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SEMILINEAR EQUATIONS AT RESONANCE WITH THE KERNEL OF AN ARBITRARY DIMENSION

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Abstract. By using the alternative method and the topological degree theory, we obtain some sufficient conditions for the existence of 2π-periodic solutions of some semilinear equations at resonance where the kernel of the linear part has an arbitrary dimension

1. Introduction. The existence problem of periodic solutions for nonlinear systems at resonance has been extensively investigated in the literature and many existence results have been obtained for nonlinear systems of first order differential equations at resonance that involve a small parameter (see Hale [1], Nagle [2] and the references therein).

Many existence results have also been obtained for some nonlinear systems whose nonlinearities satisfy the so-called Landesman-Lazer conditions. Several of these results are mentioned in [3].

In the special case where the linear part has a two-dimensional kernel, some results have also been obtained in [4]–[9]. However, considerably less is known for the case where the linear part has dimension greater than two. In this direction, an example with a three-dimensional kernel and a fourth order ordinary differential equation are considered in [8] and [10] respectively. In a recent paper [11], the authors have extended some results in [8] to semilinear equations with a three-dimensional or four-dimensional kernel. By using some fixed point theorem, [12] studied the existence of periodic solutions of the n-dimensional Duffing system at resonance

$$\ddot{x}_s+m_s^2x_s+f_s(t, x)=p_s(t)\,, \qquad s=1, 2, \dots, n$$

with unbounded perturbations $f_s(t, x)$ $(x=(x_1, x_2, \dots, x_n))$ and some additional conditions.

For some related topics, we refer to [13], [14], [15] and the references therein.

In the present paper, we are concerned with the existence of 2π-periodic solutions to the nonlinear system of first order functional differential equations of mixed type

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$$\text{(1.1)}\qquad \begin{cases} \dot{x}_i(t)=B_i x_i(t)+F_i(t, x(t+\cdot), y(t+\cdot))+p_i(t)\,, & i=1,2,\dots,n_1\,,\\ \dot{y}_j(t)=f_j(t, x(t+\cdot), y(t+\cdot))+E_j(t)\,, & j=1,2,\dots,n_2\,,\end{cases}$$

where n_1, n_2 are nonnegative integers with $n_1+n_2\geq 1$; $x_i(t)\in \boldsymbol{R}^2$, $y_j(t)\in \boldsymbol{R}$; $B_i\in \boldsymbol{R}^4$; $x(t+\cdot)\in BC(\boldsymbol{R}, \boldsymbol{R}^{2n_1})$ and $y(t+\cdot)\in BC(\boldsymbol{R}, \boldsymbol{R}^{n_2})$ are defined by $x(t+s)=(x_1(t+s), x_2(t+s), \dots, x_{n_1}(t+s))$ and $y(t+s)=(y_1(t+s), y_2(t+s), \dots, y_{n_2}(t+s))$ respectively; $p_i\in C(\boldsymbol{R}, \boldsymbol{R}^2)$ and $E_j\in C(\boldsymbol{R}, \boldsymbol{R})$ are 2π-periodic in t, and

$$F_i\colon \boldsymbol{R}\times BC(\boldsymbol{R}, \boldsymbol{R}^{2n_1})\times BC(\boldsymbol{R}, \boldsymbol{R}^{n_2})\to \boldsymbol{R}^2\,,$$
$$f_j\colon \boldsymbol{R}\times BC(\boldsymbol{R}, \boldsymbol{R}^{2n_1})\times BC(\boldsymbol{R}, \boldsymbol{R}^{n_2})\to \boldsymbol{R}\,,$$

are continuous, bounded and 2π-periodic with respect to the first variable t.

In this paper, we assume that

$$B_i=\begin{pmatrix} 0 & m_i\\ -m_i & 0\end{pmatrix}, \qquad i=1,2,\dots,n_1$$

where m_i $(i=1,2,\dots,n_1)$ are some positive integers.

2. Statement of Main Result.

We need the following two hypotheses

(F) There exists a permutation $k_1, k_2, \dots, k_{n_1}$ consisting of $1,2,\dots,n_1$ and for any i with $1\leq i\leq n_1$, there exist $\tau_i^{(1)}\in \boldsymbol{R}$, $H_i\in BC(\boldsymbol{R}^2, \boldsymbol{R}^2)$ such that the asymptotic limits $H_i(\pm, \pm)=\lim_{r,s\to\pm\infty} H_i(r, s)$ exist, and there exists $G_i\colon \boldsymbol{R}\times BC(\boldsymbol{R}, \boldsymbol{R}^{2n_1})\times BC(\boldsymbol{R}, \boldsymbol{R}^{n_2})\to \boldsymbol{R}^2$, which is continuous, bounded and 2π-periodic with respect to its first variable t, such that for any $t\in \boldsymbol{R}$, $\varphi\in BC(\boldsymbol{R}, \boldsymbol{R}^{2n_1})$ and $\psi\in BC(\boldsymbol{R}, \boldsymbol{R}^{n_2})$,

$$F_i(t, \varphi, \psi)=H_i(\varphi_{2k_i-1}(-\tau_i^{(1)}), \varphi_{2k_i}(-\tau_i^{(1)}))+G_i(t, \varphi, \psi)\,.$$

(f) There exists a permutation $l_1, l_2, \dots, l_{n_2}$ consisting of $1,2,\dots,n_2$ and for any j with $1\leq j\leq n_2$, there exist $\tau_j^{(2)}\in \boldsymbol{R}$, $h_j\in BC(\boldsymbol{R}, \boldsymbol{R})$ such that the asymptotic limits $h_j(\pm)=\lim_{r\to\pm\infty} h_j(r)$ exist, and there exists $g_j\colon \boldsymbol{R}\times BC(\boldsymbol{R}, \boldsymbol{R}^{2n_1})\times BC(\boldsymbol{R}, \boldsymbol{R}^{n_2})\to \boldsymbol{R}$, which is continuous, bounded and 2π-periodic with respect to its first variable t, such that for any $t\in \boldsymbol{R}$, $\varphi\in BC(\boldsymbol{R}, \boldsymbol{R}^{2n_1})$ and $\psi\in BC(\boldsymbol{R}, \boldsymbol{R}^{n_2})$,

$$f_j(t, \varphi, \psi)=h_j(\psi_{l_j}(-\tau_j^{(2)}))+g_j(t, \varphi, \psi)\,.$$

To state our main theorem, we also need some notation as follows. For any positive integer n, we shall denote by $|\cdot|$ the Euclidean norm in $\boldsymbol{R}^n$. Whenever the assumptions (F) and (f) are satisfied, for $i=1,2,\dots,n_1$ and $j=1,2,\dots,n_2$ we set

$$\text{(2.1)}\qquad \bar{p}_i(m_i):=\frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix}\cos m_i s & -\sin m_i s\\ \sin m_i s & \cos m_i s\end{pmatrix}p_i(s)ds\,,$$

$$\text{(2.2)}\qquad W(H_i):=\frac{\sqrt{2}}{2\pi}\left[H_i(+,+)-H_i(-,-)+\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}(H_i(+,-)-H_i(-,+))\right],$$

$$\text{(2.3)}\qquad M_{G_i}:=\sup\{|G_i(t, \varphi, \psi)|\colon t\in \boldsymbol{R},\ \varphi\in BC(\boldsymbol{R}, \boldsymbol{R}^{2n_1}),\ \psi\in BC(\boldsymbol{R}, \boldsymbol{R}^{n_2})\}\,,$$

$$\bar{E}_j := \frac{1}{2\pi} \int_0^{2\pi} E_j(s)ds\,, \tag{2.4}$$

$$M_{g_i} := \sup\{|g_j(t, \varphi, \psi)| : t \in \boldsymbol{R},\ \varphi \in BC(\boldsymbol{R}, \boldsymbol{R}^{2n_1}),\ \psi \in BC(\boldsymbol{R}, \boldsymbol{R}^{n_2})\}\,. \tag{2.5}$$

The main result in this paper is the following Theorem 2.1, which provides a sufficient condition for the existence of 2π-periodic solutions of the equation (1.1).

THEOREM 2.1. *In addition to* (F) *and* (f), *we assume that* $m_{k_i} = m_i$ $(i = 1, 2, \dots, n_1)$ *and that*

(H) $|W(H_i)| > M_{G_i} + |\bar{p}_i(m_i)|\,, \qquad i = 1, 2, \dots, n_1,$

(h) $h_j(+)h_j(-) < 0,\ |h_j(\pm)| > M_{g_j} + |\bar{E}_j|,\ j = 1, 2, \dots, n_2$

hold. Then the equation (1.1) *has at least one* 2π*-periodic solution.*

3. Preliminaries. To prove Theorem 2.1, we need to state some basic facts about the degree theory.

Let X and Z be real normed spaces and $L: \operatorname{dom} L \subset X \to Z$ be a linear Fredholm mapping of index zero, i.e. $\operatorname{Im} L$ is closed and $\dim \ker L = \operatorname{codim} \operatorname{Im} L < \infty$. It follows that there exist continuous projections $P: X \to X$ and $Q: Z \to Z$ such that $\operatorname{Im} P = \ker L$, $\operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q)$. Moreover, the restriction $L_P: \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$ of L to $\ker P$ is invertible. We denote its inverse by $K_P: \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$. We shall denote by $K_{P,Q}: Z \to \operatorname{dom} L \cap \ker P$ the generalized inverse of L defined by $K_{P,Q} = K_P(I - Q)$.

Let Ω be a bounded open subset in X such that $\operatorname{dom} L \cap \Omega \neq \emptyset$ and $N: \bar{\Omega} \to Z$ is a nonlinear mapping. The mapping N is said to be L-compact on $\bar{\Omega}$ if $QN: \bar{\Omega} \to Z$ is continuous, $QN(\bar{\Omega})$ is bounded and $K_{P,Q}N: \bar{\Omega} \to X$ is compact (i.e. it is continuous and $K_{P,Q}N(\bar{\Omega})$ is relatively compact). This definition does not depend upon the choice of P and Q.

Let $L: \operatorname{dom} L \subset X \to Z$ be a Fredholm mapping of index zero and $\Omega \subset X$ a bounded open set. In the above notation, let $C_L(\Omega)$ denote the class of mappings $F: \operatorname{dom} L \cap \bar{\Omega} \to Z$ which is of the form $F = L - N$, with $N: \bar{\Omega} \to Z$ L-compact on $\bar{\Omega}$, and which satisfies the condition $0 \notin F(\operatorname{dom} L \cap \partial\Omega)$.

We say that the mapping $D_L(\cdot, \Omega): C_L(\Omega) \to \boldsymbol{Z}$ is the degree of F in Ω relative to L if it is not identically zero, and if the following axioms are satisfied: (i) Additivity-excision axiom: If Ω_1 and Ω_2 are disjoint open subsets of Ω such that $0 \notin F(\operatorname{dom} L \cap \bar{\Omega} \setminus (\Omega_1 \cup \Omega_2))$, then

$$D_L(F, \Omega) = D_L(F, \Omega_1) + D_L(F, \Omega_2)\,.$$

(ii) Axiom of homotopy invariance: If $\bar{F}: (\operatorname{dom} L \cap \bar{\Omega}) \times [0, 1] \to Z$ is of the form $\bar{F}(x, \lambda) = Lx - \bar{N}(x, \lambda)$ with $\bar{N}: \bar{\Omega} \times [0, 1] \to Z$ L-compact on $\bar{\Omega} \times [0, 1]$, and $0 \notin \bar{F}((\operatorname{dom} L \cap \partial\Omega) \times [0, 1])$, then the mapping $\lambda \mapsto D_L(\bar{F}(\cdot, \lambda), \Omega)$ is constant on $[0, 1]$.

An important property of the degree is the following existence property: If $F \in C_L(\Omega)$ and $D_L(F, \Omega) \neq 0$, then $0 \in F(\operatorname{dom} L \cap \Omega)$, i.e. the equation

$$Fx=0 \tag{3.1}$$

has at least one solution in $\operatorname{dom} L \cap \Omega$.

To prove our main theorem, we shall use the following theorem of Borsuk proved in [17].

THEOREM 3.1 (Borsuk). *If $F \in C_L(\Omega)$ with Ω symmetric with respect to* 0 *and* $0 \in \Omega$, *and if $F(-x)=-F(x)$ for every $x \in \operatorname{dom} L \cap \partial\Omega$, then $D_L(F, \Omega) \equiv 1 (\operatorname{mod} 2)$.*

In order to use the above degree theory, we next rewrite the equation (1.1) as an equivalent operator equation.

Let n be any positive integer. Let

$$P_{2\pi}^{(n)}=\{x \in C(\boldsymbol{R}, \boldsymbol{R}^n) : x(t+2\pi)=x(t), \text{ for any } t \in \boldsymbol{R}\} .$$

$$\|x\|=\sup_{t \in \boldsymbol{R}} |x(t)|=\sup_{t \in [0,2\pi]} |x(t)| .$$

Then $P_{2\pi}^{(n)}$ is a Banach space.

In the sequel, we shall denote $P_{2\pi}^{(2n_1+n_2)}$ by $P_{2\pi}$. It is clear to see that

$$P_{2\pi}=P_{2\pi}^{(2n_1)} \times P_{2\pi}^{(n_2)} .$$

Suppose $D=\operatorname{diag}(B_1, B_2, \ldots, B_{n_1}, O_{n_2})$ is a $(2n_1+n_2) \times (2n_1+n_2)$ matrix with O_{n_2} an $n_2 \times n_2$ zero matrix. Define the operator $L: P_{2\pi} \to P_{2\pi}$ by

$$Lx(t)=\dot{x}(t)-Dx(t) , \tag{3.2}$$

$$\operatorname{dom} L=\{x \in P_{2\pi} : \dot{x}(t) \text{ exists and is continuous}\} .$$

Obviously, we have

$$\ker L=\{x \in P_{2\pi} : x(t)=e^{Dt}a,\ a \in \boldsymbol{R}^{2n_1+n_2}\} ,$$

$$\operatorname{Im} L=\left\{x \in P_{2\pi} : \int_0^{2\pi} e^{D^T t} x(t) dt=0\right\} ,$$

where D^T denotes the transpose of D. Moreover, $\operatorname{Im} L$ is closed and we have the direct sum decomposition

$$P_{2\pi}=\ker L \oplus \operatorname{Im} L$$

which implies that

$$\dim \ker L=\operatorname{codim} \operatorname{Im} L=2n_1+n_2<\infty ,$$

and thus L is a Fredholm mapping of index zero. Let $P=Q: P_{2\pi} \to P_{2\pi}$ be the projections defined by

$$Px(t)=\frac{1}{2\pi} e^{Dt} \int_0^{2\pi} e^{D^T s} x(s) ds . \tag{3.3}$$

Then we have

$$\operatorname{Im} P = \ker L\,, \qquad \ker Q = \operatorname{Im} L\,.$$

LEMMA 3.1. *Let $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ be the (unique) right inverse of L associated to P. Then K_P is a compact operator with $\|K_P\| \le 2\pi$.*

PROOF. It is easy to know that for $z \in \operatorname{Im} L$,

$$K_P z(t) = e^{Dt}\int_0^t e^{D^T s} z(s)ds - \frac{1}{2\pi}\, e^{Dt}\int_0^{2\pi}\int_0^s e^{D^T \tau} z(\tau)d\tau ds\,.$$

Since $\int_0^t e^{D^T s}z(s)ds$ is 2π-periodic, it follows that

$$|K_P z(t)| \le \pi\|z\|\,, \qquad \text{for all} \quad t \in \boldsymbol{R}\,,$$

$$|K_P z(t_1) - K_P z(t_2)| \le (1 + \pi(m_1^2 + m_2^2 + \cdots + m_{n_1}^2)^{1/2})\|z\|\,|t_1 - t_2|\,, \qquad \text{for all} \quad t_1, t_2 \in \boldsymbol{R}$$

and Lemma 3.1 is then a consequence of the Arzela-Ascoli theorem.

It is also easy to see that $H : \boldsymbol{R}^{2n_1+n_2} \to \ker L$ defined by

$$H(a) = e^{Dt}a\,, \qquad \text{for} \quad a \in \boldsymbol{R}^{2n_1+n_2}$$

is an isometry. In what follows, we identify $a \in \boldsymbol{R}^{2n_1+n_2}$ with its image $H(a) \in \ker L$, i.e., $H(a) = a$, $a \in \boldsymbol{R}^{2n_1+n_2}$.

Define the operator $N : P_{2\pi} \to P_{2\pi}$ by

$$N(x, y)(t) = (N^{(2n_1)}(x, y)(t),\, N^{(n_2)}(x, y)(t))\,, \tag{3.4}$$

$$N^{(2n_1)}(x, y)(t) = (N_1^{(2n_1)}(x, y)(t),\, N_2^{(2n_1)}(x, y)(t), \ldots, N_{n_1}^{(2n_1)}(x, y)(t))\,, \tag{3.5}$$

$$N_i^{(2n_1)}(x, y)(t) = F_i(t, x(t+\cdot), y(t+\cdot)) + p_i(t)\,, \qquad i = 1, 2, \ldots, n_1\,, \tag{3.6}$$

$$N^{(n_2)}(x, y)(t) = (N_1^{(n_2)}(x, y)(t),\, N_2^{(n_2)}(x, y)(t), \ldots, N_{n_2}^{(n_2)}(x, y)(t))\,, \tag{3.7}$$

$$N_j^{(n_2)}(x, y)(t) = f_j(t, x(t+\cdot), y(t+\cdot)) + E_j(t)\,, \qquad j = 1, 2, \ldots, n_2\,, \tag{3.8}$$

where $x \in P_{2\pi}^{(2n_1)}$, $y \in P_{2\pi}^{(n_2)}$ and $(x, y) \in P_{2\pi}$ defined by $(x, y)(t) = (x(t), y(t))$. Then N is continuous and bounded, and hence is L-compact on $\bar{\Omega}$ for any bounded open set Ω in $P_{2\pi}$ with $\operatorname{dom} L \cap \Omega \ne \varnothing$.

Let $x(t) = (x_1(t), x_2(t), \ldots, x_{n_1}(t))$ with $x_i \in P_{2\pi}^{(2)}$ $(1 \le i \le n_1)$ and $y(t) = (y_1(t), y_2(t), \ldots, \ldots, y_{n_2}(t))$ with $y_j \in P_{2\pi}^{(1)}$ $(1 \le j \le n_2)$. Then the assumptions (F) and (f) imply that

$$N_i^{(2n_1)}(x, y)(t) = H_i(x_{k_i}(t - \tau_i^{(1)})) + G_i(t, x(t+\cdot), y(t+\cdot)) + p_i(t), \quad i = 1, 2, \ldots, n_1\,, \tag{3.9}$$

and

$$N_j^{(n_2)}(x, y)(t) = h_j(y_{l_j}(t - \tau_j^{(2)})) + g_j(t, x(t+\cdot), y(t+\cdot)) + E_j(t)\,, \quad i = 1, 2, \ldots, n_2\,. \tag{3.10}$$

In the above notation, the equation (1.1) is equivalent to the operator equation

$$F(x, y)=0\,, \qquad (x, y)\in \operatorname{dom} L\,, \tag{3.11}$$

where $x\in P_{2\pi}^{(2n_1)}$, $y\in P_{2\pi}^{(n_2)}$ and $F=L-N$: $\operatorname{dom} L\subset P_{2\pi}\to P_{2\pi}$.

4. Proof of Theorem 2.1. In proving our main theorem, we also need some lemmas.

Let

$$Y=\{H\in BC(\boldsymbol{R}^2, \boldsymbol{R}^2): H(\pm, \pm)=\lim_{r,s\to\pm\infty} H(r, s) \text{ exist}\}\,,$$

$$\|H\|=\sup_{r,s\in \boldsymbol{R}} |H(r, s)|<\infty\,.$$

Then $(Y, \|\cdot\|)$ is a normed space. Define the mapping W: $Y\to \boldsymbol{R}^2$ as in (2.2). Then W is linear and continuous. Moreover, if $\hat{H}(r, s)=H(-r, -s)$, then

$$W(\hat{H})=-W(H)\,. \tag{4.1}$$

The following Lemma 4.1 is obvious.

LEMMA 4.1. *Let* $H\in Y$ *and*

$$\bar{H}(r, s)=\frac{1}{2}[H(r, s)-H(-r, -s)]\,. \tag{4.2}$$

Then

$$W(\bar{H})=W(H)\,. \tag{4.3}$$

LEMMA 4.2. *Let* $H\in Y$, $\rho\in \boldsymbol{R}$ *and* $v\in BC(\boldsymbol{R}, \boldsymbol{R}^2)$. *Let*

$$M(\rho, v)=\frac{1}{2\pi}\int_0^{2\pi} e^{A^T s}H((\rho\sin s, \rho\cos s)^T+v(s))ds\,, \tag{4.4}$$

where

$$A=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}.$$

Then

$$\lim_{\rho\to\infty} M(\rho, v)=e^{A^T(\pi/4)}W(H) \tag{4.5}$$

$$\lim_{\rho\to-\infty} M(\rho, v)=-e^{A^T(\pi/4)}W(H) \tag{4.6}$$

uniformly for $|v(t)|\le \bar{M}$, *where* $\bar{M}$ *is a constant.*

PROOF. Fixed $\varepsilon>0$ ($\varepsilon<1/4$). Let $M_0>0$ be large enough so that

$$|H(x, y)-H(+, +)|<\varepsilon, \qquad \text{for any} \quad x, y\geq M_0 .$$

Define $\rho_0=(M_0+\bar{M})/\sin\varepsilon$. Then for any $\rho\geq\rho_0$, we have

$$\begin{aligned}
&\left|\int_0^{\pi/2} e^{A^T s}H((\rho\sin s, \rho\cos s)^T+v(s))ds-\int_0^{\pi/2} e^{A^T s}H(+, +)ds\right| \\
&\quad\leq\left|\int_0^{\varepsilon} e^{A^T s}[H((\rho\sin s, \rho\cos s)^T+v(s))-H(+, +)]ds\right| \\
&\qquad+\left|\int_{\varepsilon}^{\pi/2-\varepsilon} e^{A^T s}[H((\rho\sin s, \rho\cos s)^T+v(s))-H(+, +)]ds\right| \\
&\qquad+\left|\int_{\pi/2-\varepsilon}^{\pi/2} e^{A^T s}[H((\rho\sin s, \rho\cos s)^T+v(s))-H(+, +)]ds\right| \\
&\quad\leq 4\|H\|\varepsilon+\frac{\pi}{2}\varepsilon=\left(4\|H\|+\frac{\pi}{2}\right)\varepsilon ,
\end{aligned}$$

where $\|H\|=\sup_{r,s\in\boldsymbol{R}}|H(r, s)|<\infty$. Hence

$$\lim_{\rho\to\infty}\int_0^{\pi/2} e^{A^T s}H((\rho\sin s, \rho\cos s)^T+v(s))ds=\int_0^{\pi/2} e^{A^T s}H(+, +)ds \tag{4.7}$$

uniformly for $|v(t)|\leq\bar{M}$.

A similar argument shows that

$$\lim_{\rho\to\infty}\int_{\pi/2}^{\pi} e^{A^T s}H((\rho\sin s, \rho\cos s)^T+v(s))ds=\int_{\pi/2}^{\pi} e^{A^T s}H(+, -)ds , \tag{4.8}$$

$$\lim_{\rho\to\infty}\int_{\pi}^{3\pi/2} e^{A^T s}H((\rho\sin s, \rho\cos s)^T+v(s))ds=\int_{\pi}^{3\pi/2} e^{A^T s}H(-, -)ds , \tag{4.9}$$

$$\lim_{\rho\to\infty}\int_{3\pi/2}^{2\pi} e^{A^T s}H((\rho\sin s, \rho\cos s)^T+v(s))ds=\int_{3\pi/2}^{2\pi} e^{A^T s}H(-, +)ds , \tag{4.10}$$

uniformly for $|v(t)|\leq\bar{M}$.

It follows from (4.4), (4.7)–(4.10) that

$$\begin{aligned}
\lim_{\rho\to\infty} M(\rho, v)&=\frac{1}{2\pi}\left[\int_0^{\pi/2} e^{A^T s}H(+, +)ds+\int_{\pi/2}^{\pi} e^{A^T s}H(+, -)ds\right. \\
&\qquad\left.+\int_{\pi}^{3\pi/2} e^{A^T s}H(-, -)ds+\int_{3\pi/2}^{2\pi} e^{A^T s}H(-, +)ds\right] \\
&=e^{A^T(\pi/4)}W(H) ,
\end{aligned}$$

uniformly for $|v(t)|\leq\bar{M}$.

By using a similar argument, we can show that

$$\lim_{\rho \to -\infty} M(\rho, v) = -e^{A^T(\pi/4)} W(H),$$

uniformly for $|v(t)| \le \bar{M}$, and this completes the proof.

LEMMA 4.3. *Condition* (h) *holds if and only if*

$$\frac{1}{2}|h_j(+)-h_j(-)| > \frac{1}{2}|h_j(+)+h_j(-)| + M_{g_j} + |\bar{E}_j|, \qquad j=1, 2, \ldots, n_2. \tag{4.11}$$

PROOF. Suppose that (h) holds, that is,

$$h_j(+)h_j(-) < 0, \quad |h_j(\pm)| > M_{g_j} + |\bar{E}_j|, \qquad j=1, 2, \ldots, n_2. \tag{4.12}$$

For any j with $1 \le j \le n_2$, without loss of generality, we assume that

$$h_j(+) > 0, \qquad h_j(-) < 0. \tag{4.13}$$

From (4.12) and (4.13), we find

$$\frac{1}{2}|h_j(+)-h_j(-)| > \frac{1}{2}(h_j(+)+h_j(-)) + M_{g_j} + |\bar{E}_j|, \tag{4.14}$$

and

$$\frac{1}{2}|h_j(+)-h_j(-)| > -\frac{1}{2}(h_j(+)+h_j(-)) + M_{g_j} + |\bar{E}_j|. \tag{4.15}$$

Then (4.11) follows from (4.14) and (4.15).

Conversely, suppose that (4.11) holds. Then

$$h_j(+)h_j(-) < 0.$$

Case 1. $h_j(+) > 0, h_j(-) < 0$.

If $h_j(+)+h_j(-) \ge 0$, then (4.11) implies that

$$h_j(+) \ge -h_j(-) > M_{g_j} + |\bar{E}_j|.$$

If $h_j(+)+h_j(-) < 0$, then (4.11) implies that

$$-h_j(-) \ge h_j(+) > M_{g_j} + |\bar{E}_j|.$$

Therefore, we always have

$$|h_j(\pm)| > M_{g_j} + |\bar{E}_j|.$$

Case 2. $h_j(+) < 0, h_j(-) > 0$.

A similar argument shows that

$$|h_j(\pm)| > M_{g_j} + |\bar{E}_j|.$$

The proof is complete.

We are now in a position to prove our main theorem.

PROOF OF THEOREM 2.1. Let

$$\bar{H}_i(r, s) = \frac{1}{2}(H_i(r, s) - H_i(-r, -s)), \qquad i = 1, 2, \ldots, n_1 \tag{4.16}$$

and

$$\bar{h}_j(r) = \frac{1}{2}(h_j(r) - h_j(-r)), \qquad j = 1, 2, \ldots, n_2. \tag{4.17}$$

Then

$$\bar{H}_i(-r, -s) = -\bar{H}_i(r, s), \qquad i = 1, 2, \ldots, n_1, \tag{4.18}$$

$$\bar{h}_j(-r) = -\bar{h}_j(r), \qquad j = 1, 2, \ldots, n_2. \tag{4.19}$$

Hence by virtue of Lemma 4.1, we get

$$W(\bar{H}_i) = W(H_i), \qquad i = 1, 2, \ldots, n_1. \tag{4.20}$$

Define the operator $\bar{N}$: $P_{2\pi} \times [0, 1] \to P_{2\pi}$ as follows:

$$\bar{N}(x, y, \lambda)(t) = (\bar{N}^{(2n_1)}(x, y, \lambda)(t), \bar{N}^{(n_2)}(x, y, \lambda)(t)),$$

$$\bar{N}^{(2n_1)}(x, y, \lambda)(t) = (\bar{N}_1^{(2n_1)}(x, y, \lambda)(t), \bar{N}_2^{(2n_1)}(x, y, \lambda)(t), \ldots, \bar{N}_{n_1}^{(2n_1)}(x, y, \lambda)(t)),$$

$$\bar{N}_i^{(2n_1)}(x, y, \lambda)(t) = \bar{H}_i(x_{k_i}(t - \tau_i^{(1)})) + \frac{\lambda}{2}[H_i(x_{k_i}(t - \tau_i^{(1)})) + H_i(-x_{k_i}(t - \tau_i^{(1)}))]$$

$$+ \lambda G_i(t, x(t + \cdot), y(t + \cdot)) + \lambda p_i(t), \qquad i = 1, 2, \ldots, n_1,$$

$$\bar{N}^{(n_2)}(x, y, \lambda)(t) = (\bar{N}_1^{(n_2)}(x, y, \lambda)(t), \bar{N}_2^{(n_2)}(x, y, \lambda)(t), \ldots, \bar{N}_{n_2}^{(n_2)}(x, y, \lambda)(t)),$$

$$\bar{N}_j^{(n_2)}(x, y, \lambda)(t) = \bar{h}_j(y_{l_j}(t - \tau_j^{(2)})) + \frac{\lambda}{2}[h_j(y_{l_j}(t - \tau_j^{(2)})) + h_j(-y_{l_j}(t - \tau_j^{(2)}))]$$

$$+ \lambda g_j(t, x(t + \cdot), y(t + \cdot)) + \lambda E_j(t), \qquad j = 1, 2, \ldots, n_2,$$

where $x = (x_1, x_2, \ldots, x_{n_1}) \in P_{2\pi}^{(2n_1)}$ with $x_i \in P_{2\pi}^{(2)}$ $(i = 1, 2, \ldots, n_1)$ and $y = (y_1, y_2, \ldots, y_{n_2}) \in P_{2\pi}^{(n_2)}$ with $y_j \in P_{2\pi}^{(1)}$ $(j = 1, 2, \ldots, n_2)$. Then $\bar{N}$ is continuous and bounded, and hence is L-compact on $\bar{\Omega} \times [0, 1]$ for any bounded open set Ω in $P_{2\pi}$ with $\operatorname{dom} L \cap \Omega \neq \varnothing$.

Define $\bar{F}$: $\operatorname{dom} L \times [0, 1] \to P_{2\pi}$ by

$$\bar{F}(x, y, \lambda) = L(x, y) - \bar{N}(x, y, \lambda), \tag{4.21}$$

where $x \in P_{2\pi}^{(2n_1)}$ and $y \in P_{2\pi}^{(n_2)}$. Then it is easy to see that $\bar{F}(x, y, 1) = F(x, y)$ and $F_0(x, y) := \bar{F}(x, y, 0)$ satisfies

$$(4.22)\qquad F_0(-x,-y)=-F_0(x,y)\,,\qquad \text{for any}\quad x\in P_{2\pi}^{(2n_1)}\,,\quad y\in P_{2\pi}^{(n_2)}\,.$$

Let $\rho>0$, and

$$\Omega_\rho^0=\{(u,v)\in\ker L:\ u=(r_1\rho a_1, r_2\rho a_2,\dots,r_{n_1}\rho a_{n_1}),\ a_i\in\partial B_1(0)\subset \boldsymbol{R}^2\,,\quad 0\le r_i<1\,,\ i=1,2,\dots,n_1\,,$$
$$v=(\sigma_1\rho,\sigma_2\rho,\dots,\sigma_{n_2}\rho),\ \sigma_j\in\boldsymbol{R},\ |\sigma_j|<1,\ j=1,2,\dots,n_2\}\,,$$

where $B_1(0)=\{a\in\boldsymbol{R}^2: |a|\le 1\}\subset\boldsymbol{R}^2$. Then Ω_ρ^0 is a bounded open set in $\ker L$.

Put

$$M=4\pi\left[\sum_{i=1}^{n_1}(\|H_i\|+M_{G_i}+\|p_i\|)^2+\sum_{j=1}^{n_2}(M_{h_j}+M_{g_j}+\|E_j\|)^2\right]^{1/2}+1\,,$$

where $M_{h_j}=\sup_{r\in\boldsymbol{R}}|h_j(r)|<\infty$.

Since $\|K_P(I-Q)\|\le 4\pi$, it follows from (4.16), (4.17) and the definition of $\bar N$ that

$$(4.23)\qquad \|K_P(I-Q)\bar N(x,y,\lambda)\|<M\,,$$

for any $x\in P_{2\pi}^{(2n_1)}$, $y\in P_{2\pi}^{(n_2)}$ and $\lambda\in[0,1]$.

Again set

$$\Omega_\rho=\{(x,y)\in P_{2\pi}: x\in P_{2\pi}^{(2n_1)},\ y\in P_{2\pi}^{(n_2)},\ \|(I-P)(x,y)\|<M,\ P(x,y)\in\Omega_\rho^0\}\,.$$

Then Ω_ρ is a bounded open set in $P_{2\pi}$, $0\in\Omega_\rho$ and Ω_ρ is symmetric with respect to 0. Moreover, $\partial\Omega_\rho=\Gamma_1\cup\Gamma_2$, where

$$\Gamma_1=\{(x,y)\in P_{2\pi}: x\in P_{2\pi}^{(2n_1)},\ y\in P_{2\pi}^{(n_2)},\ \|(I-P)(x,y)\|=M,\ P(x,y)\in\bar\Omega_\rho^0\}\,,$$
$$\Gamma_2=\{(x,y)\in P_{2\pi}: x\in P_{2\pi}^{(2n_1)},\ y\in P_{2\pi}^{(n_2)},\ \|(I-P)(x,y)\|\le M,\ P(x,y)\in\partial\Omega_\rho^0\}\,.$$

We claim that for ρ sufficiently large,

$$(4.24)\qquad 0\notin\bar F((\operatorname{dom}L\cap\partial\Omega_\rho)\times[0,1])\,.$$

Indeed, the equation $\bar F(x,y,\lambda)=0$ is equivalent to the system of equations

$$(4.25)\qquad Q\bar N(x,y,\lambda)=0\,,$$

$$(4.26)\qquad (I-P)(x,y)=K_P(I-Q)\bar N(x,y,\lambda)\,.$$

For any $(x,y)\in\Gamma_1$, (4.23) implies that

$$(I-P)(x,y)\ne K_P(I-Q)\bar N(x,y,\lambda)\,,\qquad\text{for any}\quad\lambda\in[0,1]$$

and hence $\bar F(x,y,\lambda)\ne 0$, for any $(x,y)\in\Gamma_1$ and $\lambda\in[0,1]$.

For any $(x,y)\in\Gamma_2$, we can assume that

$$x(t)=(r_1\rho a_1, r_2\rho a_2,\dots,r_{n_1}\rho a_{n_1})+\bar x(t),\ a_i\in\partial B_1(0)\subset\boldsymbol{R}^2\,,\quad 0\le r_i\le 1\,,\ i=1,2,\dots,n_1\,,$$

$$y(t)=(\sigma_1\rho, \sigma_2\rho, \dots, \sigma_{n_2}\rho)+\bar{y}(t),\ \sigma_j\in \boldsymbol{R},\quad |\sigma_j|<1\,,\qquad j=1, 2, \dots, n_2\,,$$

where $\bar{x}\in P_{2\pi}^{(2n_1)}$, $\bar{y}\in P_{2\pi}^{(n_2)}$, $(\bar{x}, \bar{y})\in \operatorname{Im} L$, $\|(\bar{x}, \bar{y})\|\le M$ and either $r_{k_{i_0}}=1$ $(1\le i_0\le n_1)$ or $\sigma_{l_{j_0}}=\pm 1$ $(1\le j_0\le n_2)$.

By the definition of $\bar{N}$, we find

$$QN̄(x, y, \lambda)=((Q\bar{N}(x, y, \lambda))^{(2n_1)}, (Q\bar{N}(x, y, \lambda))^{(n_2)})\,, \tag{4.27}$$

$$(Q\bar{N}(x, y, \lambda))^{(2n_1)}=((Q\bar{N}(x, y, \lambda))_1^{(2n_1)}, (Q\bar{N}(x, y, \lambda))_2^{(2n_1)}, \dots, (Q\bar{N}(x, y, \lambda))_{n_1}^{(2n_1)})\,, \tag{4.28}$$

$$(Q\bar{N}(x, y, \lambda))_i^{(2n_1)}=\frac{1}{2\pi}\int_0^{2\pi} e^{B_i^T s}\bar{N}_i^{(2n_1)}(x, y, \lambda)(s)ds\,, \qquad i=1, 2, \dots, n_1\,. \tag{4.29}$$

$$(Q\bar{N}(x, y, \lambda))^{(n_2)}=((Q\bar{N}(x, y, \lambda))_1^{(n_2)}, (Q\bar{N}(x, y, \lambda))_2^{(n_2)}, \dots, (Q\bar{N}(x, y, \lambda))_{n_2}^{(n_2)})\,, \tag{4.30}$$

$$(Q\bar{N}(x, y, \lambda))_j^{(n_2)}=\frac{1}{2\pi}\int_0^{2\pi} \bar{N}_j^{(n_2)}(x, y, \lambda)(s)ds\,, \qquad j=1, 2, \dots, n_2\,. \tag{4.31}$$

Now we consider the following two possible cases:

Case 1. $\tau_{k_{i_0}}=1$ $(1\le i_0\le n_1)$.

Since $m_{k_{i_0}}=m_{i_0}$, by (4.29) and the definition of $\bar{N}$, it is not hard to verify that

$$(Q\bar{N}(x, y, \lambda))_{i_0}^{(2n_1)}=e^{B_{i_0}^T\tau_{i_0}^{(1)}}\Phi_1(\rho, a_{k_{i_0}})+\lambda e^{B_{i_0}^T\tau_{i_0}^{(1)}}\Phi_2(\rho, a_{k_{i_0}})+\lambda X(x, y)+\lambda p_{i_0}(m_{i_0})\,, \tag{4.32}$$

where

$$\Phi_1(\rho, a_{k_{i_0}})=\frac{1}{2\pi}\int_0^{2\pi} e^{B_{i_0}^T s}\bar{H}_{i_0}(\rho e^{B_{i_0}s}a_{k_{i_0}}+\bar{x}_{k_{i_0}}(s))ds\,, \tag{4.33}$$

$$\Phi_2(\rho, a_{k_{i_0}})=\frac{1}{4\pi}\int_0^{2\pi} e^{B_{i_0}^T s}[H_{i_0}(\rho e^{B_{i_0}s}a_{k_{i_0}}+\bar{x}_{k_{i_0}}(s))+H_{i_0}(-\rho e^{B_{i_0}s}a_{k_{i_0}}-\bar{x}_{k_{i_0}}(s))]ds\,, \tag{4.34}$$

$$X(x, y)=\frac{1}{2\pi}\int_0^{2\pi} e^{B_{i_0}^T s}G_{i_0}(s, x(s+\cdot), y(s+\cdot))ds\,. \tag{4.35}$$

Let α be defined by $\sin\alpha=a_{k_{i_0}}^{(1)}$, $\cos\alpha=a_{k_{i_0}}^{(2)}$, where $a_{k_{i_0}}=(a_{k_{i_0}}^{(1)}, a_{k_{i_0}}^{(2)})^T$. Then we find

$$\begin{aligned}\Phi_1(\rho, a_{k_{i_0}})&=\frac{1}{2\pi}e^{B_{i_0}\alpha/m_{i_0}}\int_0^{2\pi} e^{B_{i_0}^T s}\bar{H}_{i_0}\left((\rho\sin m_{i_0}s, \rho\cos m_{i_0}s)^T+\bar{x}_{k_{i_0}}\left(s-\frac{\alpha}{m_{i_0}}\right)\right)ds\\ &=\frac{1}{2m_{i_0}\pi}e^{A\alpha}\sum_{k=1}^{m_{i_0}}\int_0^{2\pi} e^{A^T s}\bar{H}_{i_0}((\rho\sin s, \rho\cos s)^T+v(s))ds\,,\end{aligned}$$

$$\begin{aligned}\Phi_2(\rho, a_{k_{i_0}})=\frac{1}{4\pi}e^{B_{i_0}\alpha/m_{i_0}}\int_0^{2\pi} e^{B_{i_0}^T s}\Bigg[&H_{i_0}\left((\rho\sin m_{i_0}s, \rho\cos m_{i_0}s)^T+\bar{x}_{k_{i_0}}\left(s-\frac{\alpha}{m_{i_0}}\right)\right)\\ &+H_{i_0}\left((-\rho\sin m_{i_0}s, -\rho\cos m_{i_0}s)^T-\bar{x}_{k_{i_0}}\left(s-\frac{\alpha}{m_{i_0}}\right)\right)\Bigg]ds\end{aligned}$$

$$=\frac{1}{4m_{i_0}\pi}e^{A\alpha}\sum_{k=1}^{m_{i_0}}\int_0^{2\pi}e^{A^Ts}[H_{i_0}((\rho\sin s,\ \rho\cos s)^T+v(s))$$

$$+H_{i_0}((-\rho\sin s,\ -\rho\cos s)^T-v(s))]ds\ ,$$

where

$$v(s)=\bar{x}_{k_{i_0}}\left(\frac{s+2(k-1)\pi-\alpha}{m_{i_0}}\right).$$

Hence, by (4.20) and Lemma 4.2,

(4.36) $$\lim_{\rho\to\infty}|\Phi_1(\rho,\ a_{k_{i_0}})|=|W(\bar{H}_{i_0})|=|W(H_{i_0})|\ ,$$

(4.37) $$\lim_{\rho\to\infty}|\Phi_2(\rho,\ a_{k_{i_0}})|=\frac{1}{2}|e^{A^T(\pi/4)}W(H_{i_0})-e^{A^T(\pi/4)}W(H_{i_0})|=0\ ,$$

uniformly for any $a_{k_{i_0}}\in\partial B_1(0)\subset\boldsymbol{R}^2$ and $\bar{x}_{k_{i_0}}\in P_{2\pi}^{(2)}$ with $\|\bar{x}_{k_{i_0}}\|\leq M$.

By (4.35), we also have

(4.38) $$|X(x,\ y)|\leq M_{G_{i_0}}\ .$$

Therefore, by the assumption (H), (4.36), (4.37) and (4.38) imply that for ρ sufficiently large,

(4.39) $$|\Phi_1(\rho,\ a_{k_{i_0}})|>|\Phi_2(\rho,\ a_{k_{i_0}})|+|X(x,\ y)|+|p_{i_0}(m_{i_0})|\ ,$$

for any $a_{k_{i_0}}\in\partial B_1(0)\subset\boldsymbol{R}^2$ and $\bar{x}_{k_{i_0}}$ with $|\bar{x}_{k_{i_0}}(t)|\leq M$, which together with (4.32) yields that for ρ sufficiently large,

$$(Q\bar{N}(x,\ y,\ \lambda))_{i_0}^{(2n_1)}\neq 0\ ,\qquad \text{for any}\quad (x,\ y)\in\Gamma_2\ ,\quad \lambda\in[0,\ 1]\ ,$$

and hence $Q\bar{N}(x, y, \lambda)\neq 0$. Therefore, for ρ sufficiently large, $\bar{F}(x, y, \lambda)\neq 0$, for any $(x, y)\in\Gamma_2$ and $\lambda\in[0, 1]$.

Case 2. $\sigma_{l_{j_0}}=\pm 1$ $(1\leq j_0\leq n_2)$.

Without loss of generality, we assume that $\sigma_{l_{j_0}}=1$. The case $\sigma_{l_{j_0}}=-1$ may be treated in a similar way.

By (4.31) and the definition of $\bar{N}$, we may verify that

(4.40) $$(Q\bar{N}(x,\ y,\ \lambda))_{j_0}^{(n_2)}=\Psi_1(\rho)+\lambda\Psi_2(\rho)+\lambda Y(x,\ y)+\lambda\bar{E}_{j_0}\ ,$$

where

(4.41) $$\Psi_1(\rho)=\frac{1}{2\pi}\int_0^{2\pi}\bar{h}_{j_0}(\rho+\bar{y}_{l_{j_0}}(s))ds\ ,$$

(4.42) $$\Psi_2(\rho)=\frac{1}{4\pi}\int_0^{2\pi}[h_{j_0}(\rho+\bar{y}_{l_{j_0}}(s))+h_{j_0}(-\rho-\bar{y}_{l_{j_0}}(s))]ds\ ,$$

$$Y(x, y)=\frac{1}{2\pi}\int_0^{2\pi} g_{j_0}(s, x(s+\cdot), y(s+\cdot))ds\,. \tag{4.43}$$

Clearly, we have

$$\lim_{\rho\to\infty} \Psi_1(\rho)=\bar{h}_{j_0}(+)=\frac{1}{2}(h_{j_0}(+)-h_{j_0}(-))\,, \tag{4.44}$$

$$\lim_{\rho\to\infty} \Psi_2(\rho)=\frac{1}{2}(h_{j_0}(+)+h_{j_0}(-))\,, \tag{4.45}$$

uniformly for $\bar{y}_{l_{j_0}}$ with $|\bar{y}_{l_{j_0}}(t)|\le M$.

By (4.43), we also have

$$|Y(x, y)|\le M_{g_{j_0}}\,. \tag{4.46}$$

Therefore, by the assumption (h) and Lemma 4.3, (4.44), (4.45) and (4.46) imply that for ρ sufficiently large,

$$|\Psi_1(\rho)|>|\Psi_2(\rho)|+|Y(x, y)|+|\bar{E}_{j_0}|\,, \tag{4.47}$$

for any $\bar{y}_{l_{j_0}}$ with $|\bar{y}_{l_{j_0}}(t)|\le M$, which together with (4.40) implies that for ρ sufficiently large,

$$(Q\bar{N}(x, y, \lambda))_{j_0}^{(n_2)}\neq 0\,, \qquad \text{for any} \quad (x, y)\in\Gamma_2\,, \quad \lambda\in[0, 1]\,,$$

and hence $Q\bar{N}(x, y, \lambda)\neq 0$. Therefore, for ρ sufficiently large, $\bar{F}(x, y, \lambda)\neq 0$, for any $(x, y)\in\Gamma_2$ and $\lambda\in[0, 1]$.

Thus, we have proved that for ρ sufficiently large, (4.24) holds.

Now it follows from (4.24) that for ρ sufficiently large, the degree $D_L(\bar{F}(\cdot, \lambda), \Omega_\rho)$ is well-defined and is constant on $[0, 1]$. Therefore, by (4.22) and the Borsuk theorem, we have

$$\begin{aligned} D_L(F, \Omega_\rho)&=D_L(\bar{F}(\cdot, 1), \Omega_\rho)\\ &=D_L(\bar{F}(\cdot, 0), \Omega_\rho)\equiv 1 \pmod 2\,, \end{aligned}$$

so the existence of a solution of the equation (3.11) follows from the existence property of the degree, and thus the equation (1.1) has at least one 2π-periodic solution. The proof is complete.

5. Examples. Finally, we shall give some specific examples to illustrate our main result.

EXAMPLE 5.1. Consider the system

$$(5.1)\qquad \begin{cases} x_1' = x_2 + \arctan x_1 + x_3/(1+x_3^2) + p_1(t)\,, \\ x_2' = -x_1 + \arctan x_2 + 3\arctan x_4 + p_2(t)\,, \\ x_3' = x_4 + \sqrt{2}\,\sin x_3 + x_5 e^{-x_5^2} + p_3(t)\,, \\ x_4' = -x_3 + \sqrt{2}\,\cos x_3 - 2\arctan x_6 + p_4(t)\,, \\ x_5' = x_6 - 2\arctan x_2 + \sqrt{2}\,\arctan x_5 + p_5(t)\,, \\ x_6' = -x_5 + 2\arctan x_1 + \sqrt{2}\,\arctan x_6 + p_6(t)\,, \end{cases}$$

where $p_j (j=1, 2, \ldots, 6)$ are continuous 2π-periodic functions. In this example, $n_1=3$, $n_2=0$, $\tau_i^{(1)}=0$ $(i=1, 2, 3)$, and $(k_1, k_2, k_3)=(2, 3, 1)$, we set

$$H_1(x_3, x_4) = \begin{pmatrix} x_3/(1+x_3^2) \\ 3\arctan x_4 \end{pmatrix}, \quad G_1(t, \varphi) = \begin{pmatrix} \arctan \varphi_1(0) \\ \arctan \varphi_2(0) \end{pmatrix},$$

$$H_2(x_5, x_6) = \begin{pmatrix} x_5 e^{-x_5^2} \\ -2\arctan x_6 \end{pmatrix}, \quad G_2(t, \varphi) = \begin{pmatrix} \sqrt{2}\,\sin \varphi_3(0) \\ \sqrt{2}\,\cos \varphi_3(0) \end{pmatrix},$$

$$H_3(x_1, x_2) = \begin{pmatrix} -2\arctan x_2 \\ 2\arctan x_1 \end{pmatrix}, \quad G_3(t, \varphi) = \begin{pmatrix} \sqrt{2}\,\arctan \varphi_5(0) \\ \sqrt{2}\,\arctan \varphi_6(0) \end{pmatrix},$$

where $\varphi \in BC(\boldsymbol{R}, \boldsymbol{R}^6)$. A straightforward computation shows that

$$W(H_1) = \begin{pmatrix} 3\sqrt{2}/2 \\ 3\sqrt{2}/2 \end{pmatrix}, \quad W(H_2) = \begin{pmatrix} -\sqrt{2} \\ -\sqrt{2} \end{pmatrix}, \quad W(H_3) = \begin{pmatrix} -2\sqrt{2} \\ 2\sqrt{2} \end{pmatrix},$$

$$M_{G_1} = \frac{\sqrt{2}\,\pi}{2}, \quad M_{G_2} = \sqrt{2}\,, \quad M_{G_3} = \pi\,.$$

By Theorem 1.1, the equation (5.1) has at least one 2π-periodic solution provided

$$|c_1| < 3 - \frac{\sqrt{2}\,\pi}{2}, \quad |c_2| < 2 - \sqrt{2}\,, \quad |c_3| < 4 - \pi\,,$$

where

$$c_1 = \frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} p_1(s) \\ p_2(s) \end{pmatrix} ds\,,$$

$$c_2 = \frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} p_3(s) \\ p_4(s) \end{pmatrix} ds\,,$$

$$c_3 = \frac{1}{2\pi}\int_0^{2\pi} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} p_5(s) \\ p_6(s) \end{pmatrix} ds\,.$$

EXAMPLE 5.2. Consider the system

$$
(5.2)\qquad \begin{cases}
x_1' = 2x_2 + \sqrt{3}\arctan x_1 + x_2 e^{-x_2^2} + \dfrac{1}{2}\arctan x_4 + p_1(t)\,, \\
x_2' = -2x_1 + \dfrac{x_1}{1+x_1^2} + \dfrac{x_2}{1+x_2^2} + \dfrac{1}{2}\arctan x_3 + p_2(t)\,, \\
x_3' = 3x_4 + \dfrac{1}{2}\arctan x_1 + \arctan x_3 + \dfrac{1}{2}\arctan x_4 + p_3(t)\,, \\
x_4' = -3x_3 + \dfrac{1}{2}\arctan y + e^{-x_3^2} + \dfrac{1}{2}\arctan x_4 + p_4(t)\,, \\
y' = -\arctan y + \sin x_1 + \dfrac{x_3}{1+x_3^2} + p_5(t)\,,
\end{cases}
$$

where p_j ($j=1, 2, \ldots, 5$) are continuous 2π-periodic functions. In this example, we take $(k_1, k_2)=(1, 2)$, $l_1=1$, $\tau_1^{(1)}=\tau_2^{(1)}=\tau_1^{(2)}=0$, and

$$
H_1(x_1, x_2)=\begin{pmatrix} \sqrt{3}\arctan x_1 + x_2 e^{-x_2^2} \\ \dfrac{x_1}{1+x_1^2}+\dfrac{x_2}{1+x_2^2} \end{pmatrix}, \qquad G_1(t, \varphi, \psi)=\begin{pmatrix} \frac{1}{2}\arctan \varphi_4(0) \\ \frac{1}{2}\arctan \varphi_3(0) \end{pmatrix},
$$

$$
H_2(x_3, x_4)=\begin{pmatrix} \arctan x_3 + \frac{1}{2}\arctan x_4 \\ e^{-x_3^2}+\frac{1}{2}\arctan x_4 \end{pmatrix}, \quad G_2(t, \varphi, \psi)=\begin{pmatrix} \frac{1}{2}\arctan \varphi_1(0) \\ \frac{1}{2}\arctan \psi(0) \end{pmatrix},
$$

$$
h(y) = -\arctan y\,, \quad g(t, \varphi, \psi) = \sin\varphi_1(0) + \varphi_3(0)/(1+(\varphi_3(0))^2)\,,
$$

where $t\in \boldsymbol{R}$, $\varphi \in BC(\boldsymbol{R}, \boldsymbol{R}^4)$, $\psi \in BC(\boldsymbol{R}, \boldsymbol{R})$. A straightforward computation shows that

$$
W(H_1)=\begin{pmatrix} \sqrt{6}/2 \\ \sqrt{6}/2 \end{pmatrix}, \quad W(H_2)=\begin{pmatrix} \sqrt{2} \\ \sqrt{2}/2 \end{pmatrix},
$$

$$
M_{G_1}=M_{G_2}=\frac{\sqrt{2}\,\pi}{4}\,,
$$

$$
h(+)=-h(-)=-\frac{\pi}{2}\,, \quad M_g=\frac{3}{2}\,.
$$

By Theorem 1.1, the equation (5.2) has at least one 2π-periodic solution provided

$$
|c_1|<\sqrt{3}-\frac{\sqrt{2}\,\pi}{4}\,, \quad |c_2|<\sqrt{\frac{5}{2}}-\frac{\sqrt{2}\,\pi}{4}\,, \quad |d|<\frac{\pi}{2}-\frac{3}{2}\,,
$$

where

$$c_1=\frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix}\cos 2s & -\sin 2s\\ \sin 2s & \cos 2s\end{pmatrix}\begin{pmatrix}p_1(s)\\ p_2(s)\end{pmatrix}ds\,,$$

$$c_2=\frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix}\cos 3s & -\sin 3s\\ \sin 3s & \cos 3s\end{pmatrix}\begin{pmatrix}p_3(s)\\ p_4(s)\end{pmatrix}ds\,,$$

$$d=\frac{1}{2\pi}\int_0^{2\pi}p_5(s)ds\,.$$

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