisfies $L^{\otimes 4} \otimes \det E^\vee \cong \mathcal{O}_C(q)$. We easily check that Γ is a (-1)-curve on $S'' := (\Psi)$.

Let us look at $|4C_0 - \mu^*D|$. Clearly, $\mathrm{Bs}\,|4C_0 - \mu^*D| = \varnothing$ holds when $\deg(L_0 \otimes L_1^{-1}) = e-d \geq 2$. When $e-d=1$, a general member of $|4C_0 - \mu^*D|$ is nonsingular by Lemma 2.5. Thus a general member of $|4Y_1 - \sigma^*\mu^*D|$ is nonsingular at the base point $Y_0 \cap \sigma^{-1}(y_0)$, where y_0 is the base point of $|4C_0 - \mu^*D|$. If $e-d=0$, since we assume $L_0 \cong L_1$, we have $H^0(Y, \mathcal{O}_Y(4C_0) \otimes \mu^* \det E^\vee) \cong \boldsymbol{C}$. Thus we have $\sigma^{-1}(C') \cap Y_0 \subset \mathrm{Bs}\,|4Y_1 - \sigma^*\mu^*D|$, where C' is a section of μ with $\mu_*\mathcal{O}_Y(C') \cong F_2$. The divisor (ψ_3) on Y defined by general ψ_3 intersects C' at e_0 points transversally. Let p be one of these intersection points, (t, u) a local coordinate around p so that $t=0$ and $u=0$ are the local equations for C' and (ψ_3) around p, respectively, and denote $z_0 := Z_0/Z_\infty$, Ψ can be written as

$$\Psi = \psi_0 z_0^4 + \psi_1 z_0^3 + \psi_2 z_0^2 + \psi_3 z_0 + \psi_4 = z_0(\psi_0 z_0^3 + \psi_1 z_0^2 + \psi_2 z_0 + u) + t^4$$

near $p_0 := \sigma^{-1}(p) \cap Y_0$. This is an equation defining a rational double point of type A_3.

We have to consider the case $E \cong L \otimes (F_2 \oplus \mathcal{O}_C)$ with $L \in \mathcal{E}_C(1, 1)$. (This is the case where $3d-e=1$ and $e-d=1$ above hold at the same time.) In this case, ψ_i is contained in $H^0(Y, \mathcal{O}_Y(iC') \otimes \mu^*L)$. We have $\mathrm{Bs}\,|iC' + \Gamma_0| = \{y_0\}$ by Lemma 2.5, and hence $\mathrm{Bs}\,|4Y_1 - \sigma^*\mu^*D| = \sigma^{-1}(y_0) \cup \{(\mu \circ \sigma)^{-1}(q) \cap Y_\infty\}$. We only have to prove that it is nonsingular along $\sigma^{-1}(y_0)$. Since all the nonsingular members of $|4C' + \Gamma_0|$ have the same tangent at y_0 by Lemma 2.5, we can choose a local coordinate (t, u) around y_0 so that $t=0$ is the local equation of Γ_0 and that $u=0$ gives the tangent line of nonsingular members of $|4C' + \Gamma_0|$ at y_0. If we denote $z := Z_0/Z_\infty$, then Ψ can be written as

$$\begin{aligned}\Psi = a_0 t z^4 &+ (a_1 t + b_1 u + \iota_1(t, u)) z^3 + (a_2 t + b_2 u + \iota_2(t, u)) z^2 \\ &+ (a_3 t + b_3 u + \iota_3(t, u)) z + (b_4 u + \iota_4(t, u))\end{aligned}$$

near $\sigma^{-1}(y_0) \setminus Y_\infty$, where $a_i, b_j \in \boldsymbol{C}$, $(i=0, 1, 2, 3, j=1, 2, 3, 4)$, and $\iota_j(t, u)$, $(j=1, 2, 3, 4)$ is the sum of all the monomials with respect to t and u with degree at least two. Since Ψ is general, we may assume $a_0 \neq 0$ and $b_4 \neq 0$. If we fix a_1, a_2 and a_3, then b_1, b_2 and b_3 are uniquely determined. On the other hand, a_0 and b_4 can be chosen independently of them. Hence the two equations $\partial\Psi/\partial t = 0$ and $\partial\Psi/\partial u = 0$ do not have the same solutions, and (Ψ) is nonsingular along $\sigma^{-1}(y_0)$.

Clearly, $|4Y_1 - \sigma^*\mu^*D|$ is not composite with a pencil when $e>d$ by Lemma 2.6.

If $e=d$ holds, let S_1 and S_2 be two distinct general members of $|4Y_1-\sigma^*\mu^*D|$, and F a general fiber of $\mu\circ\sigma$, and denote $q:=F\cap\sigma^{-1}(C')\cap Y_0$. We can easily check that the intersection multiplicity of $S_1|_F$ and $S_2|_F$ at q is four. Hence $S_1|_F$ and $S_2|_F$ have other intersection points, and we see that $|4Y_1-\sigma^*\mu^*D|$ is not composite with a pencil even in the case $e=d$ by Lemma 2.6. Therefore, a general member of $|4Y_1-\sigma^*\mu^*D|$ is irreducible.

It is easily seen that $Y_\infty|_{S''}$ consists of $3d-e$ pieces of (-1)-curves, where $S'':=(\Psi)$. Hence the image of $Y_\infty\cap S''$ in W is a finite set of nonsingular points of $S':=\rho(S'')$.

(iii) The case where ($e=3d$ and $L_0\otimes L_1^{\otimes 3}\not\cong\mathcal{O}_C$), ($3d<e<4d$), or ($L_0\otimes L_1^{\otimes 2}\cong\mathcal{O}_C$). Ψ is divisible by Z_∞, i.e., the image of (Ψ) in W contains C_1. In this case, we have to consider the complete linear system of $\mathcal{O}_X(3Y_1)\otimes\sigma^*(\mathcal{O}_Y(C_0)\otimes\mu^*\det E^\vee)$. Any $\tilde{\Psi}:=\Psi/Z_\infty$ can be written as

$$\tilde{\Psi}=\sum_{j=0}^{3}\psi_{j+1}Z_0^{3-j}Z_\infty^j\,,$$

$$\psi_{j+1}\in H^0(Y,\mathcal{O}_Y((j+1)C_0)\otimes\mu^*(L^{\otimes(3-j)}\otimes\det E^\vee))\,,\qquad (j=0,\ldots,3)\,.$$

Let us look at $\mathcal{O}_Y(C_0)\otimes\mu^*(L^{\otimes 3}\otimes\det E^\vee)$. If $4d-e\geq 4$, then base points do not exist. If $4d-e=2$, then there exists a unique isolated base point on C'. If $4d-e=0$, then we have $|C'|=\{C'\}$. In each case, we easily see that a general $(\tilde{\Psi})$ is nonsingular over the base points of $\mathcal{O}_Y(C_0)\otimes\mu^*(L^{\otimes 3}\otimes\det E^\vee)$.

Let us look at $\mathcal{O}_Y(4C_0)\otimes\mu^*(\det E^\vee)\cong\mathcal{O}_Y(4C')\otimes\mu^*(L_0\otimes L_1^{-1})$. Since $\deg(L_0\otimes L_1^{-1})=e-d\geq 2$ hold, we have $\mathrm{Bs}\,|4C_0-\mu^*D|=\emptyset$.

In the same way as in the proof of (ii), we can show that $|3Y_1+\sigma^*(C_0-\mu^*D)|$ is not composite with a pencil.

By what we have seen so far, a general member of $|3Y_1+\sigma^*(C_0-\mu^*D)|$ is irreducible and nonsingular. It is easy to see that the intersection of the member with Y_∞ is irreducible, and hence its image by ρ is nonsingular.

(iv) The case where ($e=4d$ and $L_0\otimes L_1^{\otimes 2}\not\cong\mathcal{O}_C$) or ($4d<e$). (Ψ) has $2Y_\infty$ as a component, and the image of (Ψ) in W contains C_1 as a singular curve.

3.2.2. Existence in the case where e is odd. Denote $e=:2e_0+1$ (≥ 1). If we fix any $F_{2,1}\in\mathcal{E}_C(2,1)$, then there exist $L_0\in\mathcal{E}_C(1,e_0)$ and $L_1\in\mathcal{E}_C(1,d-e_0)$ with $E_0\cong L_0\otimes F_{2,1}$ and $L\cong L_0\otimes L_1$. Let $\mathcal{N}_k$ $(k=1,2,3)$ be the invertible sheaves with $\mathcal{N}_k\not\cong\mathcal{O}_C$, and $\mathcal{N}_k^{\otimes 2}\cong\mathcal{O}_C$.

THEOREM 3.10. *Let the conditions and notation be as above. Then the locally free sheaf E satisfies the condition* (A) *if and only if one of the following* (1) *and* (2) *holds*:

(1) $e=d>0$ *and* $\det F_{2,1}\otimes L_0\otimes L_1^{-1}$ *is isomorphic to one of* $\mathcal{O}_C$ *and* $\mathcal{N}_k$ $(k=1,2,3)$.

(2) $d<e<4d$.

We use the following result on symmetric products by Ashikaga [1] to prove this theorem. Since [1] is unpublished, we give the proof for the readers' convenience.

LEMMA 3.11 (cf. [1]). *If $\mathcal{N}_k$ $(k=1,2,3)$ are the three nontrivial line bundles satisfying $\mathcal{N}_k^{\otimes 2}\cong\mathcal{O}_C$, and if $F_{2,1}$ is an indecomposable locally free sheaf of rank 2 and degree 1 on an elliptic curve C, then the following hold for any nonnegative integer m:*

$$(1)\qquad S^{4m}(F_{2,1}) \cong \left(\mathcal{O}_C^{\oplus(m+1)}\oplus\left(\bigoplus_{k=1}^{3}\mathcal{N}_k\right)^{\oplus m}\right)\otimes(\det F_{2,1})^{\otimes 2m}$$

$$(2)\qquad S^{4m+2}(F_{2,1}) \cong \left(\mathcal{O}_C^{\oplus m}\oplus\left(\bigoplus_{k=1}^{3}\mathcal{N}_k\right)^{\oplus(m+1)}\right)\otimes(\det F_{2,1})^{\otimes(2m+1)}.$$

PROOF. First, we show the statement for $S^2(F_{2,1})$. We have

$$F_{2,1}\otimes F_{2,1} \cong S^2(F_{2,1})\oplus\det F_{2,1}$$
$$F_{2,1}\otimes F_{2,1} \cong (\mathcal{O}_C\oplus\mathcal{N}_1\oplus\mathcal{N}_2\oplus\mathcal{N}_3)\otimes\det F_{2,1}$$

by the Clebsch-Gordan formula [4, p. 438], and Atiyah's result [4, Theorem 14]. Hence we obtain an isomorphism

$$S^2(F_{2,1})\cong(\mathcal{N}_1\oplus\mathcal{N}_2\oplus\mathcal{N}_3)\otimes\det F_{2,1}.$$

To complete the proof, it is sufficient to show the following (i), (ii) and (iii):

(i) $S^4(F_{2,1})\cong(\mathcal{O}_C^{\oplus 2}\oplus\mathcal{N}_1\oplus\mathcal{N}_2\oplus\mathcal{N}_3)\otimes(\det F_{2,1})^{\otimes 2}$.

(ii) (1) of the lemma is true under the assumption that the lemma is true for all the even integers less than or equal to $4m-2$.

(iii) (2) of the lemma is true under the assumption that the lemma is true for all the even integers less than or equal to $4m$.

We show only (iii) here. (i) and (ii) can be shown in the same way.

We have

$$S^{4m}(F_{2,1})\otimes S^2(F_{2,1})$$
$$\cong S^{4m+2}(F_{2,1})\oplus((\det F_{2,1})\otimes S^{4m}(F_{2,1}))\oplus((\det F_{2,1})^{\otimes 2}\otimes S^{4m-2}(F_{2,1}))$$

for $m>0$ by the Clebsch-Gordan formula. On the other hand, we have

$$S^{4m}(F_{2,1})\otimes S^2(F_{2,1})\cong(\mathcal{O}_C^{\oplus 3m}\oplus(\mathcal{N}_1\oplus\mathcal{N}_2\oplus\mathcal{N}_3)^{\oplus(3m+1)})\otimes(\det F_{2,1})^{\otimes(2m+1)}$$

by the induction assumption, and furthermore, we have

$$((\det F_{2,1})\otimes S^{4m}(F_{2,1}))\oplus((\det F_{2,1})^{\otimes 2}\otimes S^{4m-2}(F_{2,1}))$$
$$\cong(\mathcal{O}_C^{\oplus 2m}\oplus(\mathcal{N}_1\oplus\mathcal{N}_2\oplus\mathcal{N}_3)^{\oplus 2m})\otimes(\det F_{2,1})^{\otimes(2m+1)}.$$

Hence we have

$$S^{4m+2}(F_{2,1})\cong(\mathcal{O}_C^{\oplus m}\oplus(\mathcal{N}_1\oplus\mathcal{N}_2\oplus\mathcal{N}_3)^{\oplus(m+1)})\otimes(\det F_{2,1})$$

by the Krull-Schmidt theorem. q.e.d.

PROOF OF THEOREM 3.10. Let Ψ and D be as in the previous section.

(i) The case where ($e<d$), or ($e=d$ and $\det F_{2,1}\otimes L_0\otimes L_1^{-1}$ is isomorphic to none of $\mathcal{O}_C$ and $\mathcal{N}_k$ ($k=1, 2, 3$)). Using Lemma 3.11, we can show that (Ψ) has Y_0 as a component in the same way as in the proof of (i) of Theorem 3.9.

(ii) The case where ($\det F_{2,1}\otimes L_0\otimes L_1^{-1}$ is isomorphic to one of $\mathcal{O}_C$ and $\mathcal{N}_k$ ($k=1, 2, 3$)), ($d<e<3d$), or ($L_0\otimes L_1^{\otimes 3}\cong\det F_{2,1}$).

We can show that E satisfies the condition (A) when $e-d\geq 2$ holds in the same way as in the proof of (ii) of Theorem 3.9. We only have to prove that a general member of $|4Y_1-\sigma^*\mu^*D|$ is nonsingular at the base points dominating the base points of $\mathcal{O}_Y(4C_0)\otimes\mu^*\det E^\vee$ when $e-d=0, 1$.

For that purpose, we need to study the structure of Y more precisely. We can show that $\dim H^0(Y, \mathcal{O}_Y(4C_0)\otimes\mu^*(\det E_0^\vee)^{\otimes 2})=2$ by using Lemma 3.11, and that this linear pencil has no base point. Let ζ: $Y\rightarrow \boldsymbol{P}^1$ be the corresponding fibration. The invertible sheaves $\mathcal{M}_k:=\mathcal{O}_Y(2C_0)\otimes\mu^*(\mathcal{N}_k\otimes\det E_0^\vee)$ ($k=1, 2, 3$) satisfy $\mathcal{M}_k^{\otimes 2}\cong\mathcal{O}_Y(4C_0)\otimes\mu^*(\det E_0^\vee)^{\otimes 2}$, and $H^0(Y, \mathcal{M}_k)\cong\boldsymbol{C}$ by Lemma 3.11, hence ζ has three multiple fibers $2\mathcal{F}_k$ ($k=1, 2, 3$) with $\mathcal{F}_k$ satisfying $\mathcal{M}_k\cong\mathcal{O}_Y(\mathcal{F}_k)$.

Next, we study the complete linear system of $\mathcal{O}_Y(4C_0)\otimes\mu^*((\det E_0^\vee)^{\otimes 2}\otimes\mathcal{N}_k)$. We obtain $H^0(Y, \mathcal{O}_Y(4C_0)\otimes\mu^*((\det E_0^\vee)^{\otimes 2}\mathcal{N}_k))\cong\boldsymbol{C}$ by Lemma 3.11. Since

$$\mathcal{O}_Y(4C_0)\otimes\mu^*((\det E_0^\vee)^{\otimes 2}\otimes\mathcal{N}_k)\cong\mathcal{M}_{\tau(k)}\otimes\mathcal{M}_{\tau^2(k)}\qquad(k=1, 2, 3)$$

where τ is a cyclic permutation, each of these three complete linear systems consists only of $\mathcal{F}_{\tau(k)}+\mathcal{F}_{\tau^2(k)}$.

If $e-d=1$, there exists a point $p\in C$ with $\det E^\vee\cong(\det E_0^\vee)^{\otimes 2}\otimes\mu^*\mathcal{O}_C(p)$, and we have $\mathrm{Bs}|4C_0-\mu^*D|\subset\Gamma:=\mu^{-1}(p)$. Let $p_k\in C$ be a point with $\mathcal{N}_k\otimes\mathcal{O}_C(p)\cong\mathcal{O}_C(p_k)$, and denote $\Gamma_k:=\mu^{-1}(p_k)$ ($k=1, 2, 3$). Then we have $\Gamma_k+\mathcal{F}_{\tau(k)}+\mathcal{F}_{\tau^2(k)}\in|4C_0-\mu^*D|$ ($k=1, 2, 3$). Since p, p_1, p_2, p_3 are pairwise distinct, and since $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ intersect Γ at distinct points, we obtain $\mathrm{Bs}|4C_0-\mu^*D|=\varnothing$.

If $e-d=0$, then $\det F_{2,1}\otimes L_0\otimes L_1^{-1}$ is isomorphic to one of $\mathcal{O}_C$ and $\mathcal{N}_k$ ($k=1, 2, 3$) by assumption. If $\det F_{2,1}\otimes L_0\otimes L_1^{-1}\cong\mathcal{O}_C$, then $\mathrm{Bs}|4C_0-\mu^*D|=\varnothing$ holds. If $\det F_{2,1}\otimes L_0\otimes L_1^{-1}\cong\mathcal{N}_k$, we have $|4C_0-\mu^*D|=\{\mathcal{F}_{\tau(k)}+\mathcal{F}_{\tau^2(k)}\}$. We can easily show that a general member of $|4Y_1-\sigma^*\mu^*D|$ is nonsingular along $\sigma^{-1}(\mathcal{F}_{\tau(k)}\cup\mathcal{F}_{\tau^2(k)})\cap Y_0$.

In the same way as in the proof of Theorem 3.9, we can show that $|4Y_1-\sigma^*\mu^*D|$ is not composite with a pencil in each case above, and hence a general member of $|4Y_1-\sigma^*\mu^*D|$ is irreducible and nonsingular.

As in the case of (ii) with e even, we can show that $(\Psi)\cap Y_\infty$ is a disjoint union of $3d-e$ pieces of (-1)-curves.

(iii) The case where ($e=3d>0$ and $L_0\otimes L_1^{\otimes 3}\not\cong\det F_{2,1}$), or ($3d<e<4d$). Using Lemma 3.11, we can show that a general member of $|3Y_1+\sigma^*(C_0-\mu^*D)|$ is irreducible and nonsingular in the same way as in the proof of (iii) of Theorem 3.9.

(iv) The case where $4d<e$. The images of all the members of $|4Y_1-\sigma^*\mu^*D|$ in

W have non-isolated singularity for the same reason as in the case (iv) with e even.

q.e.d.

REMARK. (1) We can prove Theorem 3.10 above by using an isogeny $\varphi: \tilde{C} \to C$ with $\deg \varphi = 2$ of elliptic curves as in §3.3.4 below where we use an isogeny of degree 3.

(2) The existence of the linear pencil $\zeta: Y \to \boldsymbol{P}^1$ and the multiple fibers $2\mathscr{F}_k$ ($k = 1, 2, 3$) above was proved by Suwa [21, §4]. What we mentioned in the proof of Theorem 3.10 is a re-interpretation of Suwa's result by means of Lemma 3.11 due to Ashikaga.

3.2.3. The canonical mapping. In this section, we study the canonical mappings of those surfaces whose existence was shown in §§3.9–3.10. Let E_0 and L be as above satisfying the conditions of Theorem 3.9 when e is even, and Theorem 3.10 when e is odd.

LEMMA 3.12. *If μ: $Y := \boldsymbol{P}(E_0) \to C$ is the ruled surface associated to $E_0 \in \mathscr{E}_C(2, 4)$, and C_0 is a section of μ with $\mu_* \mathcal{O}_Y(C_0) \cong E_0$, then $\Phi_{|C_0|}$ is birational onto its image.*

PROOF. Let $\delta \in \mathrm{Div}(C)$ be a divisor satisfying $L_0 \cong \mathcal{O}_C(\delta)$, and C' a section of μ with $|C_0 - \mu^*\delta| = \{C'\}$, where L_0 is an invertible sheaf with $E_0 \cong L_0 \otimes F_2$.

Let $q_1, q_2 \in Y \setminus C'$ be any pair of points contained in different fibers of μ, and Γ_1 the fiber of μ containing q_1. Since $\mathrm{Bs}\,|C_0 - \Gamma_1|$ consists of one point on C' by Lemma 2.5, there exists a member C_0' of $|C_0 - \Gamma_1|$ with $q_2 \notin C_0'$. Then $C_0' + \Gamma_1$ contains q_1 but not q_2. Hence $|C_0|$ separates q_1 and q_2, and $\Phi_{|C_0|}$ is birational onto its image. q.e.d.

PROPOSITION 3.13. *Any surface whose existence is guaranteed by Theorems* 3.9 *and* 3.10 *and the condition* (A) *is canonical if $e + d \geq 5$. If $(e, d) \neq (4, 1)$, then the canonical mapping is a morphism. If $(e, d) = (4, 1)$, then $|K_S|$ has a unique isolated base point, and its canonical image is non-normal.*

PROOF. We use the same notation as in §§3.2.1–3.2.2. We only have to prove that $\Phi_{|T|}$ is birational onto the image to show the birationality of the canonical mapping.

We can show that the restriction of $\Phi_{|T|}$ to a general fiber F of π gives an isomorphism of F onto its image as in the proof of Lemma 3.2.

If $\Phi_{|Y_1|}$ is birational onto its image, then $\Phi_{|T|}$ is also birational onto its image. Therefore, we consider $\Phi_{|Y_1|}$.

Any $\Psi \in H^0(X, \mathcal{O}_X(Y_1))$ can be written as

$$\Psi = \psi_0 Z_0 + \psi_\infty Z_\infty\,, \quad \psi_0 \in H^0(C, L)\,, \quad \psi_\infty \in H^0(Y, \mathcal{O}_Y(C_0))\,.$$

Hence it is easy to see that $\Phi_{|Y_1|}$ is birational onto its image if one of $d \geq 3$, $e \geq 6$ and $(e, d) = (5, 2)$ holds. Similarly, in the cases $(e, d) = (4, 2)$, $(4, 1)$, $\Phi_{|Y_1|}$ is birational by Lemma 3.12. If $(e, d) = (3, 2)$, then we have $\mathrm{Bs}\,|Y_1| = \varnothing$. Since $Y_1^3 = 5$ and the degree of the image of X cannot be 1, $\Phi_{|Y_1|}$ is a birational morphism.

If $e \geq 3$ and $d \geq 2$, then $\mathrm{Bs}\,|Y_1| = \varnothing$ by the equation above. The statement in the case $(e, d) = (4, 1)$ is proved in the same way as in the proof of Corollary 3.4. q.e.d.

PROPOSITION 3.14. *Let π: $W:=\boldsymbol{P}(E)\rightarrow C$ be the $\boldsymbol{P}^2$-bundle associated to a locally free sheaf $E:=E_0\oplus L$, $(E_0\in\mathscr{E}_C(2,2),\ L\in\mathscr{E}_C(1.1))$, T a tautological divisor with $\pi_*\mathcal{O}_W(T)\cong E$, and $L_0\in\mathscr{E}_C(1,1)$, $L_1\in\mathscr{E}_C(1,0)$ the invertible sheaves satisfying $E_0\cong L_0\otimes F_2$ and $L\cong L_0\otimes L_1$. We have the following for a general $S\in|4T-\pi_*D|$:*

(i) $\deg\Phi_{|K_S|}=9$, *if* $L_1^{\otimes 2}\not\cong\mathcal{O}_C$.

(ii) $\deg\Phi_{|K_S|}=8$, *if* $L_1^{\otimes 2}\cong\mathcal{O}_C$ *and* $L_1\not\cong\mathcal{O}_C$.

(iii) $\deg\Phi_{|K_S|}=4$, *if* $L_1\cong\mathcal{O}_C$.

PROOF. Let $p\in C$ be a point with $L\cong\mathcal{O}_C(p)$, and $q\in Y$ a point with $\mathrm{Bs}|C_0|=\{q\}$. (See Lemma 2.5.) $\mathrm{Bs}|Y_1|=\{(\mu\circ\sigma)^{-1}(p)\cap Y_\infty\}\cup\{\sigma^{-1}(q)\cap Y_0\}$ is proved in the same way as in the proof of Corollary 3.13.

Let C' be a section of μ with $\mathcal{O}_Y(C')\cong\mathcal{O}_Y(C_0)\otimes\mu^*L_0^{-1}$, $p_0\in C$ a point with $L_0\otimes L_1^{-1}\cong\mathcal{O}_C(p_0)$, $q'\in Y$ the intersection point of $\mu^{-1}(p_0)$ with C', and $p'\in C$ a point with $L^{\otimes 4}\otimes\det E^\vee\cong\mathcal{O}_C(p')$. We have already seen that $\mathrm{Bs}|4Y_1-\sigma^*\mu^*D|=\{\sigma^{-1}(q')\cap Y_0\}\cup\{(\mu\circ\sigma)^{-1}(p')\cap Y_\infty\}$ holds in Theorem 3.9.

q coincides with q' if and only if $L_0\cong L_0\otimes L_1^{-1}$, hence $L_1\cong\mathcal{O}_C$. p coincides with p' if and only if $L\cong L^{\otimes 4}\otimes\det E^\vee$. This is equivalent to $L_1^{\otimes 2}\cong\mathcal{O}_C$.

Hence, if $L_1^{\otimes 2}\not\cong\mathcal{O}_C$ holds, the complete linear system of the canonical divisor of a general member of $|4Y_1-\sigma^*\mu^*D|$ has no base point, and we obtain $\deg\Phi_{|K_S|}=9$.

If $L_1^{\otimes 2}\cong\mathcal{O}_C$ and $L_1\not\cong\mathcal{O}_C$ holds, then we have $q\neq q'$ and $p=p'$. Hence the canonical system of a general member of $|4T-\pi^*D|$ has one isolated base point. We have the following elementary transformation (cf. [17]):

$$\begin{tikzcd} & \bar{W} \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\ W & & W' \end{tikzcd}$$

where π': $W'\rightarrow C$ is the $\boldsymbol{P}^2$-bundle associated to a locally free sheaf $E':=E_0\oplus\mathcal{O}_C$ of rank 3 over C, ϕ is the blowing-up at the isolated base point of $|4T-\pi^*D|$, and ϕ' is the blowing-up along the line $\boldsymbol{P}(E_0\otimes_{\mathcal{O}_C}\mathcal{O}_p)\subset W'$. Let T' be a tautological divisor with $\pi'_*\mathcal{O}_{W'}(T')\cong E'$. The complete linear system $|T|$ on W is mapped to the complete linear system $|T'|$ by this elementary transformation. Furthermore, if $\bar{S}$ is the proper transform of a general member S by ϕ, and if we denote $S':=\phi'(\bar{S})$, then we have $S'\sim 4T'$ by the assumption $L_1^{\otimes 2}\cong\mathcal{O}_C$. Since $\mathrm{Bs}|T'|=\varnothing$, the complete linear system of $\mathcal{O}_{W'}(T')\otimes_{\mathcal{O}_{W'}}\mathcal{O}_{S'}$ on S' has no base point. Since $\Phi_{|K_{\bar{S}}|}$ factors as

$$\Phi_{|K_{\bar{S}}|}:\bar{S}\rightarrow S'\rightarrow\Phi_{|T'|}(S')\hookrightarrow\boldsymbol{P}^n,\qquad(n:=p_g(S)-1),$$

we have $\deg\Phi_{|K_S|}=\deg\Phi_{|K_{\bar{S}}|}=(T')^2S'=4(T')^3=4\deg E'=8$.

Finally, we consider the case (iii), i.e., the case $E\cong L\otimes(F_2\oplus\mathcal{O}_C)$. We see that $\mathrm{Bs}|T|=\mathrm{Bs}|4T-\pi^*D|$ holds, and it is a line contained in a fiber $\pi^{-1}(p)\subset W$.

We have the same elementary transformation as in the case (ii). (We use the same notation as above.) In this case, $\mathrm{Bs}|T'|$ consists of one point contained in

$\pi'^{-1}(p)$, and the image S' in W' of the proper transform of a general member $S\in|4T-\pi^*D|$ goes through this point. Let T_0' be the image of $\boldsymbol{P}(E/L)\subset W$ in W'. Regarding C_0, C' and $\mu^{-1}(p)$ as divisors on $\boldsymbol{P}(E/L)$ or T_0' in view of $Y\cong\boldsymbol{P}(E/L)\cong T_0'$, we have $\mathcal{O}_{T_0'}(T_0')\cong\mathcal{O}_{T_0'}(C_0)$ $(\cong\mathcal{O}_{T_0'}(C'+\mu^{-1}(p)))$. Since the restriction of S to $\boldsymbol{P}(E/L)$ is linearly equivalent to $4C'+\mu^{-1}(p)$, the restriction of S' to T_0' is the sum of a divisor G which is linearly equivalent to $4C'+\mu^{-1}(p)$ and $3\mu^{-1}(p)$. G goes through $q=\mu^{-1}(p)\cap C'$, and since S is generic, G is nonsingular at q. C_0 also goes through q and is nonsingular at q, and C_0 and G have different tangents by Lemma 2.5.

Let $\nu\colon \tilde{W}\to W'$ be the blowing-up at q, let $\tilde{T}$ and $\tilde{S}$ be the proper transforms of T' and S', respectively, and denote $\tilde{\mathscr{E}}:=\nu^{-1}(q)$. Since $\nu^*T'\sim\tilde{T}+\tilde{\mathscr{E}}$, we can prove $\nu^*S'\sim\tilde{S}+4\tilde{\mathscr{E}}$ by the above result. Although $|\tilde{T}|$ has one isolated base point, $\tilde{S}$ does not go through the point. Hence we have

$$\begin{aligned}\deg\Phi_{|K_S|}&=\deg(\Phi_{|\tilde{T}|}|_{\tilde{S}})=\tilde{T}^2\tilde{S}=(\nu^*T'-\tilde{\mathscr{E}})^2(\nu^*S'-4\tilde{\mathscr{E}})\\&=T'^2S'-4\tilde{\mathscr{E}}^3=4T'^3-4=4\,.\end{aligned}$$

q.e.d.

We treat the case $(e,d)=(1,1)$ in Proposition 3.30 in the next section.

In the case $(e,d)=(3,1)$, we have the following:

LEMMA 3.15. *Let $\pi\colon W:=\boldsymbol{P}(E)\to C$ be the $\boldsymbol{P}^2$-bundle associated to a locally free sheaf $E:=L\otimes(F_{2,1}\oplus L)$, $(L\in\mathscr{E}_C(1,1))$, and T a tautological divisor with $\pi_*\mathcal{O}_W(T)\cong E$. Then $\Phi_{|T|}$ is a triple covering of W over $\boldsymbol{P}^3$.*

PROOF. Let $\mu\colon Y:=\boldsymbol{P}(L\otimes F_{2,1})\to C$ be the ruled surface associated to $L\otimes F_{2,1}$, and C_0 a section of μ with $\mu_*\mathcal{O}_Y(C_0)\cong L\otimes F_{2,1}$. Then the restriction mapping $H^0(Y,\mathcal{O}_Y(C_0))\to H^0(\Gamma,\mathcal{O}_\Gamma(C_0))$ is surjective for any fiber Γ of μ, and we have $\mathrm{Bs}\,|C_0|=\varnothing$.

Consider the pull-back $|Y_1|$ of $|T|$ to X. Since $\mathrm{Bs}\,|C_0|=\varnothing$, we have $\mathrm{Bs}\,|Y_1|=Y_\infty\cap(\mu\circ\sigma)^{-1}(p)$, where $p\in C$ is the point with $L\cong\mathcal{O}_C(p)$. This curve is contracted to a point q by $\rho\colon X\to W$, and we have $\mathrm{Bs}\,|T|=\{q\}$.

Let $\pi'\colon W':=\boldsymbol{P}(E')\to C$ be the $\boldsymbol{P}^2$-bundle associated to the locally free sheaf $E':=E_0\oplus\mathcal{O}_C$, and T' a tautological divisor with $\pi'_*\mathcal{O}_{W'}(T')\cong E'$. We obtain an elementary transformation

$$\begin{tikzcd} & \bar{W} \arrow[dl, "\phi'"'] \arrow[dr, "\phi"] & \\ W' & & W \end{tikzcd}$$

as before, where ϕ is the blowing-up at q. The image under ϕ' of the proper transform of T by ϕ is linearly equivalent to T'. Clearly $\mathrm{Bs}\,|T'|=\varnothing$, and hence, $\deg\Phi_{|T|}=\deg\Phi_{|T'|}=(T')^3=3$ holds.

LEMMA 3.16. *Let μ: $Y \to C$ be the ruled surface associated to a locally free sheaf $E_0 \in \mathscr{E}_C(2, 3)$, and C_0 a section of μ with $\mu_* \mathcal{O}_Y(C_0) \cong E_0$. Then $\Phi_{|4C_0 - \mu^* D|}$ is a birational morphism onto its image for any divisor $D \in \mathrm{Div}(C)$ of degree* 4.

PROOF. It is known that Y is isomorphic to the symmetric product of C of degree 2 (cf. [7]). Let η: $C \times C \to Y$ be the quotient morphism. $C \times \{p\}$ and $\{p\} \times C$ are mapped by η to the same curve C_p on Y for any point $p \in C$. Since $(C \times \{p\} + \{p\} \times C)^2 = 2$ and $\deg \eta = 2$, this curve C_p is a section of μ with self-intersection number 1.

$|4C_0 - \mu^* D|$ contains a member of the form $\sum_{i=1}^4 C_{p_i}$, $(p_i \in C, i = 1, 2, 3, 4)$. Since $\eta^{-1}(\bigcup_{i=1}^4 C_{p_i}) = \bigcup_{i=1}^4 \{(C \times \{p_i\}) \cup (\{p_i\} \times C)\}$, there exist points $p_i \in C$ such that $\bigcup_{i=1}^4 C_{p_i}$ does not contain q for any point $q \in Y$. Hence we have $\mathrm{Bs}|4C_0 - \mu^* D| = \varnothing$.

Let $q, q^* \in Y$ be distinct points which are not contained in the image under η of the diagonal of $C \times C$, and denote $q = (p, p')$ for $p, p' \in C$ in view of the above. Then C_p and $C_{p'}$ are two distinct sections of μ. Since $C_p C_{p'} = 1$, at least one of C_p and $C_{p'}$ does not go through q^*. We may assume that C_p does not go through q^*. $|4C_0 - \pi^* D - C_p|$ contains a member of the form $\sum_{i=1}^3 C_{p_i}$, $(p_i \in C, i = 1, 2, 3)$, and there exists points $p_i \in C$ $(i = 1, 2, 3)$ such that $\sum_{i=1}^3 C_{p_i}$ does not go through q'. Hence the complete linear system $|4C_0 - \mu^* D|$ separates q and q'. q.e.d.

Proposition 3.17. *In the notation of Lemma* 3.15, *if $D \in \mathrm{Div}(C)$ is a divisor with $\mathcal{O}_C(D) \cong \det E$, then a general member $S \in |4T - \pi^* D|$ is a canonical surface.*

PROOF. By the proof of Theorem 3.10, we know that $\mathrm{Bs}|4T - \pi^* D| = \varnothing$ holds when $L \cong \det E_{2,1}$, and that $\mathrm{Bs}|4T - \pi^* D| = C_1 := \boldsymbol{P}(E/E_0) \subset W$ holds when $L \not\cong \det F_{2,1}$.

Since $\deg \Phi_{|T|} = 3$ by Lemma 3.15, we have $\deg \Phi_{|K_S|} = 1$, 2 or 3.

First, we consider the case $L \cong \det F_{2,1}$.

Assume $\deg \Phi_{|K_S|} = 2$. $\mathrm{Bs}|T|$ consists of one point $q \in W$, and S does not contain q. Let ϕ: $\bar{W} \to W$ be the blowing-up at q and $\bar{T} \subset \bar{W}$ the proper transform of T by ϕ, and denote $\mathscr{E} := \phi^{-1}(q)$. The proper transform $\bar{S}$ of S is linearly equivalent to $4\bar{T} + 4\mathscr{E} - \phi^* \pi^* D$. If ϕ': $\bar{W} \to W'$ is as in the proof of Lemma 3.15, then we have $S' := \phi'(\bar{S}) \sim 4T'$. We may identify as $S = \bar{S}$, and $\Phi_{|K_S|}$ is factored as

$$\Phi_{|K_S|}: S \to S' \to \Phi_{|T'|}(S') \quad (\subset \boldsymbol{P}^3).$$

Since $\Phi_{|T'|}^*(\Phi_{|T'|}(S')) \sim 6T'$, there exists a divisor $Q \in |2T'|$ with $\Phi_{|T'|}^*(\Phi_{|T'|}(S')) = S' + Q$. Since $\deg \Phi_{|T'|} = 3$ and $\deg \Phi_{|K_S|} = 2$ hold, Q is birationally equivalent to $\Phi_{|K_S|}(S)$. On the other hand, Q is birationally equivalent to a ruled surface over C. Thus S' is birationally equivalent to a double covering of a ruled surface over C, which is absurd.

Assume $\deg \Phi_{|K_S|} = 3$. Let $q_0 \in S' \setminus C_1$ be a point such that $\Phi_{|T'|}^{-1}(\Phi_{|T'|}(q_0))$ consists of three distinct points q_0, q_1, q_2. Since the restriction of $\Phi_{|T'|}$ to any fiber of π is an isomorphism onto its image, q_0, q_1, q_2 are contained in distinct fibers of π'. Let the notation be as in Theorem 3.10. Since $\dim H^0(X, \mathcal{O}_X(Y_1) \otimes \sigma^* \mu^* L^{-1}) = 2$, we may choose $Z_0 \in H^0(X, \mathcal{O}_X(Y_1) \otimes \sigma^* \mu^* L^{-1})$ in such a way that $Z_0(q_0) = 0$ and that the divisor (Z_0) is irreducible. Let $\Psi \in H^0(X, \mathcal{O}_X(4Y_1) \otimes \sigma^* \mu^* \det E^{\vee})$ be the global section defining the proper transform $\tilde{S}$ of S by ρ. Since $|T|$ does not separate q_0, q_1 and q_2, we have $Z_0(q_1)=Z_0(q_2)=0$. Hence we have $\Psi(q_i)=\psi_4(q_i)Z_\infty(q_i)^4$, where ψ_4 is as in the previous section. On the other hand, since $\psi_4\in H^0(Y, \mathcal{O}_Y(4C_0)\otimes\mu^*\det E^\vee)$, we have $\psi_4(q_i)\neq 0$ for a general S by Lemma 3.16, a contradiction.

Next, we consider the case $L\not\cong\det F_{2,1}$.

Since $\mathrm{Bs}\,|T|=C_1\cap\pi^{-1}(p)$, if we let the notation to be as above, we have $\bar{S}\sim 4\bar{T}+3\mathscr{E}-\phi^*\pi^*D$, and $S'\sim 4T'-\pi^*(p')$, where $S'=\phi'(\bar{S})\subset W'$, and $p'\in C$ is the point with $\mathcal{O}_C(p')\cong\det F_{2,1}$. Hence, the invertible sheaf $\mathcal{O}_{W'}(S')$ cannot be the pull-back of any invertible sheaf over $\boldsymbol{P}^3$, and we have $\deg\Phi_{|K_S|}\neq 3$.

We can prove that $\Phi_{|K_S|}$ does not give rise to a double covering onto its image as in the case $L\cong\det F_{2,1}$. Therefore, S is canonical in this case, too. q.e.d.

In the case $(e, d)=(2, 2)$, we have the following:

PROPOSITION 3.18. *Let $\pi: W\to C$ be the $\boldsymbol{P}^2$-bundle associated to E with $E_0\in\mathscr{E}_C(2, 2)$ and $L\in\mathscr{E}_C(1, 2)$, T a tautological divisor with $\pi_*\mathcal{O}_W(T)\cong E$, and $D\in\mathrm{Div}(C)$ the divisor with $\mathcal{O}_C(D)\cong\det E$. Then the minimal resolution S of a general member $S'\in|4T-\pi^*D|$ is canonical.*

PROOF. Let the notation be as above, and $y_0\in Y$ as in Lemma 2.5. We have $\mathrm{Bs}\,|Y_1|=\{q_0\}$, where $q_0:=\sigma^{-1}(y_0)\cap Y_0$. If we identify q_0 with $\rho(q_0)$ so that $q_0\in W$, then we have $\mathrm{Bs}\,|T|=\{q_0\}$. Again by Lemma 2.5, general members of $|C_0|$ have the same tangent y_0. Hence if we let $\zeta_1: W_1\to W$ to be the blowing-up at q_0, and T' the proper transform of T, then the complete linear system $|T'|$ has one base point q_0'. If we let $\zeta_2: W_2\to W_1$ to be the blowing-up at q_0', and T'' the proper transform of T', then we have $\mathrm{Bs}\,|T''|=\varnothing$, $\dim|T''|=\dim|T|=3$ and $(T'')^3=T^3-2=2$, and $\Phi_{|T''|}: Y_1''-\boldsymbol{P}^3$ is the double covering. Hence, the canonical mapping of S has degree one or two.

Since $\mathrm{Bs}\,|4T-\pi^*D|=C'$ and since S' has a rational double point of type A_3 at $q_0\in C'$ by the proof of Theorem 3.9, we have $S_1'\sim 4T'+2\mathscr{E}_1-\zeta_1^*\pi^*D$, where $\mathscr{E}_1:=\zeta_1^{-1}(q_0)$ and S_1' is the proper transform of S' by ζ_1. S_1 has a rational double point of type A_1. On the other hand, if we regard Y_0 as a divisor of W, since the support of the intersection of S' with Y_0 is C', this rational double point does not coincide with q_0'. Hence the proper transform S_2' of S_1' by ζ_2 satisfies $S_2'\sim 4T''+6\mathscr{E}_2+2\mathscr{E}_1'-\zeta_2^*\zeta_1^*\pi^*D$, where $\mathscr{E}_2$ is the exceptional divisor of ζ_2, and $\mathscr{E}_1'$ is the proper transform of $\mathscr{E}_1$ by ζ_2. Since $6\mathscr{E}_2+2\mathscr{E}_1'\not\sim\zeta_2^*\zeta_1^*\pi^*D$, we see that S_2' cannot be the pull-back of any effective divisor of $\boldsymbol{P}^3$ by $\Phi_{|T''|}$. Therefore, S is canonical. q.e.d.

3.3. *E is indecomposable.* Let E be an indecomposable locally free sheaf of rank 3 over an elliptic curve C. Denote $d:=\deg E$. We prove the following theorem in §§3.3.1–3.3.4. We consider the case $d\not\equiv 0 \pmod 3$ and $d\neq 1, 2$ in §3.3.1, the case $d\equiv 0 \pmod 3$ and $d\neq 3$ in §3.3.2, the case $d=3$ in §3.3.3, and the case $d=2$ in §3.3.4. We omit the case $d=1$ because it was investigated by Catanese and Ciliberto [7]. In §3.3.5, we study the canonical mappings of the surfaces obtained in §§3.3.1–3.3.4. The results about the canonical mappings are stated in Propositions 3.28, 3.29 and 3.30.

We only have to consider the case $d>0$ by the remark immediately before §3.1.

THEOREM 3.19. *Let* π: $W:=\boldsymbol{P}(E)\to C$ *be the* $\boldsymbol{P}^2$*-bundle associated to* $E\in\mathscr{E}_C(3, d)$, *and* T *a tautological divisor with* $\pi_*\mathcal{O}_W(T)\cong E$. *Then the locally free sheaf* E *satisfies the condition* (A) *if and only if* $d\geq 1$.

REMARK. In this case, $|4T-\pi^*D|$ turns out not to have base points except in the case $d=3$, where $D\in \mathrm{Div}(C)$ satisfies $\mathcal{O}_C(D)\cong E$. Hence its general members are irreducible and nonsingular by Bertini's theorem and Lemma 2.6. In particular, it suffices to show the following lemma when $d\geq 4$ (§§3.3.1–3.3.2).

LEMMA 3.20. *Let the notation be as in Theorem* 3.19. *Then the restriction mapping* $H^0(W, \mathcal{O}_W(4T)\otimes\pi^*\det E^\vee)\to H^0(F, \mathcal{O}_F(4T))$ *is surjective for any fiber* F *of* π *when* $d\geq 4$.

3.3.1. The proof when $\deg E\geq 4$ is not divisible by 3. Suppose $d:=\deg E\not\equiv 0$ (mod 3). By Theorem 2.4, if we choose and fix any isogeny φ: $\tilde{C}\to C$ of degree 3, there exists an invertible sheaf L_0 of degree d over $\tilde{C}$ such that $\varphi_*L_0\cong E$. Furthermore, if we denote $G:=\ker\varphi=\{0, \sigma, 2\sigma\}$ and $L_i:=T_{i\sigma}^*L_0$ $(i=1, 2)$ where $T_{i\sigma}$ is the translation by $i\sigma\in G$, then we have $\tilde{E}:=\varphi^*E\cong\bigoplus_{i=0}^2 L_i$.

Let π: $W:=\boldsymbol{P}(E)\to C$ and $\tilde{\pi}$: $\tilde{W}:=\boldsymbol{P}(\tilde{E})\to\tilde{C}$ be the $\boldsymbol{P}^2$-bundles associated to E and $\tilde{E}$, respectively. Let T and $\tilde{T}$ be tautological divisors on W and $\tilde{W}$, respectively, such that $\pi_*\mathcal{O}_W(T)\cong E$ and $\tilde{\pi}_*\mathcal{O}_{\tilde{W}}(\tilde{T})\cong\tilde{E}$. Consider the following diagram:

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{\Phi} & W \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \tilde{C} & \xrightarrow[\varphi]{} & C \end{array}$$

If we denote Φ: $\tilde{W}\to W$, then $\tilde{T}\sim\Phi^*T$.

Denote $\mathcal{N}=\mathcal{O}_W(4T-F)\otimes\pi^*\det E^\vee$, where F is any fiber of π. It suffices to show that $H^1(W, \mathcal{N})=0$ holds.

The proof of the following is immediate:

LEMMA 3.21. *If* $\{\mathcal{O}_C, \mathcal{M}, \mathcal{M}^{\otimes 2}\}$ *is the kernel* $\ker\varphi^*$ *of the isogeny* φ^*: $\mathrm{Pic}^0(C)\to \mathrm{Pic}^0(\tilde{C})$ *corresponding to* φ: $\tilde{C}\to C$, *then we have* $\varphi_*\mathcal{O}_{\tilde{C}}\cong\mathcal{O}_C\oplus\mathcal{M}\oplus\mathcal{M}^{\otimes 2}$.

By Lemma 3.21, we get $H^1(\tilde{W}, \Phi^*\mathcal{N})\cong\bigoplus_{i=0}^2 H^1(W, \mathcal{N}\otimes\mathcal{M}^{\otimes i})$. Since the action of G on $\tilde{W}$ is fixed point free, we have $H^1(W, \mathcal{N})=H^1(\tilde{W}, \Phi^*\mathcal{N})^G$ (cf., e.g., [11, p. 202, Corollaire]). On the other hand, if we denote $\tilde{F}_0+\tilde{F}_1+\tilde{F}_2:=\Phi^*F$, and $q_i:=\tilde{\pi}(\tilde{F}_i)$ $(i=0, 1, 2)$, then we have $\Phi^*\mathcal{N}\cong\mathcal{O}_{\tilde{W}}(4\tilde{T}-\sum_{i=0}^2\tilde{F}_i)\otimes\tilde{\pi}^*\det\tilde{E}^\vee$, and

$$H^1(\tilde{W}, \Phi^*\mathcal{N})\cong\bigoplus_{\substack{\alpha,\beta,\gamma>0\\ \alpha+\beta+\gamma=4}} H^1(\tilde{C}, L_0^{\otimes(\alpha-1)}\otimes L_1^{\otimes(\beta-1)}\otimes L_2^{\otimes(\gamma-1)}\otimes\mathcal{O}_{\tilde{C}}(-q_0-q_1-q_2)).$$

Since $d \geq 4$, this cohomology group vanishes, and hence $H^1(W, \mathcal{N})=0$. Therefore, Lemma 3.20 in the case $d \not\equiv 0 \pmod 3$ is proved. q.e.d.

3.3.2. The proof when $\deg E \neq 3$ is divisible by 3. If we denote $d_0 = d/3$, there exists an invertible sheaf L of degree d_0 such that $E \cong L \otimes F_3$.

Denote $p := \pi(F)$ for any fiber F of π. Since $S^4(F_3) \cong S^4(S^2(F_2)) \cong F_9 \oplus F_5 \oplus \mathcal{O}_C$ by [4, Theorem 9] and [9, p. 156], we have an isomorphism

$$H^1(W, \mathcal{O}_W(4T-F) \otimes \pi^* \det E^\vee) \cong \bigoplus_{i=0}^{2} H^1(C, F_{4i+1} \otimes L \otimes \mathcal{O}_C(-p)),$$

which vanishes if $d_0 \geq 2$. q.e.d.

3.3.3. The proof when $\deg E = 3$ holds. There exists an invertible sheaf $L \in \mathscr{E}_C(1, 1)$ with $E \cong L \otimes F_3$ for $E \in \mathscr{E}_C(3, 3)$. Let $p_0 \in C$ be a point satisfying $L \cong \mathcal{O}_C(p_0)$. Let $\pi: W \to C$ be the $\boldsymbol{P}^2$-bundle associated to E and T a tautological divisor with $\pi_* \mathcal{O}_W(T) \cong E$. Denote $F_0 := \pi^{-1}(p_0)$.

LEMMA 3.22. *Let the notation be as above, T_0 a relative hyperplane with $T_0 \sim T - F_0$, and $C_0 \subset T_0$ the section of $\mu := \pi_{|T_0}: T_0 \to C$ with $\mu_* \mathcal{O}_{T_0}(C_0) \cong F_2$. Then we have* $\mathrm{Bs}\,|4T-3F_0| = \{q_0\}$ *where $q_0 := C_0 \cap F_0$.*

PROOF. In the same way as in the proof of §3.3.2, we can show that there is no base point of $|4T-3F_0|$ on any fiber except F_0. Furthermore, the base points of $|4T-3F_0|$ exist only on the line $T_0 \cap F_0 \cong \boldsymbol{P}^1$, since $3T_0 + T \in |4T-3F_0|$.

Since $S^3(F_3) \cong F_3 \oplus F_7$ (cf. [4, Theorem 9], [9, p. 156]), the restriction mapping

$$H^0(W, \mathcal{O}_W(4T-3F_0)) \to H^0(T_0, \mathcal{O}_{T_0}(4C_0 + \Gamma_0))$$

is surjective, where $\Gamma_0 := F_0 \cap T_0$, and the statement follows from Lemma 2.5. q.e.d.

The restriction of a general member S of $|4T-3F_0|$ to T_0 is nonsingular by Lemmas 2.5 and 3.22. Hence S is irreducible and nonsingular by Lemma 2.6. q.e.d.

3.3.4. The proof when $\deg E = 2$ holds. Let the notation be an in §3.3.1, and denote $\mathscr{U} := \{\Phi^* S \in |4\tilde{T} - \tilde{\pi}^* \tilde{D}| \mid S \in |4T - \pi^* D|\}$.

G acts on $H^0(\tilde{W}, \mathcal{O}_{\tilde{W}}(4\tilde{T}) \otimes \tilde{\pi}^* \det \tilde{E}^\vee)$. Let $H^0(\tilde{W}, \mathcal{O}_{\tilde{W}}(4\tilde{T}) \otimes \tilde{\pi}^* \det \tilde{E}^\vee)^G$ be the subspace which consists of all the members which are invariant under this action.

LEMMA 3.23. *In the above notation, we have*

$$\mathscr{U} = \{(\Psi) \mid \Psi \in H^0(\tilde{W}, \mathcal{O}_{\tilde{W}}(4\tilde{T}) \otimes \tilde{\pi}^* \det \tilde{E}^\vee)^G\}.$$

PROOF. Since we have $\Phi_* \mathcal{O}_{\tilde{W}} \cong \pi^* \varphi_* \mathcal{O}_{\tilde{C}}$ by the base change theorem (cf., e.g., Mumford [18]), if we denote $\mathcal{N} := \mathcal{O}_W(4T) \otimes \pi^* \det E^\vee$, then we obtain isomorphisms

$$H^0(\tilde{W}, \Phi^* \mathcal{N}) \cong H^0(W, \mathcal{N} \otimes \pi^* \varphi_* \mathcal{O}_{\tilde{C}}) \cong \bigoplus_{i=0}^{2} H^0(W, \mathcal{N} \otimes \pi^* \mathcal{M}^{\otimes i}).$$

The eigenspace $H^0(\tilde{W}, \Phi^*\mathcal{N})^G$ of T_σ^* for the eigenvalue 1 corresponds to $H^0(W, \mathcal{N})$, and is the image of the injection $H^0(W, \mathcal{N}) \hookrightarrow H^0(\tilde{W}, \Phi^*\mathcal{N})$. q.e.d.

We describe the action of G on $H^0(\tilde{W}, \Phi^*\mathcal{N})$ to study $\mathrm{Bs}\,\mathcal{U}$.

We choose and fix $0 \neq X_i \in H^0(\tilde{W}, \mathcal{O}_{\tilde{W}}(\tilde{T}) \otimes \tilde{\pi}^* L_i^{-1})$ $(i=0, 1, 2)$ so that $X_1 = T_\sigma^* X_0$ and $X_2 = T_{2\sigma}^* X_0$ hold. Then any $\Psi \in H^0(\tilde{W}, \Phi^*\mathcal{N})$ can be written as

$$\Psi = \sum_{\substack{\alpha,\beta,\gamma \geq 0 \\ \alpha+\beta+\gamma=4}} \psi_{\alpha\beta\gamma} X_0^\alpha X_1^\beta X_2^\gamma , \qquad \psi_{\alpha\beta\gamma} \in H^0(\tilde{C}, L_0^{\otimes(\alpha-1)} \otimes L_1^{\otimes(\beta-1)} \otimes L_2^{\otimes(\gamma-1)}) .$$

Since we have

$$T_\sigma^* \Psi = \sum_{\substack{\alpha,\beta,\gamma \geq 0 \\ \alpha+\beta+\gamma=4}} (T_\sigma^* \psi_{\alpha\beta\gamma}) X_1^\alpha X_2^\beta X_0^\gamma ,$$

we get $\Psi \in H^0(\tilde{W}, \mathcal{O}_W(4\tilde{T}) \otimes \tilde{\pi}^* \det \tilde{E}^\vee)^G$ if and only if $T_\sigma^* \psi_{\alpha\beta\gamma} = \psi_{\gamma\alpha\beta} (\alpha, \beta, \gamma \geq 0, \alpha+\beta+\gamma=4)$.

Let $\Lambda\colon \tilde{C} \to \mathrm{Pic}^0(\tilde{C})$ be defined by $\Lambda(y) := T_y^* L_0 \otimes L_0^{-1}$ for $y \in C$, where T_y is the translation by y on $\tilde{C}$. Then it is a group homomorphism by the theorem of square (cf., e.g., [18]). Since $L_i = L_0 \otimes \Lambda(i\sigma)$, $(i=1, 2)$ and $\Lambda(3\sigma) = \Lambda(0) = \mathcal{O}_{\tilde{C}}$, we have isomorphisms

$$L_i \cong L_i^{\otimes 3} \otimes L_{\tau(i)}^{-1} \otimes L_{\tau^2(i)}^{-1} \cong L_{\tau(i)}^{\otimes 2} \otimes L_{\tau^2(i)}^{-1} \cong L_{\tau(i)}^{-1} \otimes L_{\tau^2(i)}^{\otimes 2} \cong L_i^{-1} \otimes L_{\tau(i)} \otimes L_{\tau^2(i)}$$

for $i=0, 1, 2$, where τ is the cyclic permutation (123). Hence we have

$$\begin{aligned}
&\psi_{400}, \psi_{211}, \psi_{130}, \psi_{103}, \psi_{022} \in H^0(\tilde{C}, L_0) \\
&\psi_{040}, \psi_{121}, \psi_{013}, \psi_{310}, \psi_{202} \in H^0(\tilde{C}, L_1) \\
&\psi_{004}, \psi_{112}, \psi_{301}, \psi_{031}, \psi_{220} \in H^0(\tilde{C}, L_2) .
\end{aligned}$$

Let $\{s_1, s_2\} \subset H^0(\tilde{C}, L_0)$ be a basis, and denote $t_j := T_\sigma^* s_j \in H^0(\tilde{C}, L_1)$, $u_j := T_{2\sigma}^* s_j \in H^0(\tilde{C}, L_2)$ $(j=1, 2)$. We can choose a basis of $H^0(\tilde{W}, \mathcal{O}_{\tilde{W}}(4\tilde{T}) \otimes \tilde{\pi}^* \det \tilde{E})^G$ consisting of the following for $j=1, 2$:

$$\begin{aligned}
\Psi_{1j} &:= s_j X_0^4 + t_j X_1^4 + u_j X_2^4 , \\
\Psi_{2j} &:= s_j X_0^2 X_1 X_2 + t_j X_0 X_1^2 X_2 + u_j X_0 X_1 X_2^2 , \\
\Psi_{3j} &:= s_j X_0 X_1^3 + t_j X_1 X_2^3 + u_j X_0^3 X_2 , \\
\Psi_{4j} &:= s_j X_0 X_2^3 + t_j X_0^3 X_1 + u_j X_1^3 X_2 , \\
\Psi_{5j} &:= s_j X_1^2 X_2^2 + t_j X_0^1 X_2^2 + u_j X_0^2 X_1^2 .
\end{aligned}$$

LEMMA 3.24. *We can choose the basis $\{s_1, s_2\}$ of $H^0(\tilde{C}, L_0)$ so that $s_j(p) t_j(p) u_j(p) \neq 0$ holds for any $p \in \tilde{C}$ and for at least one of $j=1, 2$. Furthermore, we have $s_j(p) s_j(p') s_j(p'') \neq 0$, where $p' := T_\sigma(p)$ and $p'' := T_{2\sigma}(p)$.*

PROOF. To avoid confusion in this proof, we denote by (q) the divisor on $\tilde{C}$ determined by $q \in \tilde{C}$. There exist distinct points $p_1, p_2 \in \tilde{C}$ with $L_0 \cong \mathcal{O}_{\tilde{C}}(2(p_1)) \cong \mathcal{O}_{\tilde{C}}(2(p_2))$ by Abel's theorem. If we denote $p_i' := T_{-\sigma}(p_i)$ and $p_i'' := T_{-2\sigma}(p_i)$ $(i=1, 2)$, we have

$\{p_1, p_1', p_1''\} \cap \{p_2, p_2', p_2''\} = \varnothing$.

Let $s_1, s_2 \in H^0(\tilde{C}, L_0)$ be the global sections defining the divisors $2(p_1)$, $2(p_2)$ respectively, and denote $t_j := T_\sigma^* s_j$, and $u_j := T_{2\sigma}^* s_j$ $(j=1, 2)$. Then one of s_1 and s_2 satisfies the condition of the lemma for any point. q.e.d.

We choose a basis $\{s_1, s_2\} \subset H^0(\tilde{C}, L_0)$ as in Lemma 3.24, fix any point $p \in \tilde{C}$, and denote $p' := T_\sigma(p)$ and $p'' := T_{2\sigma}(p)$. We assume that $j \in \{1, 2\}$ satisfies $s_j(p)s_j(p')s_j(p'') \neq 0$. Let us restrict Ψ_{ij} $(i=1, \dots, 5,\ j=1, 2)$ to $\tilde{\pi}^{-1}(p)$, and study if they have common solutions on it. Note $\Psi_{2j} = X_0X_1X_2(s_jX_0 + t_jX_1 + u_jX_2)$.

The following lemma is trivial.

Lemma 3.25. *If we fix $j \in \{1, 2\}$ satisfying $s_j(p)s_j(p')s_j(p'') \neq 0$, then $X_i = 0$, $\Psi_{1j} = 0$ and $\Psi_{3j} = 0$ do not have common solutions for any $i = 0, 1, 2$.*

In view of Lemma 3.25, we consider only the solutions satisfying $X_0X_1X_2 \neq 0$ in the rest of our argument. Denote $\Psi_{0j} := s_jX_0 + t_jX_1 + u_jX_2$.

Lemma 3.26. *If we fix $j \in \{1, 2\}$ with $s_j(p)s_j(p')s_j(p'') \neq 0$, then $(p, (1:a:b))$ is a common solution of $\Psi_{ij} = 0$ $(i=0, 1, 3, 4, 5)$ if any only if a, b are cube roots of 1, and $s_j(p) + at_j(p) + bu_j(p) = 0$.*

Proof. Since we have $\Psi_{1j} + \Psi_{3j} + \Psi_{4j} = \Psi_{0j}(X_0^3 + X_1^3 + X_2^3)$, we may exclude Ψ_{1j} from our consideration. We have

$$X_0^5\Psi_{0j} - X_0^2(\Psi_{3j} + \Psi_{4j}) + X_1X_2\Psi_{5j} = s_j(X_0^3 - X_1^3)(X_0^3 - X_2^3)\,.$$

If $X_1^3 = X_0^3$ holds, since we have $X_0^3\Psi_{0j} - \Psi_{4j} = s_jX_0(X_0^3 - X_2^3)$ and since $s_j(p) \neq 0$ and $X_0 \neq 0$, we obtain $X_2^3 = X_0^3$. Similarly, if we assume $X_2^3 = X_0^3$, we have $X_1^3 = X_0^3$. Hence, the common solutions are of the form $(p, (1:a:b))$ where a and b are one of 1, ω, and ω^2, and ω is a cube root of 1. If $(p, (1:1:1))$ is a common solution, we obtain $s_j(p) + t_j(p) + u_j(p) = 0$ by substituting $(p, (1:1:1))$ into $\Psi_{ij} = 0$ $(i=0, 1, 3, 4, 5)$. We can obtain the same result in the other cases. q.e.d.

Proposition 3.27. *$\mathcal{U}$ has no base point. Hence a general member of $\mathcal{U}$ is irreducible and nonsingular by Bertini's theorem and Lemma 2.6.*

Proof. Assume $(p, (1:a:b)) \in \mathrm{Bs}\,\mathcal{U}$. Let $g\colon \tilde{C} \to \boldsymbol{P}^1$ be defined by the complete linear system of L_0. Denote $p' := T_\sigma(p)$ and $p'' := T_{2\sigma}(p)$.

First assume that $g(p) = g(p')$ holds. If $s' \in H^0(\tilde{C}, L_0)$ is a global section defining the divisor $p + p'$, then $s'(p) + as'(p') + bs'(p'') \neq 0$. This contradicts Lemma 3.26. We can obtain the same results when $g(p') = g(p'')$ or $g(p'') = g(p)$ holds.

Next, assume that $g(p)$, $g(p')$ and $g(p'')$ are pairwise distinct. Let ξ_0, ξ_∞ be global sections of $\mathcal{O}_{\boldsymbol{P}^1}(1)$ defining $g(p)$, $g(p')$, respectively. Then any $\xi \in H^0(\boldsymbol{P}^1, \mathcal{O}_{\boldsymbol{P}^1}(1))$ is written as $\xi = A\xi_0 + B\xi_\infty$ $(A, B \in \boldsymbol{C})$, and clearly there exist $A, B \in \boldsymbol{C}$ such that $s := g^*\xi \in H^0(\tilde{C}, L_0)$ satisfies $s(p) + as(p') + bx(p'') \neq 0$. This contradicts Lemma 3.26. q.e.d.

Let $\tilde{S}\in\mathscr{U}$ be irreducible and nonsingular. We have $S:=\Phi(\tilde{S})\in|4T-\pi^*D|$, and $\Phi_{|\tilde{S}}$: $\tilde{S}\to S$ is the restriction of the action of G on $\tilde{W}$. Since this action has no fixed point, S is irreducible and nonsingular as well. q.e.d.

REMARK. Instead of our argument in §3.3.1, we can use the above argument also in the case $d\geq 4$ and $d\not\equiv 0 \pmod 3$.

3.3.5. The canonical mapping. In this section, we study the canonical mappings of those surfaces whose existence was shown in §§3.3.1–3.3.4.

PROPOSITION 3.28. *Let* π: $W:=\boldsymbol{P}(E)\to C$ *be the* $\boldsymbol{P}^2$*-boundle associated to* $E\in\mathscr{E}_C(3, d)$, T *a tautological divisor with* $\pi_*\mathcal{O}_W(T)\cong E$, *and* $D\in\mathrm{Div}(C)$ *satisfy* $\mathcal{O}_C(D)\cong\det E$. *Then a genral member of* $|4T-\pi^*D|$ *is canonical when* $d\geq 4$.

PROOF. Since $\dim H^1(W, \mathcal{O}_W(T-F))=0$, for any fiber F of π, the restriction mapping $H^0(W, \mathcal{O}_W(T))\to H^0(F, \mathcal{O}_F(T))$ is surjective when $d\geq 4$.

We first show that $\Phi_{|T|}$ is birational onto the image when $d\geq 5$, hence a general member of $|4T-\pi^*D|$ is canonical. We can prove this in the same way as in the proof of Lemma 3.12 when $d\geq 7$ in view of what we just saw above, and when $d=6$ in view of Lemma 3.22. If $d=5$, then since $5=T^3=\deg\Phi_{|T|}\deg\Phi_{|T|}(W)$ and $\deg\Phi_{|T|}(W)\geq 2$, we see that $\Phi_{|T|}$ is birational onto the image.

In the case where $d=4$, since $\mathrm{Bs}|T|=\varnothing$, and since $T^3=4>0$, we see that $\Phi_{|T|}$ gives a 4-fold covering of W onto $\boldsymbol{P}^3$. Hence, $\Phi_{|K_S|}$ is a morphism, and the degree of $\Phi_{|K_S|}$ is 1, 2, 3 or 4 for general $S\in|4T-\pi^*D|$.

Since $K_S^2=T^2S=12$ holds, if $\deg\Phi_{|K_S|}=4$, then $S'':=\Phi_{|K_S|}(S)\subset\boldsymbol{P}^3$ is a cubic surface. Hence, we have $\Phi_{|T|}^*S''\sim 3T$, which is absurd since $S\sim 4T-\pi^*D$.

If $\deg\Phi_{|K_S|}=3$, we have $\Phi_{|T|}^*S''\sim 4T$ as above. Therefore, there exist fibers F_1, F_2, F_3, F_4 of π satisfying $\Phi_{|T|}^*S''=S+F_1+F_2+F_3+F_4$. $\Phi_{|T|}$ is a birational morphism of $F_1\cup F_2\cup F_3\cup F_4$ onto its image, since $\deg\Phi_{|T|}=4$ and $\deg\Phi_{|K_S|}=\deg\Phi_{|K_S|}|_S=3$. This means that the image is not irreducible, a contradiction.

Finally, we show that the case $\deg\Phi_{|K_S|}=2$ does not occur. Let $p, p'\in C$ be two distinct general points. Furthermore, denote $F_p:=\pi^{-1}(p)$ and $F_{p'}:=\pi^{-1}(p')$, and let T_p and $T_{p'}$ be the relative hyperplanes of W satisfying $T\sim T_p+F_p\sim T_{p'}+F_{p'}$. Since p, p' and S are generic, $S\cap T_p\cap F_{p'}$. $S\cap T_{p'}\cap F_p$ and $S\cap T_p\cap T_{p'}$ all consist of four distinct points set-theoretically. Since any fiber of π is mapped onto its image in $\boldsymbol{P}^3$ by $\Phi_{|T|}$, if $\deg\Phi_{|K_S|}=2$, then some point of $S\cap T_p\cap F_{p'}$ and some point of $S\cap T_{p'}\cap F_p$ are mapped to the same point by $\Phi_{|T|}$. Hence if we fix any point $q\in S\cap T_p\cap F_{p'}$ and any point $q'\in S\cap T_{p'}\cap F_p$, we only have to find a member of $|T|$ containing q but not q'.

It is well-known that W is isomorphic to the symmetric product of C of degree 3 (cf. e.g., [7]). We can show that the image of $(C\times C\times\{p\})\cup(C\times\{p\}\times C)\cup(\{p\}\times C\times C)\subset C\times C\times C$ in W is a relative hyperplane with self-intersection number one in the same way as in the proof of Lemma 3.16. Therefore, for a general point of W, there exist three distinct relative hyperplanes with self-intersection number one containing the point.

Since p, p' and S are general, there exist two distinct relative hyperplanes T'_p and T''_p distinct from T_p and containing q. If F'_p and F''_p are fibers of π satisfying $T \sim T'_p + F'_p \sim T''_p + F''_p$, respectively, then one of $T'_p + F'_p$ and $T''_p + F''_p$ does not contain q'.

Hence $\Phi_{|K_S|}$ is a birational morphism onto its image. q.e.d.

Next, we investigate the canonical mapping in the case $p_g(S) = d = 3$. We use the notation of §3.3.3.

PROPOSITION 3.29. *Let the notation be as in* §3.3.3. *Then the canonical mapping of a nonsingular member* $S \in |4T - 3F_0|$ *has degree* 8.

PROOF. In the same way as in the proof of Lemma 3.22, we can show that $\mathrm{Bs}\,|T| = Bs\,|4T - 3F_0| = \{q_0\}$. Hence, the canonical system of S has one base point. If $\nu\colon \bar{W} \to W$ is the blowing-up at q_0, the complete linear system of the proper transform $\bar{T}$ of T by ν has one base point by Lemma 2.5. On the other hand, the proper transform $\bar{S}$ of S by ν does not go through the base point of $|\bar{T}|$ by Lemma 2.5. Hence, if we denote $\mathscr{E} := \nu^{-1}(q_0)$, we have $\deg \Phi_{|K_S|} = \deg \Phi_{|K_{\bar{S}}|} = \bar{T}^2(4\bar{T} + 3\mathscr{E} - 3F_0) = 8$. q.e.d.

Finally, we study the canonical mapping in the case $p_g(S) = 2$. In §3.2.2, we proved the existence of a surface S with $K_S^2 = 3p_g(S)$, $q(S) = 1$ and $p_g(S) = 2$, but did not study the canonical mapping $\Phi_{|K_S|}$ in the case $E \cong E_0 \oplus L$, $(E_0 \in \mathscr{E}_C(2, 1)$, $L \in \mathscr{E}_C(1\ 1))$. On the other hand, we showed the existence of a surface S with the same invariants in the case $E \in \mathscr{E}_C(3, 2)$. We obtain the following result in these two cases:

PROPOSITION 3.30. *Let E be one of the following*:

(i) $E := E_0 \oplus L$ *with* $E_0 \in \mathscr{E}_C(2, 1)$, $L \in \mathscr{E}_C(1, 1)$.

(ii) $E \in \mathscr{E}_C(3, 2)$.

In the same notation as in Proposition 3.28, *the canonical mapping of the minimal resolution of a general member* $S \in |4T - \pi^* D|$ *gives a linear pencil whose general fibers are irreducible nonsingular curves of genus* 7.

PROOF. Since $H^0(S, \omega_S)$ is 2-dimensional, $|K_s|$ is a linear pencil. Furthermore, we have $g(Z) = (1/2)T(2T)(4T - \pi^* D) + 1 = 7$ for a general member of Z of $|K_S|$. q.e.d.

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