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SHAPE OF SPIRALS

Dedicated to Professor Kyûya Masuda on his sixtieth birthday

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Abstract. The structure of spiral-like self-similar solutions for the plane curve shortening flow is investigated.

1. Introduction. This short note deals with the structure of spiral-like self-similar solutions for the curve shortening equation

$$v = -k\,, \tag{1}$$

where v denotes the normal velocity and k the curvature.

Suppose a spiral-like curve in the plane is given. Taking the tip as the origin, we parametrize the curve by the polar coordinate $(r, \theta) = (r(\theta, t), \theta)$.

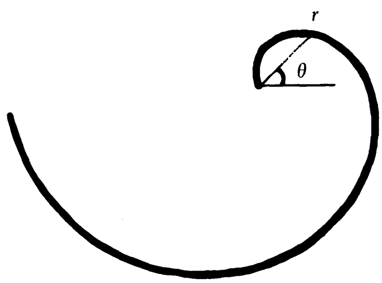

FIGURE.

Then, (1) reduces to the equation

$$\frac{rr_t}{\sqrt{r^2 + r_\theta^2}} = \frac{rr_{\theta\theta} - 2r_\theta^2 - r^2}{(r^2 + r_\theta^2)^{3/2}}\,. \tag{2}$$

We try to seek a solution to (2) of the form $r(\theta, t) = \sqrt{2(1+t)}R(\theta)$ and/or $r(\theta, t) = \sqrt{2(T-t)}R(\theta)$ for some $T > 0$; we deduce

$$\pm\frac{R^2}{\sqrt{R^2 + R_\theta^2}} = \frac{RR_{\theta\theta} - 2R_\theta^2 - R^2}{(R^2 + R_\theta^2)^{3/2}}\,, \tag{3}$$

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where the $+$ sign (resp. $-$ sign) corresponds to the former (resp. latter) one and its solution is referred to as the expanding (resp. shrinking) self-similar solution.

(3) can be written as

$$\mp\langle X, N\rangle = -k\,, \tag{4}$$

where $X=(R\cos\theta, R\sin\theta)$ denotes the position vector and N means the unit normal vector

$$N=\frac{1}{\sqrt{R^2+R_\theta^2}}(-R_\theta\sin\theta - R\cos\theta,\ R_\theta\cos\theta - R\sin\theta)\,.$$

If $k>0$ in (4), we further find the equation for k in terms of the arc-length parameter s from the origin:

$$\frac{\partial^2 k}{\partial s^2}=\frac{1}{k}\left(\frac{\partial k}{\partial s}\right)^2 \mp k - k^3\,. \tag{5}$$

See [8, §5] for details.

A strictly decreasing positive function k satisfying (5) will be defined as a spiral-like self-similar solution for (1). We now state our main result.

THEOREM. (A) *There exists a family of expanding spiral-like self-similar solutions for* (1). *Every such solution has an asymptotic behavior* $k(s)=\exp((-s^2/2)(1+o(1)))$ *as* $s\to\infty$, *and therefore, its rotation number* $n=(1/2\pi)\int_0^\infty k(s)ds$ *is finite. The values of* n *which can be attained in this family is bounded from above.*

(B) *There exists no shrinking spiral-like self-similar solution for* (1). *The solution of* (5) *with* $+$*sign must have a self-intersection unless* $k\equiv 1$. *If the curve obtained by such a solution is closed, then it is one of the homothetic solutions classified by Abresch and Langer* [1].

Before going into the proof, several remarks are in order. The field of curvature evolution is now a huge area of research and much progress has been made. See for instance a famous paper by Gage and Hamilton [5] and a recent monograph by Giga and Chen [7]. Self-similar solutions also have been studied, partly because of the aim of theoretically describing the singularity, which necessarily takes place in the curvature evolution. See Angenent [3] in this respect. A spiral-like self-similar solution might have been examined; indeed, Part I of Altschuler [2] says "We mention one other noncompact curve (besides the straight line) which moves in a self-similar manner. This is the non-convex yin-yang curve which spirals out to infinity (note that $\int|k|ds=\infty$). This curve moves by rotation." However, there seems to be no explicit presentation in the literature.

The equation (1) itself admits other family of self-similar solutions. If the curve is represented by a graph, then the structure of expanding self-similar solutions is analyzed in [9]. For other kinds of curvature evolution, we refer, for instance, to a recent paper by Dohmen, Giga, and Mizoguchi [4].

One main motivation for considering a spiral-like solution comes from material science, especially the phenomena of rotating spiral waves observed in a variety of chemical and biological excitable media. Most remarkable pattern is known in the Belousov-Zhabotinskii reagent, and to understand these patterns, several evolution equations involving the curvature are presented. We refer to the elaborate review articles by Tyson and Keener [11] and Mikhailov, Davydov, and Zykov [10]. Some equations in these papers seem to remain an attractive subject for further investigation. However, it is interesting to know whether the simple curvature evolution (1) produces a spiral-like solution or not. Another phenomenon expected to have spiral patterns is the growth of crystals. For this topic, see for instance, Giga and Giga [6] and the references therein.

This paper is dedicated to Professor Kyûya Masuda on the occasion of his sixtieth birthday, who, as an advisor, introduced the author to the world of mathematics, and as a researcher, constantly inspires the author.

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2. Proof of the theorem. The analysis of (5) is performed in an elementary fashion.

Since we are interested in a positive solution of (5), we put $k(s)=e^{l(s)}$, taking into account that the strictly decreasing property for k is equivalent to the same one for l. The equation for l becomes

$$\frac{\partial^2 l(s)}{\partial s^2}=\mp 1-e^{2l(s)}, \tag{6}$$

where the $-$ sign (resp. $+$ sign) corresponds to the expanding (resp. shrinking) case.

(6) is easy to handle and the existence of a solution is immediate. First we deal with the shrinking case, i.e., the $+$ sign in (6). This case gives a kind of nonlinear oscillation. There is an energy functional

$$E(s)=\frac{1}{2}\left(\frac{\partial l(s)}{\partial s}\right)^2+\frac{1}{2}e^{2l(s)}-l(s).$$

It is straightforward to check $\partial E(s)/\partial s=0$. We remark that $2^{-1}e^{2x}-x$ is nonnegative for all $x\in \boldsymbol{R}$ and tends to infinity as $|x|\to\infty$. The solution $l(s)$ must oscillate; the curve represented by $k(s)$ has a self-crossing. If the curve closes, then it must be one of the homothetic solutions established by Abresch and Langer [1]. Indeed, (6) with the $+$sign can be transformed into (2) of Theorem A in [1].

Next we turn our attention to the expanding case, i.e., the $-$ sign in (6).

We observe that it suffices to discuss (6) under the additional assumption

$$\frac{\partial l(0)}{\partial s}=0\,, \quad l(0)=l_0\,. \tag{7}$$

To see this, if $l'(0)<0$ (recalling that we are dealing with a decreasing function), then we solve (6) for negative s. From the inequality $l''(s)=-1-e^{2l}<-1$, we infer that $l'(0)-l'(s)<s$ for $s<0$; that is, there exists an $s_0<0$ such that $l'(s_0)=0$ and $l(s_0)$ is finite. Since the equation (6) is autonomous, the solution curve is a part of the one of (7) with a suitable l_0.

We integrate (6), now keeping (7) in mind:

$$l''(s)=-1-e^{2l(s)}<-1\,, \quad l'(s)<-s\,,$$

$$l(s)<-\frac{s^2}{2}+l_0=-\frac{s^2}{2}(1+o(1)) \qquad \text{as} \quad s\to\infty\,.$$

Using the above estimate, we have

$$l''(s)>-1-e^{-s^2+2l_0}\,,$$

$$l'(s)>-s-\int_0^s e^{-t^2+2l_0}dt\,,$$

$$\begin{aligned} l(s)&>-\frac{s^2}{2}+l_0-e^{2l_0}\int_0^s du\int_0^u e^{-t^2}dt \\ &=-\frac{s^2}{2}+l_0-e^{2l_0}\int_0^s (s-t)e^{-t^2}dt \\ &>-\frac{s^2}{2}+l_0-e^{2l_0}\left(\frac{s\sqrt{\pi}}{2}(1-e^{-2s^2})^{1/2}-\frac{1}{2}(1-e^{-s^2})\right) \\ &=-\frac{s^2}{2}(1+o(1)) \qquad \text{as} \quad s\to\infty\,. \end{aligned}$$

Finally, we give an upper bound for the rotation number n. First we derive

$$l''(s)=-1-e^{2l(s)}\geq -1-e^{2l_0}\,,$$

$$l(s)\geq -\frac{1}{2}(1+e^{2l_0})s^2+l_0\,,$$

and therefore

$$l''(s)\leq -1-e^{2l_0}\exp(-(1+e^{2l_0})s^2)\,,$$

$$l(s)\leq -\frac{s^2}{2}+l_0-e^{2l_0}\int_0^s (s-t)\exp(-(1+e^{2l_0})t^2)dt\,. \tag{8}$$

For all $l_0 \geq 100$, we estimate the last integral as follows:

$$\begin{aligned}
&\int_0^s (s-t)\exp(-(1+e^{2l_0})t^2)dt \\
&\quad > s(1+e^{2l_0})^{-1/2}\frac{\sqrt{\pi}}{2}(1-\exp(-(1+e^{2l_0})s^2))^{1/2} \\
&\quad\quad -\frac{1}{2}(1+e^{2l_0})^{-1}(1-\exp(-(1+e^{2l_0})s^2)) \\
&\quad > \frac{1}{10}e^{-l_0}s \qquad \text{for} \quad s \geq 10e^{-l_0}\,.
\end{aligned}$$

Putting this bound into (8), we obtain

$$\begin{aligned}
n &= \frac{1}{2\pi}\int_0^\infty k ds = \frac{1}{2\pi}\int_0^\infty e^{l(s)}ds \\
&< \frac{1}{2\pi}\left(\int_0^{10e^{-l_0}} e^{l_0}ds + \int_{10e^{-l_0}}^\infty \exp\left(l_0 - \frac{1}{10}e^{l_0}s\right)ds\right) \\
&= \frac{1}{2\pi}(10+10e^{-1})\,.
\end{aligned}$$

Since we have

$$n = \frac{1}{2\pi}\int_0^\infty e^{l(s)}ds < \frac{e^{2l_0}}{2\pi}\int_0^\infty e^{-s^2/2}ds \to 0 \qquad \text{as} \quad l_0 \to -\infty\,,$$

we conclude the proof.

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