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ESTIMATES OF THE FUNDAMENTAL SOLUTION FOR MAGNETIC SCHRÖDINGER OPERATORS AND THEIR APPLICATIONS

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Abstract. We study the magnetic Schrödinger operator H on $\boldsymbol{R}^n$, $n \geq 3$. We assume that the electrical potential V and the magnetic potential $\mathbf{a}$ belong to a certain reverse Hölder class, including the case that V is a non-negative polynomial and the components of $\mathbf{a}$ are polynomials. We show some estimates for operators of Schrödinger type by using estimates of the fundamental solution for H. In particular, we show that the operator $\nabla^2(-\Delta + V)^{-1}$ is a Calderón-Zygmund operator.

1. Introduction and main results. Let $V(x)$ be a non-negative potential and consider the Schrödinger operator $-\Delta + V$ on Euclidean n-space $\boldsymbol{R}^n$, $n \geq 3$. When V is a non-negative polynomial, Zhong ([Zh]) proved that the operators $\nabla^2(-\Delta + V)^{-1}$, $\nabla(-\Delta + V)^{-1/2}$ and $\nabla(-\Delta + V)^{-1}\nabla$ are Calderón-Zygmund operators. Subsequently, for the potential V belonging to the reverse Hölder class, which includes non-negative polynomials, Shen ([Sh1]) generalized Zhong's results. Actually, he proved that the operators $\nabla(-\Delta + V)^{-1/2}$ and $\nabla(-\Delta + V)^{-1}\nabla$ are Calderón-Zygmund operators and the operator $\nabla^2(-\Delta + V)^{-1}$ is bounded on L^p, $1 < p < \infty$, while it is well-known that Calderón-Zygmund operators are bounded on L^p, $1 < p < \infty$. He also proved that the operators $V(-\Delta + V)^{-1}$ and $V^{1/2}\nabla(-\Delta + V)^{-1}$ are bounded on L^p, $1 \leq p \leq \infty$.

For the operators $V(-\Delta + V)^{-1}$, $V^{1/2}\nabla(-\Delta + V)^{-1}$ and $\nabla^2(-\Delta + V)^{-1}$, in [KS] we generalized Shen's results as follows. We replace Δ by a second order uniformly elliptic operator $L_0 = -\sum_{i,j=1}^n (\partial/\partial x_i)\{a_{ij}(x)(\partial/\partial x_j)\}$ and suppose that V satisfies the same condition as above. Then we showed that the operators $V(L_0 + V)^{-1}$, $V^{1/2}\nabla(L_0 + V)^{-1}$ and $\nabla^2(L_0 + V)^{-1}$ are bounded on weighted L^p space $(1 < p < \infty)$ and Morrey spaces. (We need appropriate conditions for a_{ij} to prove the boundedness of each operator.) It should be remarked that Calderón-Zygmund operators are bounded on weighted L^p space $(1 < p < \infty)$ and Morrey spaces ([CF], [St]).

To be precise, we first recall the definitions of the reverse Hölder class (cf. [Sh2]) and the Morrey space (cf. [CF]). Throughout this paper we denote by $B_r(x)$ the ball centered at x with radius r, and the letter C stands for a constant not necessarily the same at each occurrence.

DEFINITION 1 (Reverse Hölder class). Let U be a non-negative function on $\boldsymbol{R}^n$.

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(1) For $1 < p < \infty$, we say $U \in (RH)_p$ if $U \in L^p_{loc}(\boldsymbol{R}^n)$ and there exists a constant C such that

$$\left(\frac{1}{|B_r(x)|}\int_{B_r(x)} U(y)^p dy\right)^{1/p} \leq \frac{C}{|B_r(x)|}\int_{B_r(x)} U(y)dy \tag{1}$$

holds for every $x \in \boldsymbol{R}^n$ and $0 < r < \infty$. If (1) holds for $0 < r \leq 1$, we say $U \in (RH)_{p,loc}$.

(2) We say $U \in (RH)_\infty$ if $U \in L^\infty_{loc}(\boldsymbol{R}^n)$ and there exists a constant C such that

$$\|U\|_{L^\infty(B_r(x))} \leq \frac{C}{|B_r(x)|}\int_{B_r(x)} U(y)dy \tag{2}$$

holds for every $x \in \boldsymbol{R}^n$ and $0 < r < \infty$. If (2) holds for $0 < r \leq 1$, we say $U \in (RH)_{\infty,loc}$.

REMARK 1. If $P(x)$ is a polynomial and $\alpha > 0$, then $U(x) = |P(x)|^\alpha$ belongs to $(RH)_\infty$ ([Fe]). For $1 < p < \infty$, it is easy to see $(RH)_\infty \subset (RH)_p$.

DEFINITION 2. For $0 \leq \mu < n$ and $1 \leq p < \infty$, the Morrey space $L^{p,\mu}(\boldsymbol{R}^n)$ is defined by

$$L^{p,\mu}(\boldsymbol{R}^n) = \left\{ f \in L^p_{loc}(\boldsymbol{R}^n) : \|f\|_{p,\mu} = \sup_{\substack{r>0\\ x\in\boldsymbol{R}^n}} \left(\frac{1}{r^\mu}\int_{B_r(x)} |f(y)|^p dy\right)^{1/p} < \infty \right\}.$$

Note that $L^{p,0}(\boldsymbol{R}^n) = L^p(\boldsymbol{R}^n)$.

In this paper we consider the following magnetic Schrödinger operators. Let $\mathbf{a}(x) = (a_1(x), a_2(x), \dots, a_n(x))$,

$$L_j = \frac{1}{i}\frac{\partial}{\partial x_j} - a_j(x) \quad \text{for } 1 \leq j \leq n\,,\quad n \geq 3\,,$$

and $L = (L_1, L_2, \dots, L_n)$, where $a_j \in C^2(\boldsymbol{R}^n)$. Define

$$H = H(\mathbf{a}, V) = \sum_{j=1}^n L_j^2 + V(x)\,,$$

where $V \in L^\infty_{loc}(\boldsymbol{R}^n)$ and $V \geq 0$.

We use the following notation throughout this paper. Let $\mathbf{B}(x) = (b_{jk}(x))_{1\leq j,k\leq n}$, where

$$b_{jk}(x) = \frac{\partial a_j}{\partial x_k} - \frac{\partial a_k}{\partial x_j}\,,$$

and for $1 \leq j \leq n$, $1 \leq k \leq n$, $1 \leq l \leq n$, let

$$\partial_j = \frac{\partial}{\partial x_j}\,,\quad \partial^2_{jk} = \frac{\partial^2}{\partial x_j \partial x_k}\,,\quad \partial^3_{jkl} = \frac{\partial^3}{\partial x_j \partial x_k \partial x_l}\,,\quad |Lu(x)|^2 = \sum_{j=1}^n |L_j u(x)|^2\,,$$

$$|L^2u(x)|^2 = \sum_{j,k=1}^n |L_jL_ku(x)|^2\,,\quad |L^3u(x)|^2 = \sum_{j,k,l=1}^n |L_jL_kL_lu(x)|^2$$

$$\text{and}\quad |\mathbf{B}| = |\mathbf{B}(x)| = \sum_{j,k=1}^n |b_{jk}(x)|\,.$$

For the operator H, Shen ([Sh2]) proved that the operators VH^{-1}, $V^{1/2}LH^{-1}$ and L^2H^{-1} are bounded on L^p, $1 < p < \infty$, if V and the magnetic field $\mathbf{B}$ satisfy certain conditions given in terms of the reverse Hölder inequality. These results are extensions of those in the case $\mathbf{a} \equiv \mathbf{0}$, which were shown by himself.

The purpose of this paper is to show the following two results under certain conditions on V, $\mathbf{a}$ and $\mathbf{B}$. First, we show that the operators VH^{-1}, $V^{1/2}LH^{-1}$ and L^2H^{-1} are bounded on Morrey spaces (see Theorem 1). Secondly, we show that the operator L^2H^{-1} is a Calderón-Zygmund operator (see Theorem 2) on the assumption that $\mathbf{a} \in C^4(\boldsymbol{R}^n)^n$ and $V \in C^3(\boldsymbol{R}^n)$ for the regularity of coefficients.

In his paper [Sh2], Shen established the estimates (see Theorems 5 and 6) of the fundamental solution of the Schrödinger operator by using an auxiliary function $m(x, U)$ introduced by himself. These estimates play an important role in the proof of L^p boundedness of the operators mentioned above. We also need his estimates to prove our results.

We recall the definition of the function $m(x, U)$ for later convenience.

DEFINITION 3 ([Sh1], [Sh2]). For $x \in \boldsymbol{R}^n$, the function $m(x, U)$ is defined by

$$\frac{1}{m(x,U)} = \sup\left\{ r > 0 : \frac{r^2}{|B_r(x)|}\int_{B_r(x)} U(y)dy \le 1 \right\}.$$

REMARK 2. Note that $0 < m(x, U) < \infty$ for $U \in (RH)_{n/2}$ and $1 \le m(x, U) < \infty$ for $U \in (RH)_{n/2,loc}$.

We now state Theorem 1 and Theorem 2 which are main results of this paper.

THEOREM 1. *Suppose* $\mathbf{a} \in C^2(\boldsymbol{R}^n)^n$, $V \in L^\infty_{loc}(\boldsymbol{R}^n)$, $n \ge 3$ *and* $V \ge 0$. *Assume also that*

$$\begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}, \\ V(x) \le Cm(x, |\mathbf{B}| + V)^2, \\ |\nabla \mathbf{B}(x)| \le Cm(x, |\mathbf{B}| + V)^3. \end{cases} \tag{3}$$

(1) *Let* $1 < p < \infty$ *and* $0 < \mu < n$. *Then there exist constants* C_1, C_2 *such that*

$$\|VH^{-1}f\|_{p,\mu} \le C_1\|f\|_{p,\mu} \quad \textit{for } f \in C_0^\infty(\boldsymbol{R}^n),$$

$$\|V^{1/2}LH^{-1}f\|_{p,\mu} \le C_2\|f\|_{p,\mu} \quad \textit{for } f \in C_0^\infty(\boldsymbol{R}^n).$$

(2) *Let* $1 < p < \infty$ *and* $0 < \mu < n$. *In addition, assume that*

$$|\nabla \mathbf{a}(x)| \le Cm(x, |\mathbf{B}| + V)^2, \quad |\mathbf{a}(x)| \le Cm(x, |\mathbf{B}| + V). \tag{4}$$

Then there exists a constant C such that

$$\|L^2H^{-1}f\|_{p,\mu} \le C\|f\|_{p,\mu} \quad \textit{for } f \in C_0^\infty(\boldsymbol{R}^n).$$

REMARK 3. If $V \in (RH)_\infty$, then there exists a constant C such that $V(x) \le Cm(x, V)^2$. In Theorem 1, if $\mathbf{a} \equiv \mathbf{0}$ then the conclusion was shown in [KS] under the assumption that $V \in (RH)_\infty$.

REMARK 4. Theorem 1 is also true for the case $\mu = 0$. In this case, we can replace (3) with somewhat weaker condition and do not need to assume (4) (see [Sh2, Theorems 0.9 and 3.1]). However, our proof of Theorem 1 is different from the one in [Sh2] and is based on the method of [KS]. We emphasize that the method of [Sh2] does not work in the case $\mu > 0$.

We now recall the definition of the Calderón-Zygmund operator. Let $\mathcal{D}'$ denote the space of distributions dual to $C_0^\infty(\boldsymbol{R}^n)$. An operator T taking $C_0^\infty(\boldsymbol{R}^n)$ into $\mathcal{D}'$ is called a Calderón-Zygmund operator if

(i) T extends to a bounded linear operator on $L^2(\boldsymbol{R}^n)$,

(ii) there exists a kernel K such that for every $f \in C_0^\infty(\boldsymbol{R}^n)$,

$$Tf(x) = \int_{\boldsymbol{R}^n} K(x,y)f(y)dy \quad \text{a.e. on } \{\operatorname{supp} f\}^c,$$

(iii) there exist positive constants δ and C such that for all distinct $x, y \in \boldsymbol{R}^n$ and all z such that $|x - z| < |x - y|/2$,

$$|K(x,y)| \le \frac{C}{|x-y|^n}, \tag{5}$$

$$|K(x,y) - K(z,y)| \le \frac{C|x-z|^\delta}{|x-y|^{n+\delta}}, \tag{6}$$

$$|K(y,x) - K(y,z)| \le \frac{C|x-z|^\delta}{|x-y|^{n+\delta}}, \tag{7}$$

See e.g. [Ch, page 12].

THEOREM 2. *Suppose* $\mathbf{a} \in C^4(\boldsymbol{R}^n)^n$, $V \in C^3(\boldsymbol{R}^n)$, $n \ge 3$ *and* $V \ge 0$. *Assume also that*

$$\begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}, \\ |\nabla^3 V(x)| \le Cm(x)^5, \quad |\nabla^2 V(x)| \le Cm(x)^4, \quad |\nabla V(x)| \le Cm(x)^3, \\ |\nabla^3 \mathbf{B}(x)| \le Cm(x)^5, \quad |\nabla^2 \mathbf{B}(x)| \le Cm(x)^4, \quad |\nabla \mathbf{B}(x)| \le Cm(x)^3, \\ |\mathbf{a}(x)| \le Cm(x), \end{cases} \tag{8}$$

where $m(x) = m(x, |\mathbf{B}| + V)$. *Then* $L^2 H^{-1}$ *is a Calderón-Zygmund operator.*

REMARK 5. It is known that $|\nabla \mathbf{B}(x)| \le Cm(x)^3$ implies $|\mathbf{B}(x)| \le Cm(x)^2$ (see [Sh2, Remark 1.8]). We also note that $|\nabla V(x)| \le Cm(x)^3$ implies $V(x) \le Cm(x)^2$.

REMARK 6. The condition (8) holds if the components of $\mathbf{a}$ are polynomials and V is a non-negative polynomial. This follows from the fact that, if $P(x)$ is a non-negative polynomial of degree k, then for any positive integer l there exists a constant C such that $|\nabla^l P(x)| \le Cm(x, P)^{l+2}$ (see [Sh2, page 820]). We note that Theorem 2 is an extension of Zhong's result that the operator $\nabla^2(-\Delta + V)^{-1}$ with non-negative polynomial V is a Calderón-Zygmund operator ([Zh, Proposition 3.1]). We also note that there exist potentials V which satisfy our assumptions but are not non-negative polynomials. For example, consider $V(x) = |P(x)|^\alpha$, where $P(x)$ is a polynomial and $\alpha > 0$.

We denote by $\Gamma(x, y)$ the fundamental solution for H. The operator H^{-1} is the integral operator with $\Gamma(x, y)$ as its kernel. It is known that the operator $L^2 H^{-1}$ is bounded on $L^2(\boldsymbol{R}^n)$ ([Sh2, Theorem 4.7]). We note that the estimates (6) and (7) are implied by a condition

$$|\partial_j K(x, y)| \leq \frac{C}{|x - y|^{n+1}}$$

([Ch, page 12]). Hence, to prove Theorem 2, it suffices to show that the estimates

$$|L_j L_k \Gamma(x, y)| \leq \frac{C}{|x - y|^n}, \quad |\partial_j L_k L_l \Gamma(x, y)| \leq \frac{C}{|x - y|^{n+1}}$$

hold. In fact, stronger estimates hold as the following two theorems state.

THEOREM 3. *Suppose* $\mathbf{a} \in C^3(\boldsymbol{R}^n)^n$, $V \in C^2(\boldsymbol{R}^n)$, $n \geq 3$ *and* $V \geq 0$. *Assume also that*

$$\begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}, \\ |\nabla^2 V(x)| \leq Cm(x)^4, \quad |\nabla V(x)| \leq Cm(x)^3, \\ |\nabla^2 \mathbf{B}(x)| \leq Cm(x)^4, \quad |\nabla \mathbf{B}(x)| \leq Cm(x)^3, \end{cases}$$

where $m(x) = m(x, |\mathbf{B}| + V)$. *Then for any positive integer N there exists a constant* C_N *such that*

$$|L_j L_k \Gamma(x, y)| \leq \frac{C_N}{\{1 + m(x)|x - y|\}^N} \cdot \frac{1}{|x - y|^n}.$$

For the case $V \equiv 0$, Theorem 3 was stated in [Sh2, Remark 2.9] without proof.

THEOREM 4. *Assume the same assumption as in Theorem* 2. *Then for any positive integer N there exists a constant* C_N *such that*

$$|\partial_j L_k L_l \Gamma(x, y)| \leq \frac{C_N}{\{1 + m(x)|x - y|\}^N} \cdot \frac{1}{|x - y|^{n+1}}.$$

REMARK 7. We expect that Theorem 3 would hold under the condition $V \in C^1(\boldsymbol{R}^n)$ and without the assumption $|\nabla^2 V(x)| \leq Cm(x)^4$, and that Theorem 4 (and hence Theorem 2) would hold under the condition $V \in C^2(\boldsymbol{R}^n)$ and without the assumption $|\nabla^3 V(x)| \leq Cm(x)^5$. In the proof of Theorem 6 (see [Sh2, Lemma 2.3 and Lemma 2.7]). Shen first established the estimate of $\Gamma_0(x, y)$, which is the fundamental solution to $H(\mathbf{a}, 0)$, and treated the case $V \not\equiv 0$ as a perturbation of it. We cannot take this strategy to obtain the pointwise estimate of higher order derivatives $L_j L_k \Gamma(x, y)$ because of the strong singularity of $\partial^2_{jk} \Gamma_0(x, y)$. To overcome this difficulty, in Theorem 3 (for example) we assume the additional assumptions $V \in C^2(\boldsymbol{R}^n)$ and $|\nabla^2 V(x)| \leq Cm(x)^4$ and estimate $L_j L_k \Gamma(x, y)$ directly.

We show Theorem 3 and Theorem 4 by a method similar to the one used in the proof of [Sh2, Theorem 1.13].

The plan of this paper is as follows. In Section 2, we prove Theorem 1. Section 3 is devoted to establishing Caccioppoli type inequalities, which we need to complete the proof of Theorem 3 and Theorem 4. In Section 4, we prove Theorem 3. In Section 5, we prove Theorem 4.

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2. Proof of Theorem 1. Theorem 1 is easily proved by the following pointwise estimates. These estimates generalize the results in [Zh, Lemma 3.2] and [KS, Theorem 1.3] to the magnetic Schrödinger operators. For the rest of this paper, we set $m(x) = m(x, |\mathbf{B}| + V)$.

LEMMA 1. *Suppose V, $\mathbf{a}$ and $\mathbf{B}$ satisfy the condition* (3) *assumed in Theorem* 1. *Then there exist constants C_1, C_2 such that*

$$|m(x)^2 f(x)| \le C_1 M(|H(\mathbf{a}, V)f|)(x) \quad \text{for } f \in C_0^\infty(\boldsymbol{R}^n)\,, \tag{9}$$

$$|m(x)Lf(x)| \le C_2 M(|H(\mathbf{a}, V)f|)(x) \quad \text{for } f \in C_0^\infty(\boldsymbol{R}^n)\,, \tag{10}$$

where M is the Hardy-Littlewood maximal operator.

To prove Lemma 1 we use the following estimates of the fundamental solution for H.

THEOREM 5 ([Sh2]). *Suppose $\mathbf{a} \in C^2(\boldsymbol{R}^n)^n$, $V \in L^{n/2}_{loc}(\boldsymbol{R}^n)$, $n \ge 3$ and $V \ge 0$. Assume also that*

$$\begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}\,, \\ |\nabla \mathbf{B}(x)| \le C m(x)^3\,. \end{cases}$$

Then for any positive integer N there exists a constant C_N such that

$$|\Gamma(x, y)| \le \frac{C_N}{\{1 + m(x)|x - y|\}^N} \cdot \frac{1}{|x - y|^{n-2}}\,.$$

THEOREM 6 ([Sh2]). *Suppose $\mathbf{a} \in C^2(\boldsymbol{R}^n)^n$, $V \in L^\infty_{loc}(\boldsymbol{R}^n)$, $n \ge 3$ and $V \ge 0$. Assume also that*

$$\begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}\,, \\ V(x) \le C m(x)^2\,, \\ |\nabla \mathbf{B}(x)| \le C m(x)^3\,. \end{cases}$$

Then for any positive integer N there exists a constant C_N such that

$$|L_j \Gamma(x, y)| \le \frac{C_N}{\{1 + m(x)|x - y|\}^N} \cdot \frac{1}{|x - y|^{n-1}}\,.$$

REMARK 8. For $|x - y| \le 1$, estimates of the fundamental solution for the operator $H + 1$ like above were obtained in [Sh2, Theorem 1.13, Theorem 2.8] under the assumption given in terms of the inequality (1) which holds for $0 < r \le 1$. Theorems 5 and 6 are obtained in the same manner as in the proof of Shen's theorems, since we assume $|\mathbf{B}| + V \in (RH)_{n/2}$ and the pointwise estimates which are analogues of Shen's assumptions.

PROOF OF LEMMA 1. Estimate (9) can be proved as follows. Let $u = H(\mathbf{a}, V)f$ and $r = 1/m(x)$. Then it follows from Theorem 5 that

$$\begin{aligned}
|m(x)^2 f(x)| &\le \int_{\boldsymbol{R}^n} m(x)^2 |\Gamma(x,y)|\,|u(y)|dy \\
&\le C_N \int_{\boldsymbol{R}^n} \frac{m(x)^2|u(y)|}{\{1+m(x)|x-y|\}^N |x-y|^{n-2}} dy \\
&\le C_N \sum_{j=-\infty}^{\infty} \int_{2^{j-1}r<|x-y|\le 2^j r} \frac{|u(y)|}{r^2(1+r^{-1}|x-y|)^N |x-y|^{n-2}} dy \\
&\le C_N \sum_{j=-\infty}^{\infty} \int_{|x-y|\le 2^j r} \frac{|u(y)|}{r^2(1+2^{j-1})^N (2^{j-1}r)^{n-2}} dy \\
&\le C_N \sum_{j=-\infty}^{\infty} \frac{2^{2(j-1)+n}}{(1+2^{j-1})^N} \cdot \frac{1}{(2^j r)^n} \int_{|x-y|\le 2^j r} |u(y)|dy \\
&\le CC_N \sum_{j=-\infty}^{\infty} \frac{2^{2j}}{(1+2^j)^N} M(|u|)(x)\,.
\end{aligned}$$

Therefore we obtain the desired estimate, if we take $N = 3$ for example.

The proof of (10) can be done in the same way as above by using Theorem 6. □

PROOF OF THEOREM 1(1). The boundedness of the operators VH^{-1} and $V^{1/2}LH^{-1}$ immediately follows from Lemma 1 and the fact that the Hardy-Littlewood maximal operator is bounded on Morrey spaces ([CF, Theorem 1]).

PROOF OF THEOREM 1(2). Let $f \in C_0^\infty(\boldsymbol{R}^n)$. Note that

$$L_j L_k = -\partial_{jk}^2 - a_j L_k - a_k L_j - \frac{1}{i}\partial_j a_k - a_j a_k\,, \tag{11}$$

$$H(\mathbf{a}, V) = -\Delta + V - 2\sum_{j=1}^{n} a_j L_j - \frac{1}{i}\operatorname{div}\mathbf{a} - |\mathbf{a}|^2\,. \tag{12}$$

By (11), we have

$$\|L^2 f\|_{p,\mu} \le C\|\nabla^2 f\|_{p,\mu} + C\|mLf\|_{p,\mu} + C\|m^2 f\|_{p,\mu}\,. \tag{13}$$

Here we have

$$\|\nabla^2 f\|_{p,\mu} \le C\|\Delta f\|_{p,\mu}\,, \tag{14}$$

which follows from [CF, Theorem 3], since $\nabla^2(-\Delta)^{-1}$ is a Calderón-Zygmund operator. By using (12) we can control the term $\|\Delta f\|_{p,\mu}$. Then, using Lemma 1, we arrive at the desired estimate. □

3. Caccioppoli type inequalities. In this section we establish the Caccioppoli type inequalities given in the following lemmas.

LEMMA 2 (see [Sh2, Lemma 1.2]). *Suppose $H(\mathbf{a}, V)u = 0$ in $B_R(x_0)$. Then there exists a constant C such that*

$$\int_{B_{R/2}(x_0)} |Lu(x)|^2 dx \le \frac{C}{R^2} \int_{B_R(x_0)} |u(x)|^2 dx .$$

LEMMA 3. *Suppose $H(\mathbf{a}, V)u = 0$ in $B_R(x_0)$ and*

$$\begin{cases} |\nabla V(x)| \le C m(x)^3 , \\ |\nabla \mathbf{B}(x)| \le C m(x)^3 . \end{cases}$$

Then there exist constants C, k_1 such that

$$\int_{B_{R/4}(x_0)} |L^2 u(x)|^2 dx \le \frac{C\{1 + Rm(x_0)\}^{k_1}}{R^4} \int_{B_R(x_0)} |u(x)|^2 dx .$$

LEMMA 4. *Suppose $H(\mathbf{a}, V)u = 0$ in $B_R(x_0)$ and*

$$\begin{cases} |\nabla^2 V(x)| \le C m(x)^4 , \ \ |\nabla V(x)| \le C m(x)^3 , \\ |\nabla^2 \mathbf{B}(x)| \le C m(x)^4 , \ \ |\nabla \mathbf{B}(x)| \le C m(x)^3 . \end{cases}$$

Then there exist constants C, k_2 such that

$$\int_{B_{R/8}(x_0)} |L^3 u(x)|^2 dx \le \frac{C\{1 + Rm(x_0)\}^{k_2}}{R^6} \int_{B_R(x_0)} |u(x)|^2 dx .$$

The next Lemma 5 is used in the proof of Lemmas 3 and 4, and is also used in the following sections to prove our theorems.

LEMMA 5 ([Sh1, Lemma 1.4(b), (c)]). *Suppose $U \in (RH)_{n/2}$ and $U \ge 0$. Then there exist constants C_1, C_2, k_0 such that*

$$m(y, U) \le C_1\{1 + |x - y| m(x, U)\}^{k_0} m(x, U) , \tag{15}$$

$$m(y, U) \ge \frac{C_2 m(x, U)}{\{1 + |x - y| m(x, U)\}^{k_0/(k_0+1)}} . \tag{16}$$

PROOF OF LEMMA 3. Note that, for $1 \le j \le n$, $1 \le k \le n$,

$$[L_j, L_k] = L_j L_k - L_k L_j = \frac{1}{i}(\partial_k a_j - \partial_j a_k) = \frac{1}{i} b_{jk} , \tag{17}$$

$$\begin{aligned} [L_k, L_j^2 + V] &= L_j [L_k, L_j] + [L_k, L_j] L_j + [L_k, V] \\ &= \frac{2}{i} b_{kj} L_j + \frac{1}{i} \partial_k V - \partial_j b_{kj} . \end{aligned} \tag{18}$$

Hence we have

$$\begin{aligned} H(\mathbf{a}, V) L_k u = -[L_k, H(\mathbf{a}, V)] u &= - \sum_{j=1}^{n} [L_k, L_j^2 + V] u \\ &= \sum_{j=1}^{n} \left\{ -\frac{2}{i} b_{kj} L_j u - \left(\frac{1}{i} \partial_k V - \partial_j b_{kj} \right) u \right\} . \end{aligned}$$

Let $\eta \in C_0^\infty(B_{R/2}(x_0))$ such that $\eta \equiv 1$ on $B_{R/4}(x_0)$ and $|\nabla\eta| \le C/R$. Multiplying the above equation by $\eta^2 L_k u$ and integrating over $\boldsymbol{R}^n$ by integration by parts, we have

$$\begin{aligned}\int_{\boldsymbol{R}^n} &\sum_{j=1}^n L_j(L_k u) L_j(\eta^2 L_k u)\\ &\le \int_{\boldsymbol{R}^n} \sum_{j=1}^n \left\{-\frac{2}{i} b_{kj}(L_j u)\eta^2 L_k u - \left(\frac{1}{i}\partial_k V - \partial_j b_{kj}\right) u\eta^2 L_k u\right\}.\end{aligned} \tag{19}$$

The left hand side of (19) is equal to

$$\int_{\boldsymbol{R}^n} \sum_{j=1}^n \left\{(L_j L_k u)^2 \eta^2 + \frac{2}{i}\eta(L_j L_k u)\cdot \partial_j \eta L_k u\right\}.$$

Hence we have

$$\begin{aligned}\int_{\boldsymbol{R}^n} |L^2 u(x)|^2 \eta(x)^2 dx \le & C\int_{\boldsymbol{R}^n} |\nabla\eta(x)|^2 |Lu(x)|^2 dx + C\int_{\boldsymbol{R}^n} |\mathbf{B}(x)||Lu(x)|^2 \eta(x)^2 dx\\ &+ C\int_{\boldsymbol{R}^n} (|\nabla V(x)| + |\nabla \mathbf{B}(x)|)|u(x)||Lu(x)|\eta(x)^2 dx.\end{aligned}$$

By (15) and Lemma 2, we then obtain

$$\begin{aligned}\int_{B_{R/4}(x_0)} &|L^2 u(x)|^2 dx\\ &\le \frac{C}{R^2}\int_{B_{R/2}(x_0)} |Lu(x)|^2 dx + \frac{C\{1+Rm(x_0)\}^{2(k_0+1)}}{R^2}\int_{B_{R/2}(x_0)} |Lu(x)|^2 dx\\ &\quad + \frac{C\{1+Rm(x_0)\}^{3(k_0+1)}}{R^3}\cdot R\int_{B_{R/2}(x_0)} \left(|Lu(x)|^2 + \frac{1}{R^2}|u(x)|^2\right) dx\\ &\le \frac{C\{1+Rm(x_0)\}^{k_1}}{R^4}\int_{B_R(x_0)} |u(x)|^2 dx,\end{aligned}$$

where $k_1 = 3(k_0+1)$. □

PROOF OF LEMMA 4. Note that, for $1 \le j \le n$, $1 \le k \le n$, $1 \le l \le n$,

$$\begin{aligned}[L_k L_l, L_j^2 + V] &= L_k[L_l, L_j^2 + V] + [L_k, L_j^2 + V]L_l\\ &= \frac{2}{i} b_{lj} L_k L_j + \frac{2}{i} b_{kj} L_j L_l - 2\partial_k b_{lj} L_j + \left(\frac{1}{i}\partial_l V - \partial_j b_{lj}\right) L_k\\ &\quad + \left(\frac{1}{i}\partial_k V - \partial_j b_{kj}\right) L_l - \left(\partial_{kl}^2 V + \frac{1}{i}\partial_{kj}^2 b_{lj}\right),\end{aligned} \tag{20}$$

where we used (18). Hence we have

$$
\begin{aligned}
&H(\mathbf{a}, V)L_kL_lu \\
&\quad = -[L_kL_l, H(\mathbf{a}, V)]u = -\sum_{j=1}^{n}[L_kL_l, L_j^2 + V]u \\
&\quad = \sum_{j=1}^{n}\left\{-\frac{2}{i}b_{lj}L_kL_ju - \frac{2}{i}b_{kj}L_jL_lu + 2\partial_kb_{lj}L_ju - \left(\frac{1}{i}\partial_lV - \partial_jb_{lj}\right)L_ku\right. \\
&\qquad \left. - \left(\frac{1}{i}\partial_kV - \partial_jb_{kj}\right)L_lu + \left(\partial_{kl}^2V + \frac{1}{i}\partial_{kj}^2b_{lj}\right)u\right\}.
\end{aligned} \tag{21}
$$

Let $\eta \in C_0^\infty(B_{R/4}(x_0))$ such that $\eta \equiv 1$ on $B_{R/8}(x_0)$ and $|\nabla\eta| \leq C/R$. Then as in the proof of Lemma 3, we have

$$
\begin{aligned}
\int_{\mathbf{R}^n}|L^3u(x)|^2\eta(x)^2dx \leq & C\int_{\mathbf{R}^n}|\nabla\eta(x)|^2|L^2u(x)|^2dx + C\int_{\mathbf{R}^n}|\mathbf{B}(x)||L^2u(x)|^2\eta(x)^2dx \\
& + C\int_{\mathbf{R}^n}(|\nabla V(x)| + |\nabla\mathbf{B}(x)|)|Lu(x)||L^2u(x)|\eta(x)^2dx \\
& + C\int_{\mathbf{R}^n}(|\nabla^2V(x)| + |\nabla^2\mathbf{B}(x)|)|u(x)||L^2u(x)|\eta(x)^2dx\,.
\end{aligned}
$$

By (15) and Lemmas 2 and 3, we then obtain

$$
\begin{aligned}
&\int_{B_{R/8}(x_0)}|L^3u(x)|^2dx \\
&\quad \leq \frac{C}{R^2}\int_{B_{R/4}(x_0)}|L^2u(x)|^2dx + \frac{C\{1 + Rm(x_0)\}^{2(k_0+1)}}{R^2}\int_{B_{R/4}(x_0)}|L^2u(x)|^2dx \\
&\qquad + \frac{C\{1 + Rm(x_0)\}^{3(k_0+1)}}{R^3}\cdot R\int_{B_{R/4}(x_0)}\left(|L^2u(x)|^2 + \frac{1}{R^2}|Lu(x)|^2\right)dx \\
&\qquad + \frac{C\{1 + Rm(x_0)\}^{4(k_0+1)}}{R^4}\cdot R^2\int_{B_{R/4}(x_0)}\left(|L^2u(x)|^2 + \frac{1}{R^4}|u(x)|^2\right)dx \\
&\quad \leq \frac{C\{1 + Rm(x_0)\}^{k_2}}{R^6}\int_{B_R(x_0)}|u(x)|^2dx\,,
\end{aligned}
$$

where $k_2 = k_1 + 4(k_0 + 1) = 7(k_0 + 1)$. □

4. Proof of Theorem 3. Theorem 3 follows easily from the following subsolution estimate for L^2u.

LEMMA 6. *Suppose that $H(\mathbf{a}, V)u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$ and*

$$
\begin{cases}
|\mathbf{B}| + V \in (RH)_{n/2}\,, \\
|\nabla^2V(x)| \leq Cm(x)^4\,,\ \ |\nabla V(x)| \leq Cm(x)^3\,, \\
|\nabla^2\mathbf{B}(x)| \leq Cm(x)^4\,,\ \ |\nabla\mathbf{B}(x)| \leq Cm(x)^3\,.
\end{cases}
$$

Then for any positive integer N there exists a constant C_N such that

$$(22)\qquad \sup_{y\in B_{R/2}(x_0)} |L^2u(y)| \le \frac{C_N}{\{1+Rm(x_0)\}^N}\cdot\frac{1}{R^2}\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|u(x)|^2dx\right)^{1/2}.$$

Assuming this lemma for the moment, we give

PROOF OF THEOREM 3. Note that, by the diamagnetic inequality

$$|e^{-tH(\mathbf{a},V)}f|(x)\le e^{-tH(\mathbf{0},V)}|f|(x)$$

for $t>0$ (see [Si], [LS]) and $V\ge 0$, we have

$$(23)\qquad |H(\mathbf{a},V)^{-1}f|(x)\le(-\Delta)^{-1}|f|(x)\quad \text{for } f\in C_0^\infty(\boldsymbol{R}^n).$$

Then we have

$$(24)\qquad |\Gamma(x,y)|\le\frac{C}{|x-y|^{n-2}}.$$

Fix $x_0, y_0\in\boldsymbol{R}^n$ and put $R=|x_0-y_0|$. Then $u(x)=\Gamma(x,y_0)$ is a solution of $H(\mathbf{a},V)u=0$ on $B_{R/2}(x_0)$. Hence, combining (22) with (24), we arrive at the desired estimate. □

To prove Lemma 6, we need Lemmas 3 and 5 proved in Section 3 and the following subsolution estimates.

LEMMA 7. *Suppose that $H(\mathbf{a},V)u=0$ in $B_R(x_0)$ for some $x_0\in\boldsymbol{R}^n$ and*

$$\begin{cases}|\mathbf{B}|+V\in(RH)_{n/2},\\ |\nabla\mathbf{B}(x)|\le Cm(x)^3.\end{cases}$$

Then for any positive integer N there exists a constant C_N such that

$$(25)\qquad \sup_{y\in B_{R/2}(x_0)}|u(y)|\le\frac{C_N}{\{1+Rm(x_0)\}^N}\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|u(x)|^2dx\right)^{1/2}.$$

PROOF. In the same way as in the proof of [Sh2, Lemma 1.11], for all $0<R<\infty$ we obtain the estimate for $|u(x_0)|$, that is,

$$(26)\qquad |u(x_0)|\le\frac{C_N}{\{1+Rm(x_0)\}^N}\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|u(x)|^2dx\right)^{1/2}.$$

Then, (25) follows easily from (26). Indeed, from (26) we have for all $y\in B_{R/2}(x_0)$,

$$|u(y)|\le\frac{C_N}{\{1+Rm(y)\}^N}\left(\frac{1}{|B_{R/4}(y)|}\int_{B_{R/4}(y)}|u(x)|^2dx\right)^{1/2}.$$

Then, by using (16), we have

$$\sup_{y\in B_{R/2}(x_0)}|u(y)|\le\frac{CC_N}{\{1+Rm(x_0)\}^N}\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|u(x)|^2dx\right)^{1/2}.$$

□

LEMMA 8. *Suppose that $H(\mathbf{a}, V)u = 0$ in $B_R(x_0)$ for some $x_0 \in \boldsymbol{R}^n$ and*

$$\begin{cases} |\mathbf{B}| + V \in (RH)_{n/2}\,, \\ V(x) \le Cm(x)^2\,, \\ |\nabla \mathbf{B}(x)| \le Cm(x)^3\,. \end{cases}$$

Then for any positive integer N there exists a constant C_N such that

$$\sup_{y \in B_{R/2}(x_0)} |Lu(y)| \le \frac{C_N}{\{1 + Rm(x_0)\}^N} \cdot \frac{1}{R} \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |u(x)|^2 dx \right)^{1/2}. \tag{27}$$

PROOF. In the same way as in the proof of [Sh2, Lemma 2.7], for all $0 < R < \infty$ we obtain the estimate for $|Lu(x_0)|$, that is,

$$|Lu(x_0)| \le \frac{C_N}{\{1 + Rm(x_0)\}^N} \cdot \frac{1}{R} \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |u(x)|^2 dx \right)^{1/2}. \tag{28}$$

Combining (28) with the argument in the proof of Lemma 7, we arrive at (27). □

To prove Lemma 6, we also need

LEMMA 9. *Suppose $H(\mathbf{a}, V)u = f$ in $B_R(x_0)$. Then there exists a constant C such that*

$$\begin{aligned} \left(\frac{1}{|B_{R/8}(x_0)|} \int_{B_{R/8}(x_0)} |u(x)|^q dx \right)^{1/q} \le & C \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |u(x)|^2 dx \right)^{1/2} \\ & + CR^2 \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |f(x)|^p dx \right)^{1/p}, \end{aligned}$$

where $2 \le p \le q \le \infty$ and $1/q > 1/p - 2/n$.

See [Sh2, Lemma 1.3] for the proof. Now we are ready to give

PROOF OF LEMMA 6. Let $2 \le p \le q \le \infty$ and $1/q > 1/p - 2/n$. Then it follows from (21) and Lemma 9 that

$$\begin{aligned} & \left(\frac{1}{|B_{R/64}(x_0)|} \int_{B_{R/64}(x_0)} |L^2 u(x)|^q dx \right)^{1/q} \\ & \le C \left(\frac{1}{|B_{R/8}(x_0)|} \int_{B_{R/8}(x_0)} |L^2 u(x)|^2 dx \right)^{1/2} \\ & + CR^2 \left(\frac{1}{|B_{R/8}(x_0)|} \int_{B_{R/8}(x_0)} \{|\mathbf{B}(x)||L^2 u(x)|\}^p dx \right)^{1/p} \\ & + CR^2 \left(\frac{1}{|B_{R/8}(x_0)|} \int_{B_{R/8}(x_0)} \{(|\nabla V(x)| + |\nabla \mathbf{B}(x)|)|Lu(x)|\}^p dx \right)^{1/p} \end{aligned}$$

$$
\begin{aligned}
&+ CR^2 \left(\frac{1}{|B_{R/8}(x_0)|} \int_{B_{R/8}(x_0)} \{(|\nabla^2 V(x)| + |\nabla^2 \mathbf{B}(x)|)|u(x)|\}^p dx \right)^{1/p} \\
&\leq \frac{C\{1 + Rm(x_0)\}^{k_1/2}}{R^2} \left(\frac{1}{|B_{R/2}(x_0)|} \int_{B_{R/2}(x_0)} |u(x)|^2 dx \right)^{1/2} \\
&+ CR^2\{1 + Rm(x_0)\}^{2k_0} m(x_0)^2 \left(\frac{1}{|B_{R/8}(x_0)|} \int_{B_{R/8}(x_0)} |L^2 u(x)|^p dx \right)^{1/p} \\
&+ CR^2\{1 + Rm(x_0)\}^{3k_0} m(x_0)^3 \left(\frac{1}{|B_{R/8}(x_0)|} \int_{B_{R/8}(x_0)} |Lu(x)|^p dx \right)^{1/p} \\
&+ CR^2\{1 + Rm(x_0)\}^{4k_0} m(x_0)^4 \left(\frac{1}{|B_{R/8}(x_0)|} \int_{B_{R/8}(x_0)} |u(x)|^p dx \right)^{1/p} \\
&\leq \frac{C\{1 + Rm(x_0)\}^{k_3}}{R^2} \left(\frac{1}{|B_{R/2}(x_0)|} \int_{B_{R/2}(x_0)} |u(x)|^2 dx \right)^{1/2} \\
&+ C\{1 + Rm(x_0)\}^{2(k_0+1)} \left(\frac{1}{|B_{R/8}(x_0)|} \int_{B_{R/8}(x_0)} |L^2 u(x)|^p dx \right)^{1/p},
\end{aligned}
$$

where k_3 is a constant depending only on k_0 and we have used (15) and Lemmas 3, 7 and 8. A bootstrap argument combined with Lemmas 3 and 7 then yields that

$$
\begin{aligned}
|L^2 u(x_0)| &\leq \frac{C\{1 + Rm(x_0)\}^{k_4}}{R^2} \left(\frac{1}{|B_{R/2}(x_0)|} \int_{B_{R/2}(x_0)} |u(x)|^2 dx \right)^{1/2} \\
&+ C\{1 + Rm(x_0)\}^{k_4} \left(\frac{1}{|B_{R/8}(x_0)|} \int_{B_{R/8}(x_0)} |L^2 u(x)|^2 dx \right)^{1/2} \\
&\leq \frac{C\{1 + Rm(x_0)\}^{k_1/2 + k_4}}{R^2} \left(\frac{1}{|B_{R/2}(x_0)|} \int_{B_{R/2}(x_0)} |u(x)|^2 dx \right)^{1/2} \\
&\leq \frac{C_N}{\{1 + Rm(x_0)\}^N} \cdot \frac{1}{R^2} \left(\frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} |u(x)|^2 dx \right)^{1/2},
\end{aligned}
$$

where k_4 is a constant depending only on n and k_0. □

5. Proof of Theorem 4. We need the following lemma to prove Theorem 4.

LEMMA 10. *Suppose that $H(\mathbf{a}, V)u = 0$ in $B_R(x_0)$ for some $x_0 \in \mathbf{R}^n$ and*

$$
\begin{cases}
|\mathbf{B}| + V \in (RH)_{n/2}, \\
|\nabla^3 V(x)| \leq Cm(x)^5, \quad |\nabla^2 V(x)| \leq Cm(x)^4, \quad |\nabla V(x)| \leq Cm(x)^3, \\
|\nabla^3 \mathbf{B}(x)| \leq Cm(x)^5, \quad |\nabla^2 \mathbf{B}(x)| \leq Cm(x)^4, \quad |\nabla \mathbf{B}(x)| \leq Cm(x)^3.
\end{cases}
$$

Then for any positive integer N there exists a constant C_N such that

$$\sup_{y\in B_{R/2}(x_0)} |L^3u(y)| \le \frac{C_N}{\{1+Rm(x_0)\}^N}\cdot\frac{1}{R^3}\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|u(x)|^2dx\right)^{1/2}. \tag{29}$$

PROOF OF THEOREM 4. Fix $x_0, y_0 \in \boldsymbol{R}^n$ and put $R = |x_0 - y_0|$. Applying (29) to $u(x) = \Gamma(x, y_0)$ and using (24), we obtain the estimate

$$|L_jL_kL_l\Gamma(x,y)| \le \frac{C_N}{\{1+m(x)|x-y|\}^N}\cdot\frac{1}{|x-y|^{n+1}}.$$

Then, using the assumption on $\mathbf{a}$ and Theorem 3, we arrive at the desired estimate. □

PROOF OF LEMMA 10. Let $1 \le j \le n$, $1 \le k \le n$, $1 \le l \le n$, $1 \le m \le n$. By (18) and (20) we then have

$$\begin{aligned}
&H(\mathbf{a},V)L_kL_lL_mu = -[L_kL_lL_m, H(\mathbf{a},V)]u = -\sum_{j=1}^n [L_kL_lL_m, L_j^2+V]u\\
&= -\sum_{j=1}^n \{L_k[L_lL_m, L_j^2+V]u + [L_k, L_j^2+V]L_lL_mu\}\\
&= \sum_{j=1}^n \Bigg\{ -\frac{2}{i}b_{mj}L_kL_lL_ju - \frac{2}{i}b_{lj}L_kL_jL_mu - \frac{2}{i}b_{kj}L_jL_lL_mu + 2\partial_kb_{mj}L_lL_ju\\
&\quad +2\partial_kb_{lj}L_jL_mu + 2\partial_lb_{mj}L_kL_ju - \left(\frac{1}{i}\partial_mV - \partial_jb_{mj}\right)L_kL_lu\\
&\quad -\left(\frac{1}{i}\partial_lV - \partial_jb_{lj}\right)L_kL_mu - \left(\frac{1}{i}\partial_kV - \partial_jb_{kj}\right)L_lL_mu + \frac{2}{i}\partial_{kl}^2b_{mj}L_ju\\
&\quad +\left(\partial_{km}^2V + \frac{1}{i}\partial_{kj}^2b_{mj}\right)L_lu + \left(\partial_{kl}^2V + \frac{1}{i}\partial_{kj}^2b_{lj}\right)L_mu\\
&\quad +\left(\partial_{lm}^2V + \frac{1}{i}\partial_{lj}^2b_{mj}\right)L_ku + \left(\frac{1}{i}\partial_{klm}^3V - \partial_{klj}^3b_{mj}\right)u\Bigg\}.
\end{aligned} \tag{30}$$

Let $2 \le p \le q \le \infty$ and $1/q > 1/p - 2/n$. Then it follows from (30) and Lemma 9 that

$$\begin{aligned}
&\left(\frac{1}{|B_{R/128}(x_0)|}\int_{B_{R/128}(x_0)}|L^3u(x)|^qdx\right)^{1/q}\\
&\le C\left(\frac{1}{|B_{R/16}(x_0)|}\int_{B_{R/16}(x_0)}|L^3u(x)|^2dx\right)^{1/2}\\
&\quad + CR^2\left(\frac{1}{|B_{R/16}(x_0)|}\int_{B_{R/16}(x_0)}\{|\mathbf{B}(x)||L^3u(x)|\}^pdx\right)^{1/p}\\
&\quad + CR^2\left(\frac{1}{|B_{R/8}(x_0)|}\int_{B_{R/8}(x_0)}\{(|\nabla V(x)|+|\nabla\mathbf{B}(x)|)|L^2u(x)|\}^pdx\right)^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&+CR^2\left(\frac{1}{|B_{R/16}(x_0)|}\int_{B_{R/16}(x_0)}\{(|\nabla^2 V(x)|+|\nabla^2\mathbf{B}(x)|)|Lu(x)|\}^p dx\right)^{1/p}\\
&+CR^2\left(\frac{1}{|B_{R/16}(x_0)|}\int_{B_{R/16}(x_0)}\{(|\nabla^3 V(x)|+|\nabla^3\mathbf{B}(x)|)|u(x)|\}^p dx\right)^{1/p}\\
&\le \frac{C\{1+Rm(x_0)\}^{k_2/2}}{R^3}\left(\frac{1}{|B_{R/2}(x_0)|}\int_{B_{R/2}(x_0)}|u(x)|^2 dx\right)^{1/2}\\
&+CR^2\{1+Rm(x_0)\}^{2k_0}m(x_0)^2\left(\frac{1}{|B_{R/16}(x_0)|}\int_{B_{R/16}(x_0)}|L^3u(x)|^p dx\right)^{1/p}\\
&+CR^2\{1+Rm(x_0)\}^{3k_0}m(x_0)^3\left(\frac{1}{|B_{R/16}(x_0)|}\int_{B_{R/16}(x_0)}|L^2u(x)|^p dx\right)^{1/p}\\
&+CR^2\{1+Rm(x_0)\}^{4k_0}m(x_0)^4\left(\frac{1}{|B_{R/16}(x_0)|}\int_{B_{R/16}(x_0)}|Lu(x)|^p dx\right)^{1/p}\\
&+CR^2\{1+Rm(x_0)\}^{5k_0}m(x_0)^5\left(\frac{1}{|B_{R/16}(x_0)|}\int_{B_{R/16}(x_0)}|u(x)|^p dx\right)^{1/p}\\
&\le \frac{C\{1+Rm(x_0)\}^{k_5}}{R^3}\left(\frac{1}{|B_{R/2}(x_0)|}\int_{B_{R/2}(x_0)}|u(x)|^2 dx\right)^{1/2}\\
&+C\{1+Rm(x_0)\}^{2(k_0+1)}\left(\frac{1}{|B_{R/16}(x_0)|}\int_{B_{R/16}(x_0)}|L^3u(x)|^p dx\right)^{1/p},
\end{aligned}
$$

where k_5 is a constant depending only on k_0 and we have used (15) and Lemmas 4, 6, 7 and 8. A bootstrap argument combined with Lemmas 4 and 7 then yields that

$$
\begin{aligned}
|L^3u(x_0)| &\le \frac{C\{1+Rm(x_0)\}^{k_6}}{R^3}\left(\frac{1}{|B_{R/2}(x_0)|}\int_{B_{R/2}(x_0)}|u(x)|^2 dx\right)^{1/2}\\
&\quad + C\{1+Rm(x_0)\}^{k_6}\left(\frac{1}{|B_{R/16}(x_0)|}\int_{B_{R/16}(x_0)}|L^3u(x)|^2 dx\right)^{1/2}\\
&\le \frac{C\{1+Rm(x_0)^{k_2/2+k_6}}{R^3}\left(\frac{1}{|B_{R/2}(x_0)|}\int_{B_{R/2}(x_0)}|u(x)|^2 dx\right)^{1/2}\\
&\le \frac{C_N}{\{1+Rm(x_0)\}^N}\cdot\frac{1}{R^3}\left(\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)}|u(x)|^2 dx\right)^{1/2},
\end{aligned}
$$

where k_6 is a constant depending only on n and k_0. □

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