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The Hausdorff dimension of non-uniquely ergodic directions in $H(2)$ is almost everywhere $\frac{1}{2}$

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We show that for almost every (with respect to Masur–Veech measure) translation surface $\omega \in \mathcal{H}(2)$, the set of angles $\theta \in [0, 2\pi)$ such that $e^{i\theta}\omega$ has non-uniquely ergodic vertical foliation has Hausdorff dimension (and codimension) $\frac{1}{2}$. We show this by proving that the Hausdorff codimension of the set of non-uniquely ergodic interval exchange transformations (IETs) in the Rauzy class of (4321) is also $\frac{1}{2}$.

37E05, 37E35

1 Introduction

A genus-g *translation surface* (X, ω) is a compact, genus-g Riemann surface together with a holomorphic one-form ω. This gives a structure of a flat metric away from a finite number of singular points, as integrating the one-form ω gives charts (away from zeros of ω) to $\mathbb{C}$ where the transition functions between charts are translations. The zeros of ω are singular points of the metric, and have cone angles $2\pi(k+1)$ at a zero of order k. Translation surfaces inherit a straight line unit-speed flow in each direction $\theta \in [0, 2\pi)$ (corresponding to the foliation $\mathrm{Re}(e^{i\theta}\omega) = 0$). These flows preserve Lebesgue measure on the surface. A key result on the ergodic properties of these flows was proved by Kerckhoff, Masur and Smillie [4]:

Theorem [4, Theorem 2] *For every translation surface, the flow in almost every direction is uniquely ergodic with respect to Lebesgue measure.*

Moduli spaces of translation surfaces are stratified by their genus g and the combinatorics of their singularities. We say a singularity has order k if the angle is $2\pi(k+1)$. The Gauss–Bonnet theorem implies that the sum of orders of singularities on a genus-g surface is $2g-2$. Given a partition $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$, $\sum \alpha_i = 2g-2$, we define the stratum $\mathcal{H} = \mathcal{H}(\alpha)$ to be the moduli space of (unit-area) translation surfaces with singularity pattern α. On each stratum $\mathcal{H}$, there are coordinate charts to an appropriate Euclidean space, and pulling back Lebesgue measure yields a natural measure μ_{MV},

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known as *Masur–Veech* measure. Similarly, pulling back Euclidean distance yields a (local) metric on flat surfaces in a given stratum. For each translation surface, there is a countable set of directions where the flow is not minimal (that is, there are non-dense infinite trajectories). Moreover, by a theorem of Masur and Smillie [9], for almost every translation surface there is an uncountable set of non-uniquely ergodic directions. Given a translation surface ω, let

$$\mathrm{NUE}(\omega) := \{\theta : \text{vertical flow on } e^{i\theta}\omega \text{ is non-uniquely ergodic}\}.$$

Theorem [9, Main theorem] *In every stratum of translation surfaces $\mathcal{H}(\alpha)$ of surfaces of genus at least 2 there is a constant $c = c(\alpha) > 0$ such that for μ_{MV}–almost every flat surface $\omega \in \mathcal{H}$,*

$$\mathrm{Hdim}(\mathrm{NUE}(\omega)) = c.$$

We call the constant $c = c(\alpha)$ the *Masur–Smillie constant* of the stratum of $\mathcal{H}(\alpha)$. Masur [8] showed that $c(\alpha) \leq \frac{1}{2}$ for all α. The main result of this paper is that for $\mathcal{H}(2)$, the Masur–Smillie constant is $\frac{1}{2}$.

Theorem 1.1 *For μ_{MV}–almost every $\omega \in \mathcal{H}(2)$,*

$$\mathrm{Hdim}(\mathrm{NUE}(\omega)) = \tfrac{1}{2} = 1 - \tfrac{1}{2}.$$

By our methods we also obtain the Hausdorff dimension of the set of translation surfaces in $\mathcal{H}(2)$ where the vertical flow is non-uniquely ergodic. The real dimension of $\mathcal{H}(2)$ is 7, and we have:

Theorem 1.2

$$\mathrm{Hdim}(\{\omega \in \mathcal{H}(2) : \textit{vertical flow on } \omega \textit{ is non-uniquely ergodic}\}) = \tfrac{13}{2} = 7 - \tfrac{1}{2}.$$

Remark 1.3 This is the first time the Masur–Smillie constant for a stratum has been identified. Earlier, Cheung, Hubert, and Masur [2] identified the Hausdorff dimension of non-uniquely ergodic directions for the historically important example of two symmetric tori glued along a slit. The Hausdorff dimension is either $\frac{1}{2}$ or 0 and they gave an explicit description of these two cases based on the diophantine properties of the length of the slit. Earlier, Cheung [1] had found an example of two symmetric tori glued along a slit where the set of non-uniquely ergodic directions has Hausdorff dimension $\frac{1}{2}$. In the paper's appendix, Boshernitzan showed a residual set of these examples have that the set of non-uniquely ergodic directions has Hausdorff dimension 0. All of these results deal with a measure zero subset of the stratum $\mathcal{H}(1,1)$.

Prior to the paper [4], Masur [7] and Veech [15] independently showed that for almost every flat surface the flow in almost every direction was uniquely ergodic with respect to Lebesgue measure. Constructions of non-uniquely ergodic IETs are due to Sataev [12], Keane [3] and Keynes and Newton [5], and, anachronistically, Veech [13].

To prove Theorems 1.1 and 1.2, we establish a related theorem for interval exchange transformations (IETs) (see Section 2 for the definition of IETs). Given a permutation $\pi \in S_m$ on m letters, we parametrize the set of IETs which have π as a permutation by the unit simplex

$$\Delta_m := \left\{\lambda \in \mathbb{R}_+^m : \sum_{i=1}^m \lambda_i = 1\right\}.$$

We denote the IET with length vector λ and permutation π by $T_{\lambda,\pi}$. Note that the real dimension of Δ_m is $m-1$.

Theorem 1.4 *The Hausdorff dimension of the set* NUE(4321) *of non-uniquely ergodic* 4*–IETs of* $[0,1)$ *with permutation* $\pi_0 = (4321)$ *is* $\frac{5}{2}$*. That is,*

$$\mathrm{Hdim}(\mathrm{NUE}(4321)) = \mathrm{Hdim}(\{\lambda \in \Delta_4 : T_{\lambda,\pi_0} \text{ is non-uniquely ergodic }\}) = \tfrac{5}{2} = 3 - \tfrac{1}{2}.$$

Remark 1.5 Theorems 1.1, 1.2 and 1.4 all state that the *Hausdorff codimension* of non-uniquely ergodic objects is $\frac{1}{2}$.

By combining work of Masur [8], Minsky and Weiss [11] and a result in metric geometry [10] we obtain the following general result.

Theorem 1.6 *The set of non-uniquely ergodic* n*–IETs has Hausdorff codimension at least* $\frac{1}{2}$*. The set of flat surfaces in any (connected component of any) stratum whose vertical flow is not uniquely ergodic has Hausdorff codimension at least* $\frac{1}{2}$*.*

1.1 Outline of proof

Masur's theorem [8], together with standard results in metric geometry, provides the upper bound. For the lower bound, we generate specific paths in Rauzy induction (Section 2.2) that by a criterion of Veech [14] (Lemma 4.1 in this paper) give minimal and non-uniquely ergodic IETs. The cylinder sets of these paths have nice geometric properties (Section 4.2). This allows us to construct a measure supported on this subset of non-uniquely ergodic IETs with appropriate decay properties. Using Frostman's lemma, we obtain that this set has Hausdorff dimension at least $\frac{5}{2}$; in Section 3 we prove an abstract result on measures supported on intersections of simplices and in Section 5 we use this construction to build our measure on the set of non-uniquely ergodic IETs. To prove Theorem 1.2, we appeal to a standard decomposition of the stratum into stable and unstable foliations. The property that the vertical flow is non-uniquely ergodic depends only on the unstable coordinate and reduces the problem to IETs. To prove Theorem 1.1 we use a result of Minsky and Weiss [11] that varying along a horocycle in the space of flat surfaces gives lines in IET space. By a standard correspondence between the horocycle and changing directions on a fixed flat surface this implies the theorem. Having enough lines implies by standard results in metric geometry that many of the lines intersect non-uniquely ergodic IETs in Hausdorff dimension $\frac{1}{2}$. This establishes Theorem 1.1 for a positive measure set of flat surfaces. There is an $\mathrm{SL}_2(\mathbb{R})$ action on each stratum which is ergodic (on connected components of strata). The Hausdorff dimension of non-uniquely ergodic directions is invariant under this action. The ergodicity of the $\mathrm{SL}_2(\mathbb{R})$ action lets us go from positive to full measure and proves Theorem 1.1.

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2 Background material

2.1 Our spaces

This section recalls standard material which is treated in, for example, Zorich's survey [17]. A translation surface can be given by a union of polygons $P_1 \cup \cdots \cup P_n$, where each $P_i \subset \mathbb{C}$, so that each side of each P_i is glued to exactly one other (parallel) side by a translation, and the total resulting angle at each vertex is an integer multiple of 2π. Translation surfaces can be organized by the number and order of these singularities, that is by integer partitions α of $2g-2$, where g is the genus of the surface, yielding *strata* $\mathcal{H}(\alpha)$. Since translations are holomorphic, and preserve the one-form dz, we obtain a complex structure and a holomorphic differential ω on the identified surface, which away from zeros is locally dz. The zeroes of the differential will be at the identified vertices with total angle greater than 2π, and the order of the zero is equal to the excess angle, that is $\omega = z^k dz$ in a neighborhood of a point with total angle $2\pi(k+1)$. This paper concerns translation surfaces in the stratum $\mathcal{H}(2)$, that is, genus 2 surfaces with one singularity with angle 6π. Kontsevich and Zorich [6] classified the connected components of strata. There are at most three and in our case there is only one; that is, $\mathcal{H}(2)$ is connected.

By varying the sides of the polygons P_i one changes the flat surface. This gives *local coordinates* on strata (modeled on relative cohomology of the surface with respect to the singularities). Pulling back Lebesgue measure in these coordinates gives the *Masur–Veech* measure μ_{MV} on each stratum. $\mathrm{SL}_2(\mathbb{R})$ acts on strata via the natural linear action on the polygons P_i. Masur [7] and Veech [15] showed that this action is ergodic with respect to μ_{MV} on connected components of the stratum. On any translation surface, we have the straight line flow given by flow in the vertical direction in $\mathbb{C}$, and the flow in direction θ, which is the vertical flow on the surface $e^{i\theta}\omega$. For any translation surface, the first return map of the flow in a fixed direction to a transverse interval gives a special map of the interval, known as an *interval exchange transformation*.

Definition 2.1 *Given* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ *where* $\lambda_i > 0$, *we obtain* d *sub-intervals of the interval* $[0, \sum_{i=1}^d \lambda_i)$,

$$I_1 = [0, \lambda_1), \quad I_2 = [\lambda_1, \lambda_1 + \lambda_2), \quad \dots, \quad I_d = \left[\sum_{i=1}^{d-1} \lambda_i, \sum_{i=1}^{d} \lambda_i\right).$$

Given a permutation π *on the set* $\{1, 2, \dots, d\}$, *we obtain a* d–Interval Exchange Transformation *(IET),* $T\colon [0, \sum_{i=1}^d \lambda_i) \to [0, \sum_{i=1}^d \lambda_i)$, *which exchanges the intervals* I_i *according to* π. *That is, if* $x \in I_j$ *then*

$$T(x) = x - \sum_{k<j} \lambda_k + \sum_{\pi(k')<\pi(j)} \lambda_{k'}.$$

For a small enough neighborhood in the space of translation surfaces U, one can locally fix a horizontal transversal where the IET has $2g+k-1$ intervals, where g is the genus and k is the number of singularities of the translation surface. This provides a map $\mathcal{T}\colon U \to \mathbb{R}_+^{2g+k-1}$. This is a (locally) Lipschitz map from U with the metric given by coordinates to $\mathbb{R}_+^{2g+k-1}$ with the Euclidean metric. In fact, it is still locally

Lipschitz if we compose it with the natural map $\lambda \mapsto \lambda/|\lambda|$, where $|\lambda| = \sum \lambda_i$, to obtain a map from U to the simplex $\Delta_{2g+k-1} := \{\lambda \in \mathbb{R}_+^{2g+k-1} : \sum \lambda_i = 1\}$.

2.2 Rauzy induction

The proof of our main result (and indeed many results on ergodic properties of IETs) uses in a crucial fashion the Rauzy induction renormalization procedures for IETs, involving induced maps on certain subintervals, and closely related to Teichmüller geodesic flow. Our treatment of Rauzy induction will be the same as in [15, Section 7]. For further details of the procedure (and much more on IETs) we refer the interested reader to [16] for an excellent survey.

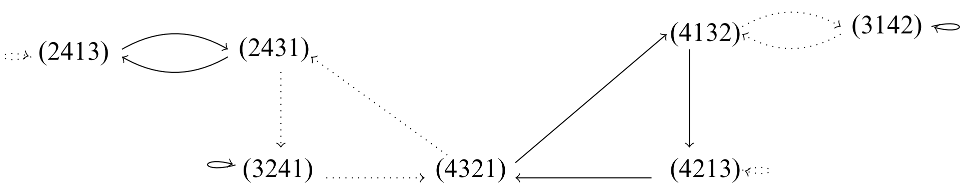

Figure 1: The Rauzy class of (4321). Dashed arrows represent A–moves, and solid B–moves.

The Rauzy induction map R is defined for all but a codimension 1 set of IETs and associates to an interval exchange map $T = T_{\lambda,\pi}$ (now we restrict to $\lambda \in \Delta_m$, the unit simplex in $\mathbb{R}_+^m$, $\pi \in S_m$) a new interval exchange map $R(T) = T_{\lambda',\pi'}$, by considering the induced map of T on the subinterval $[0, 1 - \min(\lambda_m, \lambda_{\pi^{-1}m}))$, and renormalizing the lengths so $R(T)$ is again a map of $[0, 1)$. λ' is related to λ via a projective linear transformation defined below. Rauzy induction is only defined if $\lambda_m \neq \lambda_{\pi^{-1}m}$. The *Rauzy class* $\mathcal{R}$ of a permutation π is the subset of S_m that contains all the forward images of π under Rauzy induction. The permutations in the Rauzy class form the vertices of the *Rauzy graph*, a directed graph with two edges emanating from each permutation $\sigma \in \mathcal{R}$, corresponding to the permutations obtained by inducing on $[0, 1 - \lambda_m)$ and $[0, 1 - \lambda_{\sigma^{-1}m})$ respectively.

If $\lambda_m = \min(\lambda_m, \lambda_{\pi^{-1}m})$ we say the first step in Rauzy induction is A. In this case the permutation of $R(T)$ is given by

$$\pi'(j) = \begin{cases} \pi(j) & j \le \pi^{-1}(d), \\ \pi(d) & j = \pi^{-1}(d) + 1, \\ \pi(j-1) & \text{otherwise.} \end{cases}$$

Right side		Left side	
Edge	Matrix	Edge	Matrix
(4321) → (4132)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$	(4321) → (2431)	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
(4132) → (4213)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$	(2431) → (3241)	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
(4213) → (4321)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$	(3241) → (4321)	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
(4132) → (3142)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	(2431) → (2413)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$
(3142) → (3142)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$	(2413) → (2413)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
(3142) → (4132)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	(2413) → (2431)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$
(4213) → (4213)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	(3241) → (3241)	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

Figure 2: Rauzy matrices for the edges in the Rauzy graph of (4321). We divide into left and right sides and note the symmetries.

We record the action of Rauzy induction by the elementary matrix $M(T,1)$, where

$$M(T,1)[ij] = \begin{cases} \delta_{i,j} & j \le \pi^{-1}(d), \\ \delta_{i,j-1} & j > \pi^{-1}(d) \text{ and } i \neq d, \\ \delta_{\pi^{-1}(d)+1,j} & i = d. \end{cases}$$

If $\lambda_{\pi^{-1}m} = \min(\lambda_m, \lambda_{\pi^{-1}m})$ we say the first step in Rauzy induction is B. In this case the permutation of $R(T)$ is given by

$$\pi'(j) = \begin{cases} \pi(j) & \pi(j) \leq \pi(d), \\ \pi(j)+1 & \pi(d) < \pi(j) < d, \\ \pi(d)+1 & \pi(j) = d, \end{cases}$$

and the associated matrix is

$$M(T,1)[ij] = \begin{cases} 1 & i = d \text{ and } j = \pi^{-1}(d), \\ \delta_{i,j} & \text{otherwise.} \end{cases}$$

We have

$$\lambda = \frac{M(T,1)\lambda'}{|M(T,1)\lambda'|}.$$

Here, $|\cdot|$ denotes the L_1 norm on $\mathbb{R}^d$. $M(T,1)$ depends on whether the step is A or B and the permutation π. We define the n^{th} matrix of Rauzy induction by

$$M(T,n) = M(T,n-1)M(R^{n-1}(T),1).$$

If $R^n(T) = \lambda^{(n)}$, we have

$$\lambda = \frac{M(T,n)\lambda^{(n)}}{|M(T,n)\lambda^{(n)}|}.$$

Given a matrix M, we write

$$M\Delta = M\mathbb{R}_d^+ \cap \Delta_d = \left\{ \frac{Mv}{|Mv|} : v \in \Delta_d \right\}.$$

Observe that if $T = T_{\lambda,\pi}$, $S = S_{\eta,\pi}$ are IETs with $\eta \in M(T,k)\Delta$, then

$$M(S,k) = M(T,k).$$

That is, the IETs T and S have the same first k steps of Rauzy induction.

We will be working with the Rauzy class of the permutation (4321) on 4 letters. We record the graph (Figure 1) and the associated matrices (Figure 2).

3 Measures on subsimplices

Our main abstract result on Hausdorff dimension is the following axiomatic result, essentially a consequence of Frostman's lemma:

Proposition 3.1 *Let $\{S_i\}$ be a sequence of finite collections of disjoint affine 3–simplices in Δ_3. We assume the collection is* nested*: for each simplex $J \in S_{i+1}$, there is a $J' \in S_i$ so that $J \subset J'$. We denote*

$$\mathcal{S}_i := \bigcup_{J \in S_i} J.$$

Suppose $\{S_i\}$ satisfies:

(1) *There exists a constant $c > 0$ such that each $J \in S_k$ has one side of length at least c.*

(2) *There exists a constant $\rho > 0$ and a quadratic polynomial $p(x)$ with leading coefficient $a > 0$ such that each simplex $J \in S_k$ contains $\rho 10^{p(k)}$ simplices in S_{k+1}.*

(3) *There exist $q \in \mathbb{N}$ and $h(x)$, a cubic polynomial with leading coefficient $-b$, $b > 0$, such that if*

$$r < 10^{h(k)} \quad \text{and} \quad p \in \Delta_3,$$

then there is at most one simplex $J \in S_{k+q}$ that intersects $B(p,r)$. In particular, for all $q' \geq q$,

$$B(p,r) \cap \mathcal{S}_{k+q'} = B(p,r) \cap J \cap \mathcal{S}_{k+q'}.$$

Then

$$H_{dim}\left(\bigcap_{i=1}^{\infty} \mathcal{S}_i\right) \geq 1 + \frac{a}{3b}.$$

In particular if $a = 24$ and $b = \frac{16}{3}$ then

$$H_{dim}\left(\bigcap_{i=1}^{\infty} \mathcal{S}_i\right) \geq \frac{5}{2}.$$

To prove Proposition 3.1 we use Frostman's lemma:

Theorem 3.2 (Frostman's lemma) *Let A be a Borel set, $s > 0$, μ be a measure on A with $\mu(A) > 0$, and $C \in \mathbb{R}$ so that for all x, r we have*

$$\mu(B(x,r)) \leq C r^s.$$

Then

$$\mathrm{Hdim}(A) \geq s.$$

Proof of Proposition 3.1 We build a sequence of measures supported on the $\mathcal{S}_k$ whose weak-* limits are supported on $\bigcap_{k=1}^{\infty} \mathcal{S}_k$. Let μ_1 be defined to be the probability measure such that:

(1) It gives equal mass to each element $J \in S_1$.

(2) It is (a scalar multiple of) Lebesgue measure when restricted to any element $J \in S_1$.

Given μ_{k-1} which is a probability measure which restricted to each $J \in S_{k-1}$ is a scalar multiple of Lebesgue measure, we inductively define μ_k to be the probability measure such that on each element $J \in S_k$ it is a scalar multiple of Lebesgue satisfying the following:

$$\text{if } J, J' \subset I \in S_{k-1}, \quad \text{then } \mu_k(J) = \mu_k(J').$$

That is, we evenly divide the mass in I among its *descendants* in S_k (we say $J' \in S_k$ is a descendant of $J \in S_{k-1}$ if $J' \subset J$). Note that each element of S_{k-1} has the same number of descendants. Let μ_∞ be a weak-* limit of these measures (it is unique but this is not important for our purposes).

Since our inductive process divides measure evenly among descendants and the elements of S_k are disjoint, we have the following *stability condition*: For all $J, J' \in S_k$ and $L \geq k$,

$$\mu_k(J) = \mu_L(J) = \mu_L(J') = \mu_k(J'). \tag{3-1}$$

In the statement of the next lemma $|\cdot|$ denotes cardinality.

Lemma 3.3 *For any x, r we have*

$$\mu_L(B(x,r)) \leq \frac{|\{J \in S_L : B(x,r) \cap J \neq \varnothing\}|}{|S_L|} \, g \frac{r}{c},$$

where g depends only on dimension.

Proof We use condition (1) of Proposition 3.1 to bound $\mu_L(B(x,r) \cap J)$ from above. If $J \in S_k$, consider slices of J by parallel hyperplanes perpendicular to the long side of J. Because J is a simplex and simplices are bounded by hyperplanes (with constant slope) there exists a constant $e > 0$ such that for a segment of the long side of length c/e we have that the hyperplanes intersect J in area at least $\frac{1}{2}$ of the maximal area of such a hyperplane. By Fubini's theorem we have

$$\frac{\lambda(B(x,r) \cap J)}{\lambda(J)} \leq 2 \, \frac{2r}{c/e}.$$

Let $g = 4e$. □

Corollary 3.4 *Let $B(x,r) \cap S_{k+L} \subset J \in S_k$. Then*

$$\mu_\infty(B(x,r)) \leq g \, \frac{r}{c} \, |S_k|^{-1} = g \, \frac{r}{c} \left(\prod_{i=1}^{k} \left(\rho 10^{p(i)}\right) \right)^{-1}.$$

If P is a cubic polynomial with leading term $-a/3$, then since h is a cubic polynomial with leading coefficient $-b$, we have that for every $\epsilon > 0$ there exists a C such that

$$10^{P(k)} < C\big(10^{h(k)}\big)^{(a/3b)-\epsilon}. \tag{3-2}$$

We now complete the proof of Proposition 3.1. For each r let $k_r = \max\{L : r < 10^{h(L)}\}$. By condition (3) of Proposition 3.1 there exists a unique $J \in S_{k_r}$ such that $\mu_\infty(B(x,r)) = \mu_\infty(B(x,r) \cap J)$. So by the previous corollary and the stability condition (3-1),

$$\mu_\infty(B(x,r)) \le g\frac{r}{c}|S_{k_r}|^{-1}.$$

By condition (2) of the proposition, this is at most

$$\frac{r}{c}\rho^{k_r-1}10^{-\sum_{i=1}^{k_r-1} p(i)} = \frac{r}{c}\rho^{k_r-1}10^{P(k_r-1)},$$

where $P(x)$ is a cubic polynomial with leading coefficient $-a/3$. It follows by our observation (3-2) above that for every ϵ there exists $\widehat{C}$ such that

$$\mu_\infty(B(x,r)) < \frac{r}{c}\widehat{C}\big(10^{h(k_r-1)}\big)^{(a/3b)-\epsilon} \le \frac{1}{c}\widehat{C}r^{1+(a/3b)-\epsilon}. \qquad \square$$

We also record a technical lemma that will allow us to verify the conditions of the proposition more easily:

Lemma 3.5 *To verify condition (3) of Proposition 3.1 it suffices to show that there exists a cubic polynomial g with leading coefficient $-b$ such that the elements of S_{k+2} avoid a $10^{g(k)}$ neighborhood of the boundary of S_k.*

Proof Let $u \in \mathbb{N}$ be such that $10^{g(k+u)} < \frac{1}{2}10^{g(k)}$. We claim that we can choose $h(k)$ in condition 3 to be $g(k+u)$. If

$$r < 10^{g(k+u)} < \tfrac{1}{2}10^{g(k)}$$

and $B(p,r)$ intersects two elements of S_k then it must lie in a $10^{g(k)}$ neighborhood of the boundary of any of the elements of S_k that it intersects. Thus by our assumption on g it does not intersect any element of S_{k+2}. This establishes that condition (3) holds with $q = 2$ and $h(k) = g(k+u)$. Notice $h(k)$ has leading coefficient $-b$. $\square$

4 The paths we take

Rauzy induction provides a criterion (due to Veech) for non-unique ergodicity that is crucial for our construction.

Lemma 4.1 [14, Section 1 and Proposition 3.22] *Let $T = T_{\lambda,\pi}$ be an IET such that $R^k(T)$ is defined for all T. If T has exactly r ergodic probability measures then*

$$\Delta_\infty(T) := \bigcap_{k=1}^{\infty} M(T,k)\Delta$$

is a subsimplex of dimension $r-1$. The simplex $\Delta_\infty(T)$ is in bijective correspondence with the set of invariant measures for T, as an invariant measure for T is specified by the weights it gives to the basic subintervals $[\sum_{j=1}^{i-1}\lambda_j, \sum_{j=1}^{i}\lambda_j)$ of T.

We will use this criterion to build a large set of IETs T with at least 2 invariant measures. For this, we need to consider some very specific paths. First, define the matrix $L_1(n)$ by going from (4321) to (4132) to (4213) and back to (4321) n times. We have

$$L_1(n) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ n & n & n & 1 \end{pmatrix}.$$

Similarly, define the matrix $U_1(n)$ by going from (4321) to (4132) to (4213) then looping at (4213) n times and then going back to (4321). We have

$$U_1(n) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & n+1 & n \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Given $A \in \mathrm{SL}_2(\mathbb{Z}_+)$, we can write

$$A = H_1^{p_1} H_2^{p_2} H_1^{p_3} \cdots H_1^{p_k} \tag{4-1}$$

for nonnegative integers $p_1, \dots, p_r$, and where

$$H_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Notice the interactions of the 3rd and 4th columns under $L_1(n)$ and $U_1(n)$ are H_1^n and $H_2^n H_1$ respectively. This motivates the definition

$$N_1(A,m) = L_1^{p_1} U_1^{p_2} L_1^{p_3-1} U_1^{p_4} \cdots L_1^{p_k-1} U_1^m.$$

Similarly we define $L_2(n), U_2(n)$ and $N_2(A,m)$ on the left hand side of the Rauzy graph. We will be especially concerned with A and m satisfying

$$|A| \in I_k := [10^{k^2-k}, 2\cdot 10^{k^2-k}] \quad \text{and} \quad m \in J_k := [10^{(k+1)^2+k}, 2\cdot 10^{(k+1)^2+k}].$$

4.0.1 Notation Given a matrix M, let $C_j(M)$ denote the j^{th} column, $|C_j(M)|$ denote the sum of the entries in the i^{th} column. Recall that given a metric d on a space X, we can define a pseudo-metric on subsets of X via

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

We will use this where d is the metric on the simplex Δ induced by angles between vectors. If v, w are vectors let

$$\operatorname{span}_\Delta(v, w) = \{av + bw : a, b \geq 0\} \cap \Delta.$$

In general this can be empty but if $v, w \in \mathbb{R}^4_+$ it will not be. In the section below we will alternately view columns of matrices as elements of the simplex or as vectors with non-negative integer entries when we calculate their size.

4.0.2 A technical lemma

Lemma 4.2 *For any $M \in \mathrm{SL}_2(\mathbb{Z})$ and $m \in \mathbb{N}$ we have*

$$\begin{aligned} C_i(MN_1(A, m)) &= C_i(MN_1(A, 1)) \quad \textit{for } i \in \{1, 2\}, \\ C_i(N_2(A, m)) &= C_i(N_2(A, 1)) \qquad \textit{for } i \in \{3, 4\}. \end{aligned}$$

4.1 Matrices and subsimplices

We consider matrices M_k of the form

$$\begin{aligned} M_k &= M_k\big(\{A_i\}_{i=1}^{2k}, \{m_i\}_{i=1}^{2k+1}\big) \\ &= N_1(A_1, m_2)N_2(A_2, m_3)\cdots N_1(A_{2k-1}, m_{2k})N_2(A_{2k}, m_{2k+1}), \end{aligned} \tag{4-2}$$

where $|A_i| \in I_i$ and $m_i \in J_i$.

4.2 Verifying the axioms

Recall a matrix is called *D–balanced* if $\max_{i,j} |C_i|/|C_j| < D$. In this section we will prove that if M_k has the form given above, where the A_i are all D–balanced ($D > 9$) for all $i \leq 2k + 2$ then there exist:

- A cubic polynomial f with leading coefficient $-\frac{16}{3}$.
- Quadratic polynomials p, q.
- A quadratic polynomial H with leading coefficient 24, satisfying:

(1) (Proposition 4.9) $d\big(C_i(M_k), C_j(M_k)\big) > \frac{1}{900}$ for $i \in \{1, 2\}$ and $j \in \{3, 4\}$.

(2) (Proposition 4.3) $d\big(C_1(M_k), C_2(M_k)\big), d\big(C_3(M_k), C_4(M_k)\big) \in [10^{p(k)}10^{f(k)}, 10^{q(k)}10^{f(k)}]$.

(3) (Section 4.5) There is a $\rho > 0$ such that for each such M_{k-1} there are at least $\rho^k H(k)$ such M_k with the form

$$M_{k-1}N_1(A_{2k-1}, m_{2k})N_2(A_{2k}, m_{2k+1}).$$

Note the connection between conclusions (1) and (3) and conditions (1) and (2) of Proposition 3.1 respectively.

4.3 Angle bounds

This section is devoted to proving:

Proposition 4.3 *Let M_k be as given in Equation (4-2), where all of the A_i are D–balanced. There exist a cubic polynomial f with leading coefficient $-\frac{16}{3}$ and two quadratic polynomials p, q such that*

$$d(C_1(M_k), C_2(M_k)), d(C_3(M_k), C_4(M_k)) \in \big[10^{p(k)}10^{f(k)}, 10^{q(k)}10^{f(k)}\big].$$

We first estimate the sizes of the columns in M_k.

Lemma 4.4 *If M_k is a matrix described above, then*

$$|C_j(M_k)| \in \left[\prod_{i=2}^{2k+1} 10^{i^2}, 2^{2k}\cdot 2\prod_{i=4}^{2k+3} 10^{i^2}\right] \quad \text{for } j = 1, 2,$$

$$|C_j(M_k)| \in \left[\prod_{i=1}^{2k} 10^{i^2}, 2^{2k}\cdot 2\prod_{i=3}^{2k+2} 10^{i^2}\right] \quad \text{for } j = 3, 4.$$

Proof By Lemma 4.2, we have

$$|C_j(MN_1(A,m))| \le |C_j(M)|\cdot(|A|+1)\max\{|C_3(M)|, |C_4(M)|\}$$

for $j = 1, 2$. Similarly,

$$|C_i(MN_1(A,m))| \ge |A| r \min\{|C_3(M)|, |C_4(M)|\}$$

for $j = 3, 4$. Similar inequalities hold for N_2. The lemma follows by induction with the extra factor of 2 absorbing N_1's contribution to C_1, C_2 or N_2's contribution to C_3, C_4. □

Our next lemma describes how the angle between vectors changes under addition.

Lemma 4.5 *Let $v, w \in \mathbb{R}^n$, and let θ_0 denote the angle between v and w. If θ_1 denotes the angle between $v+w$ and w, we have*

$$|\sin\theta_1| = \frac{\|v\|}{\|v+w\|}|\sin\theta_0|.$$

In particular, we have

$$|\sin\theta_1| \le \frac{\|v\|}{\|v+w\|}.$$

Proof If v and w are linearly dependent then both sides are zero. If not, let w' denote the vector w rotated by $\pi/2$ in the plane spanned by v, w. Then

$$|\sin\theta_0| = \frac{|\langle v, w'\rangle|}{\|v\|\|w\|},$$

and

$$|\sin\theta_1| = \frac{|\langle v+w, w'\rangle|}{\|v+w\|\|w\|} = \frac{|\langle v, w'\rangle|}{\|v+w\|\|w\|},$$

proving the result. □

Lemma 4.6 *Let $D > 1$, and suppose $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}_+)$ is D–balanced. Then if θ is the angle between $\left(\begin{smallmatrix} a \\ c\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} b \\ d\end{smallmatrix}\right)$, we have*

$$\frac{1}{D'|A|^2} < |\theta| < \frac{D'}{|A|^2},$$

where D' depends quadratically on D.

Proof We have

$$\sin\theta = \frac{ad-bc}{\left\|\begin{pmatrix} a \\ c\end{pmatrix}\right\|\left\|\begin{pmatrix} b \\ d\end{pmatrix}\right\|} = \frac{1}{\left\|\begin{pmatrix} a \\ c\end{pmatrix}\right\|\left\|\begin{pmatrix} b \\ d\end{pmatrix}\right\|}.$$

By the balancedness condition the norms

$$\left\|\begin{pmatrix} a \\ c\end{pmatrix}\right\|, \left\|\begin{pmatrix} cb \\ d\end{pmatrix}\right\|, |A|$$

are all comparable up to a factor of D and a universal factor ($\sqrt{2}$) from comparison of the L^1 and L^2 norms on $\mathbb{R}^2$. The other factor going into D' is the Lipschitz constant of the arcsin function on $[-\frac{1}{2}, \frac{1}{2}]$. □

Lemma 4.7 *Let $A \in \mathrm{SL}_2(\mathbb{Z}_+)$ be a D–balanced matrix. Then there exists a constant D' depending only on D such that*

$$\frac{d(C_3(M),C_4(M))}{D'|A|^2m^2} \le d(C_3(MN_1(A,m)),C_4(MN_1(A,m))) \le D'\,\frac{d(C_3(M),C_4(M))}{|A|^2m^2}.$$

Proof Observe that $AH_2^mH_1$ is $(D+1)$–balanced and $AH_2^mH_1$ has norm satisfying

$$\frac{m|A|}{D} \le |AH_2^mH_1| \le (m+1)D.$$

By the previous lemma there exists D'' depending only on D such that

$$\frac{d(C_3(M),C_4(M))}{D''|A|^2m^2} \le d(C_3(MN_1(A,0)),C_4(MN_1(A,0))) \le D''\,\frac{d(C_3(M),C_4(M))}{|A|^2m^2}.$$

□

Lemma 4.8 *Let $A \subset \mathrm{SL}_2(\mathbb{Z}_+)$ be D–balanced. Then there exists D' depending only on D such that*

$$\frac{d(C_1(M),C_2(M))}{D'|A|^2r^2} \le d(C_1(MN_2(A,r)),C_2(MN_2(A,r))) \le D'\,\frac{d(C_1(M),C_2(M))}{|A|^2r^2}.$$

This is similar to the proof of Lemma 4.7.

Proof of Proposition 4.3 This follows from the previous two lemmas and induction. Indeed, let M_k and $M_{k+1} = M_kN_1(A_{2k+1},m_{2_k+2})N_2(A_{2k+2},m_{2k+3})$ satisfy our assumptions. By Lemma 4.7,

$$\frac{1}{4D'}\big(10^{-((2k+1)^2+(2k+2)^2)}\big)^2 < \frac{d(C_3(M_{k+1}),C_4(M_{k+1}))}{d(C_3(M_k),C_4(M_k))} < 4D'\big(10^{-((2k+1)^2+(2k+2)^2)}\big)^2.$$

The leading term in the exponent is $-16k^2$. It follows that there exist quadratic polynomials p,q such that

$$10^{p(k+1)}10^{-\sum_{j=1}^{k+1}16j^2} < d(C_3(M_{k+1}),C_4(M_{k+1})) < 10^{q(k+1)}10^{-\sum_{i=1}^{k+1}16j^2}.$$

Since $f(k) = -\sum_{i=1}^{k+1}16j^2$ is a cubic polynomial with leading coefficient $-\frac{16}{3}$, the proposition is proved for C_3,C_4. The argument for C_1,C_2 is similar. □

4.4 Non-unique ergodicity

Proposition 4.9 *Under the assumptions in Proposition 4.3 on A_i and m_i,*

$$\bigcap_{k=1}^{\infty} M_k \Delta$$

is a non-degenerate line segment with length at least $\frac{1}{900}$.

Let

$$U_k = \mathrm{span}_\Delta\{C_1(M_k), C_2(M_k))\} \quad \text{and} \quad V_k = \mathrm{span}_\Delta\{C_3(M_k), C_4(M_k))\}.$$

In the following lemma we repeatedly use Lemma 4.2.

Lemma 4.10 $$d(U_1, V_1) > \tfrac{1}{10}.$$

Proof Let $A, A' \in \mathrm{SL}_2(\mathbb{Z}_+)$ have $|A| \in I_1, |A'| \in I_2$ and $m \in J_2, m' \in J_3$. Recall that

$$\max\{|C_i(N_1(A,m)|\}_{i=1,2} \le 2.$$

Then if

$$u \in \mathrm{span}_\Delta\{C_1(N_1(A,m), C_2(N_1(A,m))\},$$

by Lemma 4.2 the sum of the first and second coordinates of u satisfies

$$u_1 + u_2 \ge \tfrac{1}{3},$$

and if

$$v \in \mathrm{span}_\Delta\{C_3(N_1(A,m), C_4(N_1(A,m))\}$$

then the first and second coordinates of v vanish:

$$v_1 = v_2 = 0.$$

Thus

$$d\big(\mathrm{span}_\Delta\{C_1(N_1(A,m), C_2(N_1(A,m))\}, \mathrm{span}_\Delta\{C_3(N_1(A,m), C_4(N_1(A,m))\}\big) \ge \frac{1}{3\sqrt{2}}.$$

We have

$$\mathrm{span}_\Delta\{C_1(N_1(A,m)N_2(A',m')), C_2(N_1(A,m)N_2(A',m'))\} \subset \mathrm{span}_\Delta\{C_1(N_1(A,m)), C_2(N_1(A,m)))\}.$$

Moreover, if

$$v = aC_3(N_1(A,m)N_2(A',m')) + bC_4(N_1(A,m)N_2(A',m'))\},$$

where $a, b \geq 0$ and $a+b=1$, then $v = v' + u$, where

$$v' \in \mathrm{span}_\Delta\{C_3(N_1(A,m), C_4(N_1(A,m))\}, \quad |v'| \geq \min_{i=3,4} |C_i(N_1(A,m))| \geq 10^4,$$

and by Lemma 4.2,

$$|u| \leq \max_{i=1,2} C_i(N_1(A,m)N_2(A',1))| \leq 2\|A'\| \leq 2\cdot 2\cdot 10^{4-2}.$$

By Lemma 4.5,

$$\begin{aligned}\text{(4-3)} \quad \max\bigl\{d(y, \mathrm{span}_\Delta\{C_3(N_1(A,m), C_4(N_1(A,m))\}) :\\ y \in \mathrm{span}_\Delta\{C_3(N_1(A,m)N_2(A',m')), C_4(N_1(A,m)N_2(A',m'))\}\bigr\} \leq \tfrac{1}{10}.\end{aligned}$$

This completes the proof. □

Lemma 4.11 $d(U_{k+1}, V_{k+1}) > d(U_k, V_k) - 2\cdot \frac{2^{2k}}{10^{2k}}.$

Proof By a similar argument to the second paragraph of the previous lemma,

$$\text{(4-4)} \qquad \max\{d(y, U_k) : y \in U_{k+1}\} \leq \frac{\max_{i=3,4} |C_i(M_k N_1(A_{2i+1}, 1))|}{\min_{i=1,2} |C_i(M_k)|} \leq \frac{2^{2k}}{10^{2k}}.$$

The last inequality uses Lemma 4.4. Similarly,

$$\max\{d(y, V_k) : y \in V_{k+1}\} \leq \frac{2^{2k}}{10^{2k}}.$$

Because

$$\begin{aligned} d(U_{k+1}, V_{k+1}) \\ \geq d(U_k, V_k) - \bigl(\max\{d(y,U_k) : y \in U_{k+1}\} + \max\{d(y, V_k) : y \in V_{k+1}\}\bigr),\end{aligned}$$

the lemma follows. □

Proof of Proposition 4.9 It is straightforward to see that

$$\mathrm{length}\Bigl(\bigcap_{k=1}^{\infty} M_k \Delta\Bigr) \geq \lim_{k\to\infty} d(U_k, V_k).$$

By the previous two lemmas,

$$\lim_{k\to\infty} d(U_k, V_k) \geq \frac{1}{10} - 2\sum_{k=2}^{\infty} \frac{2^{2k}}{10^{2k}} > \frac{1}{900}.$$ □

4.5 Number of described matrices

Proposition 4.12 *There exists a quadratic polynomial H with leading coefficient 24 and a constant $\rho > 0$ such that for each matrix M_{k-1} satisfying the assumptions of Proposition 4.3 there are at least $\rho^k H(k)$ matrices of the form*

$$M_{k-1} N_1(A_{2k-1}, m_{2k}) N_2(A_{2k}, m_{2k+1})$$

satisfying the assumptions of Proposition 4.3.

Lemma 4.13 *There exist a $D > 0$ and a polynomial ϕ of degree 2 and leading coefficient 12 such that set of D–balanced matrices $N_i(A, m)$ satisfying*

- $|A| \in I_{2j+i}$, *and A is D–balanced,*
- $m \in J_{2j+i+1}$,

has cardinality $10^{\phi(j)}$.

Proof There exists a $c > 0$ such that the number of choices for A is at least $c(10^{(2j)^2-2j})^2$, since the number of positive matrices in $\mathrm{SL}_2(\mathbb{Z}_+)$ with norm between R and $2R$ is proportional to R^2, and if $M \in \mathrm{SL}_2(\mathbb{Z}^+)$ then

$$M \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

is 2–balanced. The lemma follows as there are at least $10^{(2j)^2+j}$ choices for m. □

Proof of Proposition 4.12 By applying the previous lemma to N_1 and N_2 we observe that there is a quadratic polynomial $\widehat{\phi}$ with leading coefficient 24 such that for each matrix

$$\begin{aligned} M_k &= M_k\big(\{A_i\}_{i=1}^{2k}, \{m_i\}_{i=1}^{2k+1}\big) \\ &= N_1(A_1, m_2) N_2(A_2, m_3) \cdots N_1(A_{2k-1}, m_{2k}) N_2(A_{2k}, m_{2k+1}), \end{aligned}$$

where

- $|A_i| \in I_i$, A_i is D–balanced,
- $m_i \in J_i$,

there are at least $10^{\widehat{\phi}(k+1)}$ choices of M_{k+1} such that

- $|A_{2k+1}| \in I_{2k+1}, |A_{2k+2}| \in I_{2k+2}$ are both D–balanced,
- $m_{2k+2} \in J_{2k+2}$ and $m_{2k+3} \in J_{2k+3}$. □

5 Measures on subsimplices

In this section we verify that (a slight modification of) the simplices $M_k\Delta$ given in the previous section satisfy the assumptions of Proposition 3.1, thereby proving our result.

Our subsimplices are parallelepipeds contained in the 3–dimensional simplex Δ_3. They have 4 long sides connecting C_i and C_j for $i \in \{1,2\}$ and $j \in \{3,4\}$. There are two short sides: one connecting C_1 to C_2 and another connecting C_3 to C_4.

They satisfy the conditions of Proposition 3.1 except condition (3). When $k = 5$ we insert some additional conditions, in order to achieve the separation we require, which geometrically can be thought of as "chopping off the ends" of the long sides of the parallelepiped:

(1) We delete a 10^{-5} neighborhood of

$$\mathrm{span}_\Delta(C_1(M_5), C_2(M_5)) \quad \text{and} \quad \mathrm{span}_\Delta(C_3(M_5), C_4(M_5)).$$

(2) For each $k \geq 5$, inductively consider

$$M_k N_1(A_{2k+1}, r_{2k+2}) N_2(A_{2k+2}, r_{2k+3})\Delta$$

given as in the previous section. Remove all of these simplices that contain the two A_{2k+2} closest to the end points of $\mathrm{span}_\Delta(C_1(M_k), C_2(M_k))$ and the two A_{2k+1} closest to the end points of $\mathrm{span}_\Delta(C_3(M_k), C_4(M_k))$.

Call the sets remaining after these deletions S_k. We claim that this sequence of sets satisfies condition (3) of Proposition 3.1 for all $k \geq 5$.

Lemma 5.1 *S_k satisfies the assumptions of Lemma 3.5.*

Let M_k be as in Proposition 4.3 and such that

$$M_k\Delta \cap S_k \neq \varnothing,$$

and let

$$M' = M_k N_1(A_{2k+1}, m_{2k+2}) N_2(A_{2k+2}, m_{2k+3})$$

be also as in Proposition 4.3.

Proof Every $x \in M'\Delta \subset M_k\Delta$ has the form

$$x = \frac{\sum_{i=1}^4 a_i C_i(M_k)}{\left|\sum_{i=1}^4 a_i C_i(M_k)\right|}.$$

If x is also in S_{k+1} then by (1) in the construction of S_k and Lemma 4.11,

$$\text{(5-1)} \qquad \min\left\{ \frac{|a_1C_1(M_k)+a_2C_2(M_k)|}{|\sum_{i=1}^4 a_iC_i(M_k)|}, \frac{|a_3C_3(M_k)+a_4C_4(M_k)|}{|\sum_{i=1}^4 a_iC_i(M_k)|} \right\} > \frac{1}{10^6}.$$

Additionally $x = \alpha u + \beta v$, where

$$u \in \operatorname{span}_\Delta\{C_1(M'), C_2(M')\} \quad \text{and} \quad v \in \operatorname{span}_\Delta\{C_3(M'), C_4(M')\}.$$

Let

$$u = \frac{\sum_{i=1}^4 b_iC_i(M_k)}{|\sum_{i=1}^4 b_iC_i(M_k)|} \quad \text{and} \quad v = \frac{\sum_{i=1}^4 c_iC_i(M_k)}{|\sum_{i=1}^4 c_iC_i(M_k)|}.$$

By (2) in the construction of S_k we have

$$\min\{b_1, b_2\} > d(C_1(M), C_2(M))\,\frac{1}{D'|A_{2k+2}|^2},$$
$$\min\{c_3, c_4\} > d(C_3(M), C_4(M))\,\frac{1}{D'|A_{2k+1}|^2}.$$

Combining these equations and Proposition 4.3, if $x \in M'\Delta \cap S_{k+1}$ then for all i

$$a_i \geq \frac{1}{10^6} 10^{\hat{g}(k)},$$

where $\hat{g}$ is a cubic polynomial with leading coefficient $-\frac{16}{3}$. Thus there is a cubic polynomial g with leading coefficient $-\frac{16}{3}$ such that $M'\Delta \cap S_{k+1}$ avoids a $10^{g(k)}$ neighborhood of the boundary of S_k. □

Thus, we may invoke Proposition 3.1 to show the *lower bound*:

Corollary 5.2 *The set of non-uniquely ergodic 4–IETs* NUE(4321) $\subset \Delta$ *satisfies*

$$\operatorname{Hdim}(\mathrm{NUE}(4321)) \geq \tfrac{5}{2}.$$

Proof By Proposition 4.9 and Veech's Lemma 4.1,

$$\mathcal{S}_\infty := \bigcap_{k=1}^{\infty} \mathcal{S}_k \subset \mathrm{NUE}(4321).$$

By Proposition 4.9 the sequence $\mathcal{S}_k$ satisfies condition (1) of Proposition 3.1. By Proposition 4.12 and the fact that condition (2) of the construction of S_k removes at most $4|J_{2k+1}|\cdot|J_{2k+2}|$ of the descendants of each element of S_k, this description satisfies condition (2) of Proposition 3.1. By Lemmas 3.5 and 5.1 this description of S_k satisfies condition (3) of Proposition 3.1. □

6 Bounds for Hausdorff dimension

6.1 Technical lemmas

Before we prove the upper bound for Hausdorff dimension, we require some technical lemmas.

6.1.1 Lines and measures In this subsection, we collect some technical lemmas on lines and measures in Δ and $\mathcal{H}(2)$. First, we state a proposition relating Masur–Veech measure on the stratum $\mathcal{H}(2)$ to the measure class on the *set of line segments* in the simplex Δ_4, which we view as $\Delta_4 \times \mathbb{R}^4$, and endow it with the Lebesgue measure class $m_\ell = m_\Delta \times m$, where m_Δ is the Lebesgue measure class on Δ_4, and m is the Lebesgue measure on $\mathbb{R}^4$.

Recall from Section 2 we have the map $\mathcal{T}: \mathcal{H}(2) \to \Delta_4$, associating the normalized return map to the horizontal transversal to the vertical flow on ω. Let

$$h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

Let $U \subset \mathcal{H}(2)$, and consider the set of lines

$$L(U) = \bigl\{\{\mathcal{T}(h_s\omega)\}_{s\in[0,\epsilon]} : \omega \in U, \epsilon > 0\bigr\}$$

in Δ_4 associated to horocycle trajectories based in U. Recall the description of the space of flat surfaces in Section 2.1. Fixing a horizontal transversal, locally $\mathcal{T}(h_s\omega)$ gives a line segment of IETs with slope given by the heights of the sides of the polygons. Let μ_{MV} denote Masur–Veech measure on the stratum $\mathcal{H}(2)$. It has been a long-standing problem to understand the image of L, that is, to understand the set of line segments in Δ_4 which arise as projections of horocycle trajectories. This problem was solved in full generality by Minsky and Weiss [11, Section 5], who showed [11, Theorem 5.3], that given $(\boldsymbol{a}, \boldsymbol{b}) \in \Delta_4 \times \mathbb{R}^4$, the line segment defined by

$$\{\boldsymbol{a} + s\boldsymbol{b} : s \in [0, 1]\}$$

is in the image of L if and only if a certain quadratic form Q evaluated at $\boldsymbol{a}, \boldsymbol{b}$ is positive, and thus there is an open set of such pairs. Moreover they show that for each pair

$$(\boldsymbol{a}_0, \boldsymbol{b}_0) \quad \text{with} \quad Q(\boldsymbol{a}_0, \boldsymbol{b}_0) > 0,$$

there is an open neighborhood U' of $(\boldsymbol{a}_0, \boldsymbol{b}_0)$ and a local *affine* inverse map to the map L. Thus, we have:

Proposition 6.1 *Let $U \subset \mathcal{H}(2)$ be such that $L(U)$ is open. Suppose $A \subset L(U)$ with $m_\ell(A) > 0$. Then $\mu_{MV}(L^{-1}(A)) > 0$. In particular, if $\mu_{MV}(U) > 0$, then $m_\ell(L(U)) > 0$.*

Second, we require the following:

Lemma 6.2 *If $U \subset \Delta_4 \times \mathbb{R}^4$ is an open set in the space of lines then there exist an open subset V of Δ_4 and an open set of directions $\Theta \subset \mathbb{R}^4$ such that for every $\theta \in \Theta$ and $v \in V$, the line*

$$L(v, \theta) := \{v + t\theta : t \in (0, 1)\}$$

in the direction θ through v is in U.

Proof Let Θ be a set of directions such that $\overline{\Theta}$ is compact and for every $\theta \in \overline{\Theta}$ we have $L(v, \theta) \in U$. For each $\theta \in \overline{\Theta}$ there exists an $\epsilon_\theta > 0$ such that if $d(w, v) < \epsilon_\theta$ we have $L(w, \theta) \in U$. By compactness there exists an $\epsilon > 0$ such that for all $\theta \in \overline{\Theta}$ and w with $d(w, v) < \epsilon$ we have $L(w, \theta) \in U$. Let $V = B(v, \epsilon)$. □

6.1.2 Metric geometry We also require some more general technical lemmas on Hausdorff dimension. Let $\mathcal{H}^t$ be the t–dimensional Hausdorff measure. Let C_t be the Riesz t–capacity (see [10, Definition 8.4] for the definition). Let $\mathrm{Grass}(n, n-m)$ denote the Grassmannian on $n-m$ planes in n space. Let $\gamma_{n,n-m}$ denote the natural measure class on $\mathrm{Grass}(n, n-m)$. If $W \in \mathrm{Grass}(n, n-m)$ and $a \in W^\perp$ let W_a be the translate of W by a.

Theorem 6.3 [10, Theorem 10.8] *Let $m \le t \le n$ and $A \subset \mathbb{R}^n$ with $C_t(A) > 0$ then for $\gamma_{n,n-m}$ almost every $W \in \mathrm{Grass}(n, n-m)$,*

$$\mathcal{H}^m(\{a \in W^\perp : C_{t-m}(A \cap W_a) > 0\}) > 0.$$

By [10, Theorem 8.9(3)], for Borel sets, capacity dimension and Hausdorff dimension are the same so we obtain the following corollary:

Corollary 6.4 *If $A \subset \mathbb{R}^n$ is a Borel set and $\mathrm{Hdim}(A) \ge t > n-1$ then a positive measure set of lines in $\mathbb{R}^n$ intersect A in a set with Hausdorff dimension at least $t-(n-1)$.*

Lemma 6.5 *Let V be a non-empty open subset of Δ_4 and assume that the Hausdorff dimension of the set $\mathrm{NUE}(4321)$ of not uniquely ergodic 4–IETs is at least $3-c$ as a subset of Δ_4. Then*

$$\mathrm{Hdim}(\mathrm{NUE}(4321) \cap V) \ge 3 - c.$$

Proof There exists a matrix of Rauzy induction M such that $M\Delta \subset V$ (pick a matrix so that the subsimplex $M\Delta$ is contained in a ball inside V). Being non-uniquely ergodic is Rauzy induction invariant, so

$$M(\mathrm{NUE}(4321)) \subset \mathrm{NUE}(4321).$$

M is a bilipshitz map. (Note that the various M are not uniformly bilipshitz but each individual one is bilipshitz.) Since

$$M(\mathrm{NUE}(4321)) \subset (\mathrm{NUE}(4321) \cap V),$$

we have

$$\mathrm{Hdim}(\mathrm{NUE}(4321) \cap V) \geq \mathrm{Hdim}(M(\mathrm{NUE}(4321))) = \mathrm{Hdim}(\mathrm{NUE}(4321)) \geq 3-c. \quad \square$$

6.2 Upper bound

In this section we prove:

Theorem 6.6 *The set of minimal and not uniquely ergodic 4–IETs* $\mathrm{NUE}(4321) \subset \Delta_4$ *has Hausdorff dimension at most* $\frac{5}{2}$,

$$\mathrm{Hdim}(\mathrm{NUE}(4321)) \leq \tfrac{5}{2}.$$

This result and Corollary 5.2 establishes Theorem 1.4.

Proposition 6.7 *Let* $c < 1$ *and suppose* $h = \mathrm{Hdim}(\mathrm{NUE}(4321)) > 3 - c$. *Then for* μ_{MV}*–almost every abelian differential* ω,

$$\mathrm{Hdim}(\mathrm{NUE}(\omega)) \geq 1 - c.$$

Before we prove this result, we need to understand how to move between horocycles and rotations. Write

$$r_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, \quad \hat{h}_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \quad \text{and} \quad g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

Lemma 6.8 *Let* $r_\theta = \hat{h}_u g_a h_s$. *Then:*

- *For every* $\epsilon > 0$, *the map* $\phi : (-\frac{\pi}{2} + \epsilon, \frac{\pi}{2} - \epsilon) \to \mathbb{R}$ *defined by*

  $$\phi(\theta) = s, \quad \text{where} \quad r_\theta = \hat{h}_u g_a h_s,$$

  is a bilipshitz map onto its image.
- *The vertical flow on* $h_s\omega$ *is not uniquely ergodic if and only if the vertical on* $r_\theta\omega$ *is not uniquely ergodic.*

Proof The first assertion is a direct computation. For the second assertion, note that because $\widehat{h}_u$ acts isometrically on vertical lines the vertical flow on $\widehat{h}_u g_a h_s \omega$ is isomorphic to the vertical flow on $g_a h_s \omega$. Now if F^t_ω denotes the vertical flow on ω then $F^{e^a t}_\omega$ is isomorphic to $F^t_{g_a \omega}$. Since the non-unique ergodicity of F^t_ω is equivalent to the non-unique ergodicity of the time-scaled flow F^{ct}_ω (for any, and thus for all $c > 0$), the lemma follows. □

Proof of Proposition 6.7 Let U be an open set in the space of lines given by Minsky and Weiss [11, Theorem 5.3]. Let Θ and V be given by Lemma 6.2. By Theorem 6.3 and Lemma 6.5 almost every element of

$$\{(v:\theta) : v \in V, \theta \in \Theta\}$$

has a translate that intersects $V \cap \mathrm{NUE}$ in Hausdorff dimension at least $1-c$. By our assumption (Lemma 6.2) these lines are in U. The set of flat surfaces they come from has positive μ_{MV} measure. By Lemma 6.8 the Hausdorff dimension of non-uniquely ergodic directions on each of these lines (which corresponds to a horocycle orbit) equals the Hausdorff dimension of the non-uniquely ergodic directions on the corresponding flat surface. Because for any $A \in \mathrm{SL}_2(\mathbb{R})$ we have

$$\mathrm{Hdim}(\mathrm{NUE}(\omega)) = \mathrm{Hdim}(\mathrm{NUE}(A\omega)),$$

the fact that μ_{MV} is an ergodic measure for $\mathrm{SL}_2(\mathbb{R})$ implies

$$\mathrm{Hdim}(\mathrm{NUE}(\omega)) \geq 1-c$$

for μ_{MV}–almost every ω. □

Recall Masur's upper bound:

Theorem 6.9 [8, Main theorem] *For every abelian differential ω, the Hausdorff dimension of the set of not uniquely ergodic directions is at most $\frac{1}{2}$.*

Proof of Theorem 6.6 By Masur's upper bound,

$$\mathrm{Hdim}(\mathrm{NUE}(\omega)) \leq \tfrac{1}{2},$$

and thus, by Proposition 6.7, $h \leq \frac{5}{2}$. □

6.3 Corollaries

Finally, we prove Theorems 1.2 and 1.1 using Theorem 1.4. The key result is the fundamental property of Hausdorff dimension:

$$\mathrm{Hdim}(X \times Y) \geq \mathrm{Hdim}(X) + \mathrm{Hdim}(Y).$$

Proof of Theorem 1.2 $\mathcal{H}(2)$ is a fiber bundle over Δ_4 via the map $\mathcal{T}$, and the vertical foliation of ω is non-uniquely ergodic if and only if $\mathcal{T}(\omega)$ is a non-uniquely ergodic IET. The fibers of $\mathcal{T}$ are 4–dimensional and in fact $\mathcal{T}$ defines a local product structure on $\mathcal{H}$. Thus we have that the Hausdorff dimension of the set of ω with a non-uniquely ergodic vertical foliation is given by

$$4 + \tfrac{5}{2} = 7 - \tfrac{1}{2}$$

as claimed. □

Proof of Theorem 1.1 Combining Theorem 6.9 for the upper bound and Corollary 5.2 with Proposition 6.7 for the lower bound, we have that the set of flat surfaces ω satisfying

$$\mathrm{Hdim}(\mathrm{NUE}(\omega)) = \tfrac{1}{2}$$

has positive μ_{MV}–measure. By μ_{MV}–ergodicity of $SL_2(\mathbb{R})$, it must have full μ_{MV}–measure. A similar argument using only the upper bound yields Theorem 1.6. □

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