

Indefinite Morse 2–functions: Broken fibrations and generalizations

DAVID T GAY
ROBION KIRBY

A Morse 2–function is a generic smooth map from a smooth manifold to a surface. In the absence of definite folds (in which case we say that the Morse 2–function is indefinite), these are natural generalizations of broken (Lefschetz) fibrations. We prove existence and uniqueness results for indefinite Morse 2–functions mapping to arbitrary compact, oriented surfaces. “Uniqueness” means there is a set of moves which are sufficient to go between two homotopic indefinite Morse 2–functions while remaining indefinite throughout. We extend the existence and uniqueness results to indefinite, Morse 2–functions with connected fibers.

57M50; 57R17

1 Introduction

A Morse 2–function on a smooth n –manifold X is a generic smooth map from X to a 2–manifold, just as an ordinary Morse function is a generic smooth map to a 1–manifold. The singularities are *folds* and *cusps*. Folds look locally like $(t, x_1, \dots, x_{n-1}) \mapsto (t, f(x_1, \dots, x_{n-1}))$ for a standard Morse singularity f , and cusps look locally like $(t, x_1, \dots, x_{n-1}) \mapsto (t, f_t(x_1, \dots, x_{n-1}))$ for a standard birth f_t of a cancelling pair of Morse singularities.

We develop techniques for working with Morse 2–functions and generic homotopies between them, paying particular attention to (1) avoiding *definite folds*, in which the modeling function f is a definite Morse singularity, ie a local extremum, and (2) guaranteeing connected fibers. When definite folds are avoided, we say that the Morse 2–function (or generic homotopy) is *indefinite*. When fibers are connected, we say that the function or homotopy is *fiber-connected*. The same adjectives also describe Morse functions and their homotopies, when definite singularities (local extrema) are avoided and when level sets are connected.

Let X be a compact, connected, oriented, smooth (C^∞) n –manifold and let Σ be a compact, connected, oriented surface (with possibly empty, possibly disconnected, boundaries). We leave the nonoriented case for others to think about.

Theorem 1.1 (Existence) *Let $g: \partial X \rightarrow \partial \Sigma$ be an indefinite, surjective, Morse function which extends to a map $G': X \rightarrow \Sigma$. If $n > 2$ and $G'_*(\pi_1(X))$ has finite index in $\pi_1(\Sigma)$, then G' is homotopic rel boundary to an indefinite Morse 2–function $G: X \rightarrow \Sigma$. When $n > 3$, if g is fiber-connected and $G'_*(\pi_1(X)) = \pi_1(\Sigma)$ then we can arrange that G is fiber-connected.*

Theorem 1.2 (Uniqueness) *Let $G_0, G_1: X \rightarrow \Sigma$ be indefinite Morse 2–functions which agree on ∂X and are homotopic rel boundary. If $n > 3$ then G_0 and G_1 are homotopic through an indefinite generic homotopy G_s . If in addition G_0 and G_1 are fiber-connected then we can also arrange that G_s is fiber-connected.*

These results are analogs of the following facts in ordinary Morse theory:

Theorem 1.3 (Existence) *A compact, connected m –dimensional cobordism M between $F_0 \neq \emptyset$ and $F_1 \neq \emptyset$ supports an indefinite Morse function $g: M \rightarrow I = [0, 1] = B^1$. If $m > 2$ and F_0 and F_1 are connected then we can also arrange that g is fiber-connected. Any homotopically nontrivial map g' from a closed connected m –manifold M to S^1 is homotopic to an indefinite Morse function $g: M \rightarrow S^1$. If $m > 2$ and $g'_*(\pi_1(M)) = \pi_1(S^1)$ then we can also arrange that g is fiber-connected.*

Theorem 1.4 (Uniqueness) *In both cases above, two homotopic (rel ∂) indefinite Morse functions $g_0, g_1: M \rightarrow N^1$, where $N = B^1$ or $N = S^1$, are homotopic through an indefinite generic homotopy g_s , and if $m > 2$ and g_0 and g_1 are fiber-connected then we can arrange that g_s is fiber-connected for all s .*

Remark 1.5 Because our proofs of the above theorems begin with the case of maps to $I \times I$ or I and are completed with Thom–Pontrjagin-type arguments, all the homotopies constructed can be taken to be homotopic to given homotopies.

The motivation for generalizing Morse functions and Cerf theory [6] from dimension one to dimension two comes originally from the importance of Lefschetz fibrations in complex and symplectic geometry, and new ideas around broken Lefschetz fibrations. After LeBrun [14] and Honda [10; 11] showed that a smooth 4–manifold X^4 with $b_2^+ > 0$ has a *near symplectic form* (a closed 2–form ω with $\omega \wedge \omega \geq 0$ and zero only on a smooth embedded 1–manifold Z in X), Auroux, Donaldson and Katzarkov [3] proved that such 4–manifolds are Lefschetz pencils in the complement of Z , where Z mapped onto latitudes of S^2 . Gay and Kirby [7], Lekili [15], Baykur [4] and Akbulut and Karakurt [1] extended this result to all smooth, oriented, compact 4–manifolds as broken fibrations (not just pencils).

One aim was to define invariants by counting pseudoholomorphic curves in X^4 which limit on Z , as had been done in the symplectic case by Taubes [19] and Usher [21]. To do this, Perutz [16; 17] defined his Lagrangian matching invariants for a broken Lefschetz fibration, but to get invariants of the underlying smooth 4–manifold, one needs *moves* between broken Lefschetz fibrations, preserving connectedness of fibers, under which the Lagrangian matching invariants are preserved. Our uniqueness theorem is intended to provide these moves, and thus provide purely topological definitions of invariants.

Note that maps $X^4 \rightarrow S^2$ are partitioned according to their homotopy class into the elements of the cohomotopy set $\pi^2(X^4)$, calculated homotopically by Taylor [20] and geometrically by Kirby, Melvin and Teichner [13]. It is not clear how this partitioning relates to known invariants, but all elements are realized by indefinite, fiber connected, Morse 2–functions.

Theorem 1.1, in the case of $\Sigma = S^2$ and without fiber-connectedness, is originally due to Saeki [18], who also pointed out that the finiteness of the index $[G'_*(\pi_1(X)) : \pi_1(\Sigma)]$ is a necessary condition. A short proof of existence for closed X to S^2 is sketched by Gay and Kirby in [8]. A significant step forward in the uniqueness case was provided by Lekili [15] when he reintroduced singularity theory into the subject and showed how to go back and forth between Lefschetz singularities and cusps on fold curves.

Of course there is an extensive history behind this paper in the world of singularity theory, which is too long to present, and an extensive history in complex algebraic geometry in the study of honest Lefschetz fibrations. A purely topological precedent lies in the study of round handles; see [2; 5] for example.

Theorem 1.2, when $n = 4$ and without fiber-connectedness, was originally proved by Williams [23]. **Theorem 1.3** is standard, with some of it proved in [18]. The B^1 –valued (cobordism) case of **Theorem 1.4** is an essential ingredient in developing the calculus of framed links for 3–manifolds and thus appears in [12]. It seems that the fiber-connected assertion in the S^1 –valued case of **Theorem 1.4** is a new result, and was originally posed to us as a question by Katrin Wehrheim and Chris Woodward.

If we remove the adjectives “indefinite” and “fiber-connected” from the above theorems then the theorems become simply the facts that Morse functions, Morse 2–functions and generic homotopies between them are, in fact, generic. Although in the above discussion we simply stated that Morse functions, Morse 2–functions and the homotopies we are calling “generic” are actually generic, in fact the definitions we prefer are in terms of local models, and the fact that maps and homotopies with these local models are generic (and in fact stable) is a standard result in singularity theory.

To prove Theorems 1.1 and 1.2, we spend most of our time on the case where $\Sigma = B^2$, the disk, and in fact think of B^2 as the square $I^2 = I \times I$. Here the natural structure on the n -dimensional domain X of a Morse 2-function $G: X \rightarrow I^2$ is that of a cobordism with sides from M_0 to M_1 , where M_0 is an $(n-1)$ -dimensional cobordism from F_{00} to F_{01} and M_1 is an $(n-1)$ -dimensional cobordism from F_{10} to F_{11} , with $F_{00} \cong F_{10}$ and $F_{01} \cong F_{11}$. We ask that this cobordism structure should be mapped to the cobordism structure on I^2 as a cobordism from I to I , and the boundary data comes in the form of I -valued Morse functions on M_0 and M_1 . See Figure 1.

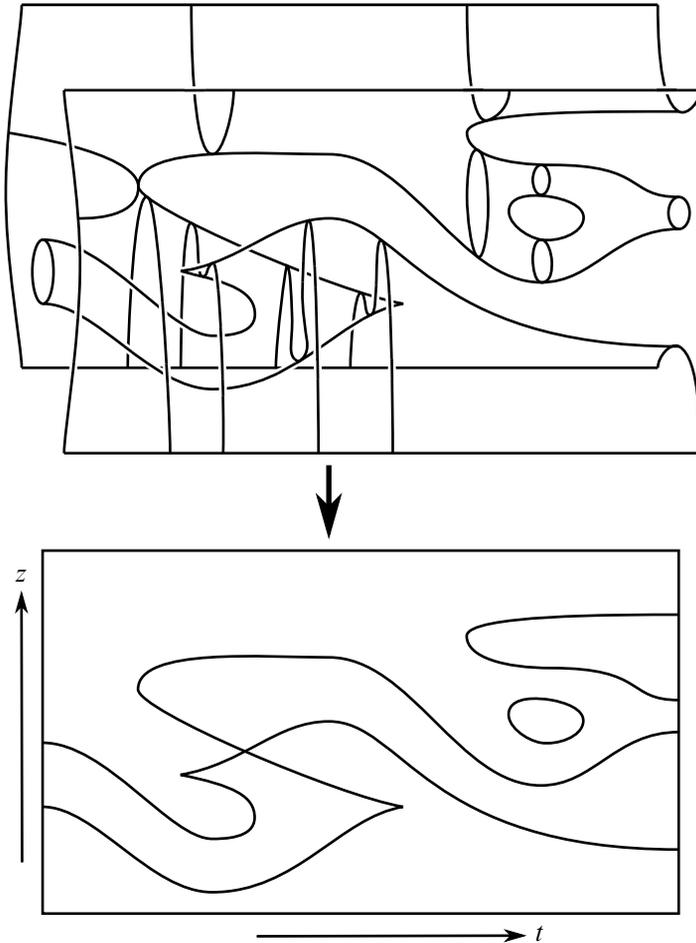


Figure 1: A Morse 2-function on a surface, mapping to the square $I \times I$

Consider the special case where $X = [0, 1] \times M$ and $G(t, p) = (t, g_t(p))$, a generic homotopy between Morse functions $g_0, g_1: M \rightarrow [0, 1]$. Then removing definite folds

from G is the same as removing definite critical points from g_t , and this is done in Section 4 and also in [12]. Fibers remain connected, and therefore existence is done in this special case. For uniqueness, suppose we have a generic 2-parameter family $g_{s,t}$ between $g_{0,t}$ and $g_{1,t}$, giving a generic homotopy $G_s(t, p) = g_{s,t}(p)$. The 2-dimensional definite folds are shown to be removable in Section 4 by use of singularity theory, in particular, use of the codimension-two singularities called the *butterfly* and the *elliptic umbilic*.

For existence in the general case of a cobordism $(X, M_0, M_1) \rightarrow (I \times I, \{0\} \times I, \{1\} \times I)$, choose a Morse function $\tau: X \rightarrow I$ with no definite critical points. We will construct the indefinite Morse 2-function G so that $t \circ G = \tau$, where (t, z) are coordinates on $I \times I$. Choose times t_a and t_b just before and after a critical value of τ . At the critical point there is a standard function in local coordinates giving the descending sphere. Choose a z -valued Morse function ζ_a on the slice $\tau^{-1}(t_a)$ such that the descending sphere lies in a level set $\zeta_a^{-1}(z_a)$, which will mean that the descending sphere lies in the fiber of G over (t_a, z_a) . Away from the local coordinates, we essentially have a product between $\tau^{-1}(t_a)$ and $\tau^{-1}(t_b)$ and the Morse function on $\tau^{-1}(t_a)$ determines a Morse function on $\tau^{-1}(t_b)$.

We finish the existence outline by filling in the gaps between the strips around critical values by choosing Cerf graphics without definite folds; ie appealing to the existence of generic homotopies between ordinary Morse functions without definite critical points. Thus we get images of the folds in $I \times I$ as Figure 2.

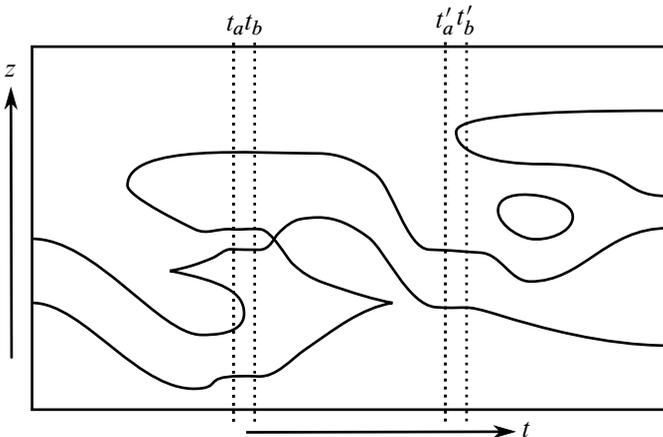


Figure 2: An example illustrating a Cerf graphic in between two critical values of $\tau = t \circ G$

In our proof, we would produce the Morse function ζ_a mentioned above and the Cerf graphic that connects it to an earlier ζ via a very general argument which, in any

particular application, should be replaced by something more explicit. An interesting and illustrative case is that of a (horizontal) Morse function $\tau: X^4 \rightarrow R$, a critical point p of index 2, a given Morse function $\zeta: M_a^3 \rightarrow R$, where M_a is the level set of τ at t_a just below $\tau(p)$, and the attaching circle C of the descending disk from p lying in M_a , but not in a level set of ζ . But we wish it to lie in a level set, for that will be a fiber of the eventual Morse 2–function. In other words, before attaching the handle associated to this critical point, we need to isotope C and construct a Cerf graphic from the given ζ to a new Morse function ζ_a such that C lies in a level set. See Figure 3.

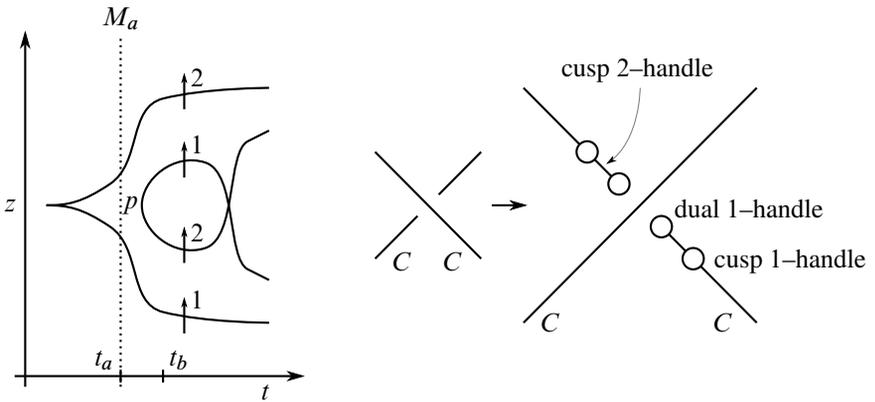


Figure 3: Resolving crossings

First we can isotope C into a region $[-1, 1] \times F$, where F is a level set of M_a with respect to ζ . After generically projecting C into $0 \times F = F$, we get crossings, and these have to be resolved somehow. For each crossing a cusp consisting of a cancelling 1–2–pair of critical points must be created at a lower level of τ , that is, before trying to embed C in F , so that they are available to remove the crossing.

Then to construct ζ on $M_b = \tau^{-1}(t_b)$ near the crossing, we modify the ζ on M_a as follows. The 1–handle is attached first, on either side of the arc of C which is the under crossing. Then a 2–handle is attached along C , with the crossing having been resolved by sending one strand of C over the cusp 1–handle. Next we must add the dual 1–handle of C . This is attached to the 0–sphere bundle (which is the boundary of the normal line bundle to C in F), and this 0–sphere can be placed where convenient. In our case it is as drawn in Figure 3. Finally, the 2–handle of the cusp, which must cancel its 1–handle, goes over each of the 1–handles once, as drawn. This describes how ζ changes locally at each crossing while shifting from M_a to M_b . At each crossing of C in F , a cusp is added and the genus of the fiber F is raised by one.

Returning to the main outline, the same ideas are used for uniqueness in Section 5. We are given two maps $G_0, G_1: (X, M_0, M_1) \rightarrow (I \times I, \{0\} \times I, \{1\} \times I)$ and $\tau_0 = t \circ G_0$ and $\tau_1 = t \circ G_1$ are homotopic indefinite Morse functions. Appealing again to the existence of indefinite homotopies between Morse functions, we get a generic homotopy τ_s from τ_0 to τ_1 with no definite critical points. This homotopy is a sequence of indefinite births, followed by changes of heights of critical values, followed by deaths. We first construct the homotopy G_s for $s \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$ so as to arrange that $t \circ G_s$ realizes the appropriate births and height changes for $s \in [0, \frac{1}{4}]$ and the appropriate deaths for $s \in [\frac{3}{4}, 1]$. For example, a birth is achieved by the introduction of an eye followed by a kink, as in Figure 34 (see Section 5). Then we have arranged that $t \circ G_{1/4} = t \circ G_{3/4}$ and we construct G_s for $s \in [\frac{1}{4}, \frac{3}{4}]$ keeping $t \circ G_s$ fixed. Here we end up appealing again to the existence of indefinite 2–parameter homotopies between ordinary Morse functions, as in Section 4, where now the Morse functions are of the form $z \circ G|_{G^{-1}(\{t\} \times I)}$.

Once the case of image equal to $I \times I$ is done, it is not hard using a zigzag argument to extend the theorems to other surfaces (see Section 6). Extra care with Thom–Pontrjagin-type arguments is needed to keep fibers connected.

Although our motivation comes from the $n = 4$ case, the arguments all generalize in a straightforward manner to all dimensions, save occasionally troubles with dimensions less than or equal to 3. It is hoped that in other dimensions, even 3, the theorems will be useful.

Remark 1.6 In this paper we have assumed X is 0–connected and we have removed definite folds (0– and $(n - 1)$ –folds) when $n \geq 3$, and removed them in 1–parameter families when $n \geq 4$. One could speculate that, if X is 1–connected, we could remove 0–, 1–, $(n - 2)$ – and $(n - 1)$ –folds for $n \geq 5$ (existence) or $n \geq 6$ (uniqueness). In particular, simply connected 5–manifolds would have only 2–folds and no cusps. This speculation would further generalize to k –connected n –manifolds with $n \geq 2k + 3$ (existence) or $n \geq 2k + 4$ (uniqueness).

Acknowledgements We would like to thank Denis Auroux, İnanç Baykur, Michael Freedman, Yankı Lekili, Kevin Walker, Katrin Wehrheim and Jonathan Williams for helpful discussions during the preparation of this paper. We are also extremely grateful for the care and time which our anonymous referee put into the paper, which has led to greatly improved exposition and eliminated some significant mistakes.

This work was partially supported by a grant from the Simons Foundation (number 210381 to David Gay), by South African National Research Foundation Focus Area grant FA2007042500033 and United States National Science Foundation grants EMSW21-RTG and DMS-1207721.

2 Definitions and basic results

We begin with the usual definition of Morse functions in terms of local models, as a warm-up to the succeeding definitions, and also add a few slightly less standard terms to this setting.

Definition 2.1 The *standard index- k Morse model in dimension m* is the function $\mu_k^m(x_1, \dots, x_m) = -x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_m^2$. When the ambient dimension m is understood we will write μ_k instead of μ_k^m . We will also abbreviate $\mu_k(x_1, \dots, x_m)$ as $\mu_k(\mathbf{x})$.

Definition 2.2 Given an m -manifold M and an oriented 1-manifold N , a smooth function $g: M \rightarrow N$ is *locally Morse* if there exist coordinates in a neighborhood of each critical point p together with coordinates in a neighborhood of $g(p)$ with respect to which $g(x_1, \dots, x_m) = \mu_k(\mathbf{x})$, where k is the *index* of p . A *Morse function* is a proper map $g: M \rightarrow N$ which is locally Morse with the additional property that distinct critical points map to distinct critical values. When g is a Morse function from M to $I = [0, 1]$ we imply that M is given as a cobordism from F_0 to F_1 and that $g^{-1}(0) = F_0$ and $g^{-1}(1) = F_1$.

It is a standard fact that Morse functions are stable and generic. Next we will discuss homotopies and homotopies of homotopies between Morse functions, and also make similar statements that homotopies satisfying certain properties are stable and generic. These facts are only slightly less standard, and are discussed in many different references on singularity theory and Cerf theory. Probably the most comprehensive reference for the facts we mention is Hatcher and Wagoner [9]. To see these results in the more general context of singularity theory, look at Wassermann [22]. For a modern exposition explicitly in a low-dimensional setting, which also explains much of the motivation for this paper, we recommend [15].

We want to discuss homotopies $g_t: M \rightarrow N$ between Morse functions which are not necessarily Morse at intermediate times, in which case it is useful to discuss also the associated function $G: I \times M \rightarrow I \times N$ defined by $G(t, p) = (t, g_t(p))$, and its singular locus Z_G , the trajectory of the critical points of g_t . Given an m -manifold M and a 1-manifold N with two Morse functions $g_0, g_1: M \rightarrow N$, we are interested in homotopies $g_t: M \rightarrow N$ satisfying the following properties. The functions g_t should be Morse for all but finitely many values of t and, at those values t_* when g_{t_*} is not Morse exactly one of the following events should occur, possibly with the t parameter reversed (Figure 4 illustrates these by drawing the Cerf graphic $G(Z_G)$ for a typical generic homotopy):

Definition 2.4 An arc of Morse functions is a homotopy g_t which is Morse for all t .

Next we discuss homotopies $g_{s,t}: M \rightarrow N$ between generic homotopies between Morse functions, which are not necessarily generic homotopies for certain fixed values of s . In this case it is useful to consider the associated functions $G_s: I \times M \rightarrow I \times N$ defined by $(t, p) \mapsto (t, g_{s,t}(p))$ and $\mathcal{G}: I \times I \times M \rightarrow I \times I \times N$ defined by $(s, t, p) \mapsto (s, t, g_{s,t}(p))$, and their singular loci $Z_{G_s} \subset I \times M$ and $Z_{\mathcal{G}} \subset I \times I \times M$. Given an m -manifold M and a 1 -manifold N , with one generic homotopy $g_{0,t}: M \rightarrow N$ between Morse functions $g_{0,0}$ and $g_{0,1}$ and another generic homotopy $g_{1,t}: M \rightarrow N$ between Morse functions $g_{1,0}$ and $g_{1,1}$, we are interested in connecting these through a 2 -parameter family $g_{s,t}: M \rightarrow N$, with $s, t \in I$, satisfying the following conditions.

- (1) $g_{s,0}$ is an arc of Morse functions from $g_{0,0}$ to $g_{1,0}$ and $g_{s,1}$ is an arc of Morse functions from $g_{0,1}$ to $g_{1,1}$.
- (2) For all but finitely many fixed values of s , $g_{s,t}$ is, in the parameter t , a generic homotopy between the Morse functions $g_{s,0}$ and $g_{s,1}$.
- (3) At those values s_* when $g_{s_*,t}$ is not a generic homotopy there is a single value t_* such that $g_{s_*,t}$ is a generic homotopy for $t \in [0, t_*)$ and for $t \in (t_*, 1]$.
- (4) At each of these points $(s_*, t_*) \in I \times I$ exactly one of the following events occurs, possibly with either the s or t parameter reversed (some of which are illustrated in figures below by drawing sequences of Cerf graphics $G_s(Z_{G_s})$).

- (a) (This event is not particularly important to us but we list it for completeness.) The function g_{s_*,t_* is locally Morse (or has a birth or death in the parameter t) but the 1 -parameter family $g_{s_*,t}$ does not meet the requirements to be a generic homotopy because exactly two of the events listed in [Definition 2.3](#) occur simultaneously at $t = t_*$. For example, a birth singularity may happen at the same time t_* as a crossing. This phenomenon should be transverse in the obvious sense; for example, for $s < s_*$, the birth might happen before the crossing, and for $s > s_*$, the birth would then happen after the crossing. We call this event a *2-parameter coincidence*.
- (b) The function g_{s_*,t_* is locally Morse but the 1 -parameter family $g_{s_*,t}$ does not meet the requirements to be a generic homotopy because the singular locus $Z_{G_{s_*}} \cap ([t_* - \epsilon, t_* + \epsilon] \times M)$ is mapped into $I \times N$ via G_{s_*} with a single nontransverse quadratic double point at $t = t_*$. However, we require here that the singular locus $Z_{\mathcal{G}} \cap ([t_* - \epsilon, t_* + \epsilon] \times [s_* - \epsilon, s_* + \epsilon] \times M)$ is a collection of disjoint squares and is mapped into $I \times I \times N$ via \mathcal{G} with a single arc of transverse double points. In other words, the image of Z_{G_s} in $I \times N$ changes via a Reidemeister-II type move at $s = s_*$. See [Figure 5](#); we call this event a *Reidemeister-II fold crossing*.

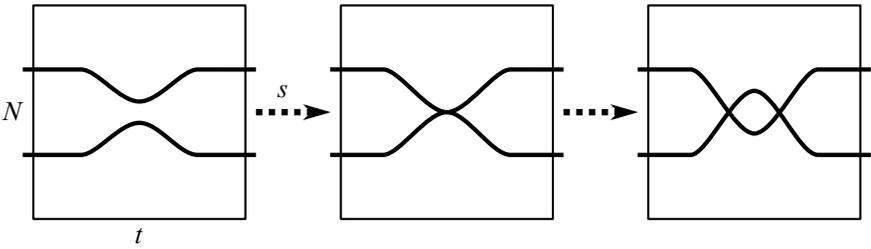


Figure 5: Nontransverse double point in the singular locus for a generic homotopy between generic homotopies between Morse functions: note that in general the indices of the two critical points involved can be arbitrary.

- (c) The function g_{s_*,t_*} is locally Morse but the 1-parameter family $g_{s_*,t}$ does not meet the requirements to be a generic homotopy because the singular locus $Z_{G_{s_*}} \cap ([t_* - \epsilon, t_* + \epsilon] \times M)$ is mapped into $I \times N$ via G_{s_*} with a single transverse triple point. However, we require here that the singular locus $Z_{\mathcal{G}} \cap ([t_* - \epsilon, t_* + \epsilon] \times [s_* - \epsilon, s_* + \epsilon] \times M)$ is a collection of disjoint squares and is mapped into $I \times I \times N$ via \mathcal{G} with three arcs of double points which meet transversely at the triple point. In other words, the image of Z_{G_s} in $I \times N$ is modified via a Reidemeister-III type move. See Figure 6; we call this event a *Reidemeister-III fold crossing*.

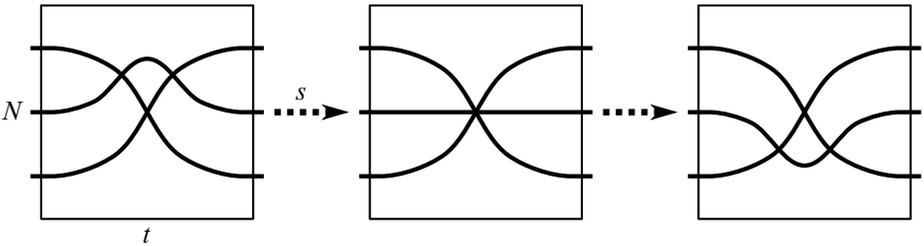


Figure 6: Transverse triple point in the singular locus for a generic homotopy between generic homotopies between Morse functions: again, the indices involved can be arbitrary.

- (d) The 1-parameter family $g_{s_*,t}$ fails to be a generic homotopy because a birth (or death) occurs at time t_* at a point $p \in M$ at the same value as another Morse critical point q ; $g_{s_*,t_*}(p) = g_{s_*,t_*}(q)$. In other words, G_{s_*} maps $Z_{G_{s_*}}$ into $I \times N$ in such a way that a nontransverse double point occurs between a cusp and a noncusp point. However, here we require that the 1-dimensional cusp locus $C_{\mathcal{G}} \subset I \times I \times M$ and the 2-dimensional singular locus $Z_{\mathcal{G}} \subset I \times I \times M$ are mapped into $I \times I \times N$ via \mathcal{G} with a transverse intersection at this point. See Figure 7; we call this event a *cusp-fold crossing*.

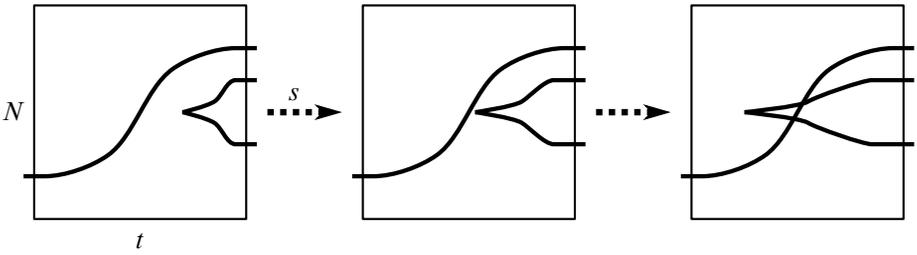


Figure 7: Nontransverse double point involving a cusp, occurring in a generic homotopy between generic homotopies between Morse functions: the only constraint on indices is that coming from the cusp, namely that the two critical points born at the cusp are of successive index.

- (e) The function g_{s_*,t_*} is Morse away from a point $p \in M$, and in neighborhoods of p and $g_{s_*,t_*}(p)$ we have coordinates with respect to which, for $|s - s_*| < \delta$ and $|t - t_*| < \epsilon$, $g_{s,t}$ is given by $g_{s,t}(x_1, \dots, x_m) = -x_1^2 - \dots - x_k^2 + x_{k+1}^3 + (t - t_*)^2 x_{k+1} - (s - s_*)x_{k+1} + x_{k+2}^2 + \dots + x_m^2$. Furthermore, for these (s, t) there are no other singularities of $g_{s,t}$ in the inverse image of a small neighborhood of $g_{s_*,t_*}(p)$. Geometrically, this is the birth of a pair of cusps joined in an “eye” shape, involving a birth and a death of a pair of cancelling critical points. See Figure 8; we call this event an *eye birth singularity* (or *death* when s is reversed).

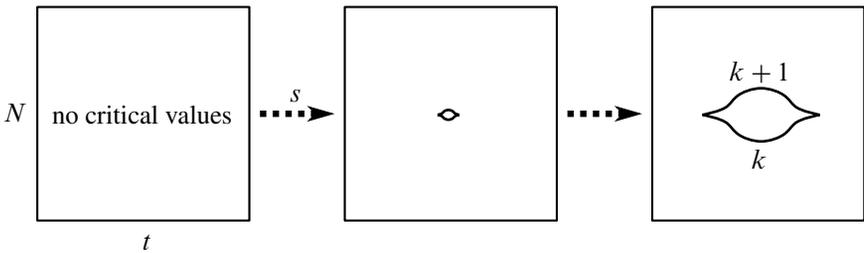


Figure 8: The birth of a birth–death pair (eye birth) involving a pair of cancelling critical points of successive index

- (f) The function g_{s_*,t_*} is Morse away from a point $p \in M$, and in neighborhoods of p and $g_{s_*,t_*}(p)$ we have coordinates with respect to which, for $|s - s_*| < \delta$ and $|t - t_*| < \epsilon$, $g_{s,t}$ is given by $g_{s,t}(x_1, \dots, x_m) = -x_1^2 - \dots - x_k^2 + x_{k+1}^3 - (t - t_*)^2 x_{k+1} - (s - s_*)x_{k+1} + x_{k+2}^2 + \dots + x_m^2$. Furthermore, for these (s, t) there are no other singularities of $g_{s,t}$ in the inverse image of a small neighborhood of $g_{s_*,t_*}(p)$. Here a death and a birth of a cancelling pair merge together, so that afterwards there is no cancellation. See Figure 9; we call this event a *merge singularity* (or *unmerge* when s is reversed).

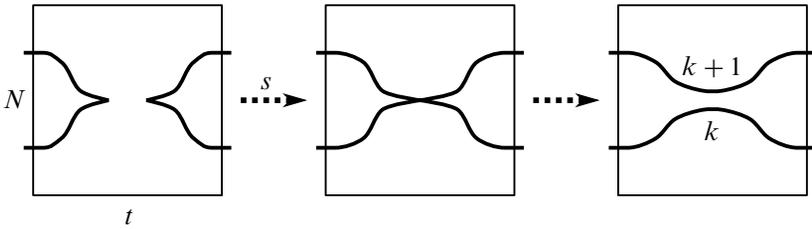


Figure 9: The merge of a death–birth pair involving a pair of cancelling critical points of successive index

- (g) The function g_{s_*,t_*} is Morse away from a point $p \in M$, and in neighborhoods of p and $g_{s_*,t_*}(p)$ we have coordinates with respect to which, for $|s - s_*| < \delta$ and $|t - t_*| < \epsilon$, $g_{s,t}$ is given by $g_{s,t}(x_1, \dots, x_m) = -x_1^2 - \dots - x_k^2 + x_{k+1}^4 - (s - s_*)x_{k+1}^2 + (t - t_*)x_{k+1} + x_{k+2}^2 + \dots + x_m^2$. Furthermore, for these (s, t) there are no other singularities of $g_{s,t}$ in the inverse image of a small neighborhood of $g_{s_*,t_*}(p)$. This singularity is known as a swallowtail. See Figure 10; we call this event a *swallowtail birth singularity* (or *death* when s is reversed).

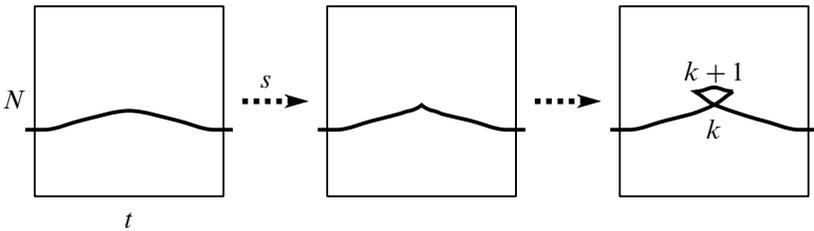


Figure 10: Birth of a swallowtail; an upside down version also occurs

Note that, besides the coincidence event, we have two types of events: *2-parameter crossings* (Reidemeister-II’s, Reidemeister-III’s and cusp-folds) and *2-parameter singularities* (eye births and deaths, merges and unmerges, and swallowtail births and deaths). (As a technical point, note also that, in the definitions of the 2-parameter singularities, the coordinates in which the homotopy of homotopies takes on the standard models may vary with s and t , and also the parametrization of t may depend on s .)

It is also standard that such homotopies of homotopies are generic and stable, and so for this reason:

Definition 2.5 A homotopy $g_{s,t}$ between generic homotopies $g_{0,t}$ and $g_{1,t}$ is a *generic homotopy of homotopies* if it satisfies the properties described above. If $g_{0,0} = g_{1,0}$ and $g_{0,1} = g_{1,1}$, we can also ask that $g_{s,0} = g_{0,0}$ and $g_{s,1} = g_{0,1}$ for all s , in which case we say that $g_{s,t}$ is a *generic homotopy with fixed endpoints*.

Again, we distinguish this from the following:

Definition 2.6 An *arc of generic homotopies* is a homotopy of homotopies $g_{s,t}$ which, for each fixed value of s , is a generic homotopy in the parameter t .

Definition 2.7 Given an n -manifold X and a 2-manifold Σ , a smooth proper map $G: X \rightarrow \Sigma$ is a *Morse 2-function* if for each $q \in \Sigma$ there is a compact neighborhood S of q with a diffeomorphism $\psi: S \rightarrow I \times I$ and a diffeomorphism $\phi: G^{-1}(S) \rightarrow I \times M$, for an $(n-1)$ -manifold M , such that $\psi \circ G \circ \phi^{-1}: I \times M \rightarrow I \times I$ is of the form $(t, p) \mapsto (t, g_t(p))$ for some generic homotopy between Morse functions $g_t: M \rightarrow I$. A singular point for G is called a *fold point* if the homotopy used to model G at that point can actually be taken to be Morse, and is called a *cusp point* if the homotopy has a birth or death at that point. An arc of fold points is called a *fold*. When Σ is given as a cobordism between 1-manifolds N_0 and N_1 then X should be given as a cobordism between $(n-1)$ -manifolds M_0 and M_1 , with $G^{-1}(N_i) = M_i$ and with $G|_{M_i}: M_i \rightarrow N_i$ a Morse function. When Σ is given as a cobordism between cobordisms (in particular, when $\Sigma = I^2$, a cobordism from I to I , with I being a cobordism from $\{0\}$ to $\{1\}$), then X should also be given as such a relative cobordism, with all the cobordism structure preserved by G . For us the structure of a relative cobordism includes an explicit product structure on the sides, and this should also be respected by G . In particular, there should be no critical points along the side of the cobordism.

Remark 2.8 The important thing to understand here is that Morse 2-functions look locally like generic homotopies between Morse functions, but that there is no global time direction. Note that the *index* of a fold is not well defined, but that if we choose a transverse direction to the fold, and consider local models $(t, p) \mapsto (t, g_t(p))$ in which the second coordinate in the range is given by this transverse direction, then we do have a well-defined index. In figures, we will indicate this by drawing a small arrow transverse to the fold and labeling it with the index. If, however, we are drawing a Cerf graphic, then it is understood that the transverse direction is up, and we will label folds (arcs of critical points) with indices without indicating the arrow. We illustrate these conventions in [Figure 11](#), which show the images of the singular locus for, on the left, a hypothetical Morse 2-function mapping to a genus-2 surface and, on the right, a generic homotopy between Morse functions.

Definition 2.9 A 1-parameter family $G_s: X \rightarrow \Sigma$ is a *generic homotopy between Morse 2-functions* if, for each $q \in \Sigma$ and each $s_* \in I$ there is an $\epsilon > 0$ and a compact neighborhood S of q with a diffeomorphism $\psi: S \rightarrow I \times I$ and a 1-parameter family

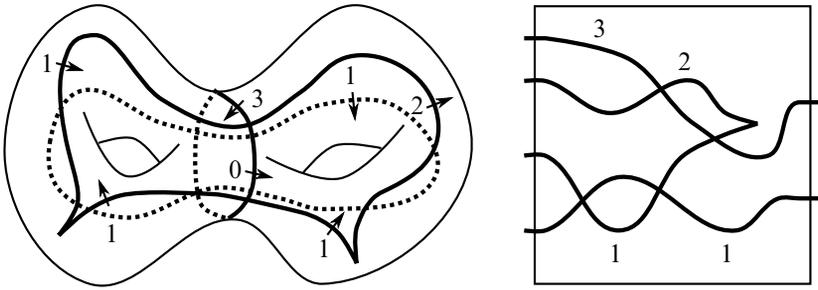


Figure 11: Morse 2–functions versus Cerf graphics, and index labeling conventions: on the left, we are mapping from a 4–manifold to a genus-2 surface; on the right, we are illustrating a generic homotopy between I –valued Morse functions on a 3–manifold.

of diffeomorphisms $\phi_s: G_s^{-1}(S) \rightarrow I \times M$, for an $(n - 1)$ –manifold M and for $|s - s_*| < \epsilon$, such that $\psi \circ G_s \circ \phi_s^{-1}: I \times M \rightarrow I \times I$ is of the form $(t, p) \mapsto (t, g_{s,t}(p))$ for some generic homotopy of homotopies $g_{s,t}: M \rightarrow I$. Generic homotopies of Morse 2–functions $G_s: X \rightarrow \Sigma$ are expected to be constant (independent of s) on ∂X .

Again, although our terminology is not standard, it is a standard fact that Morse 2–functions and generic homotopies of Morse 2–functions are stable and generic. This is mostly explained in Section 4 and [15, Appendix A].

We will be interested, for most of this paper, in the special case of Morse 2–functions mapping to I^2 , seen as a cobordism from $\{0\} \times I$ to $\{1\} \times I$. We use coordinates (t, z) on I^2 , ie t is the horizontal axis. Here it is useful to impose one extra genericity condition:

Definition 2.10 Suppose X^n is a cobordism from M_0 to M_1 , with each M_i a cobordism from F_{i0} to F_{i1} . A *square Morse 2–function* on X is a Morse 2–function $G: X \rightarrow I^2$, respecting the cobordism structure, with no critical values in $I \times \{0, 1\}$, such that the projection onto the horizontal axis, $t \circ G: X \rightarrow I$, is itself a Morse function. In particular, there is a parametrization of the sides $G^{-1}(I \times \{0, 1\})$ as $I \times (F_{00} \amalg F_{10})$ with respect to which the horizontal projection $t \circ G$ restricts as projection to I . Homotopies between square Morse 2–functions are assumed to maintain this last condition.

It is not hard to see that, amongst Morse 2–functions mapping to I^2 respecting the cobordism structures on domain and range, square Morse 2–functions are generic and stable.

Definition 2.11 A Morse function is *indefinite* if there are no critical points of minimal or maximal index, ie no critical points with the local model $(x_1, \dots, x_m) \mapsto \pm(x_1^2 + \dots + x_m^2)$. A generic homotopy, or generic homotopy of homotopies, of Morse functions is *indefinite* if it is indefinite at all parameter values at which it is Morse. A Morse 2–function, resp. generic homotopy of Morse 2–functions, is *indefinite* if it can always be locally modeled, as in the definition, by an indefinite generic homotopy, resp. generic homotopy of homotopies.

The following definition will be useful when we want to make assertions about the connectedness of fibers:

Definition 2.12 A Morse function $g: M \rightarrow I$ is *ordered* if, given two critical points $p, q \in M$ with indices i, j , respectively, if $i < j$ then $g(p) < g(q)$. A generic homotopy or generic homotopy of homotopies is ordered if it is ordered at all parameter values at which it is Morse. A Morse function (or homotopy or homotopy of homotopies) is *almost ordered* if, whenever $i < j - 1$, we have $g(p) < g(q)$.

(Note that it is not clear how to generalize this definition to Morse 2–functions.) We leave the proof of the following observation to the reader:

Lemma 2.13 Consider a Morse function $g: M^m \rightarrow I$, with M a cobordism from F_0 to F_1 , and with F_0 and F_1 both connected. If $m \geq 3$ and g is indefinite and ordered then all level sets of g will be connected. If $m \geq 4$ and g is indefinite and almost ordered then the level sets will all be connected.

3 An extended example

An important example in dimension 4, first described in [3, Section 8.2], concerns Morse 2–functions over S^2 with some fibers being torus fibers. Because $\text{Diff}(S^1 \times S^1)$ is not simply connected, a neighborhood $B^2 \times S^1 \times S^1$ of a torus fiber can be removed and glued back in via a nontrivial loop in $\text{Diff}(S^1 \times S^1)$, ie by performing a logarithmic transform on the fiber. (See also [5].) Thus we may change the 4–manifold without changing the data of folds, fibers and attaching maps on S^2 . The example in [3] involves, in particular, an indefinite Morse 2–function $S^4 \rightarrow S^2$ which is homotopically trivial and can be obtained from a definite Morse 2–function $S^4 \rightarrow B^4 \rightarrow B^2 \hookrightarrow S^2$ by flipping a circle of 0–folds to a circle of 1–folds as illustrated in Figure 16. In this section, we show in detail how this happens in dimension 3, in which case the fiber in question is $S^0 \times S^1$, we show how the nontrivial loop arises, and explain that the example generalizes to arbitrary dimensions $n \geq 3$, with fiber $S^{n-3} \times S^1$. This example

also illustrates the important ideas, used throughout the paper, associated with thinking of a disk B^2 in the base as $I \times I$.

We begin with the following simple example of a Morse 2-function $G: S^1 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Using cartesian coordinates on \mathbb{R}^2 in $S^1 \times \mathbb{R}^2$ and polar coordinates on the range \mathbb{R}^2 , G is defined by $G(\theta, x_1, x_2) = (\frac{1}{2} + (x_1^2 + x_2^2)/2, \theta)$. The singular set is a single circle of definite folds at $S^1 \times \{(0, 0)\}$ and is embedded via G into \mathbb{R}^2 as the circle of radius $\frac{1}{2}$. [Figure 12](#) illustrates this map by showing the image of the fold locus as a dark circle, with paraboloid fibers over rays emanating from the origin, showing clearly that the total space is $S^1 \times \mathbb{R}^2$.

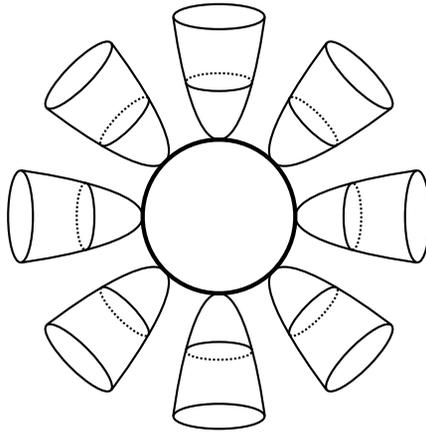


Figure 12: A Morse 2-function $G: S^1 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with a single definite fold circle

Now let B be the square $[-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ and let $X = G^{-1}(B)$. Then X is a solid torus seen as a cobordism from $G^{-1}(\{-1\} \times [-1, 1])$ to $G^{-1}(\{1\} \times [-1, 1])$, both of which are diffeomorphic to $[-1, 1] \times S^1$, and G is a Morse 2-function from X to B . The first row in [Figure 13](#) illustrates “vertical slices” of this map, ie the inverse images of vertical line segments $\{t\} \times [-1, 1]$; the reader should take a moment to reconcile this with [Figure 12](#), which shows the inverse images of rays from the origin. [Figures 13](#) and [14](#) then illustrate a homotopy G_s beginning with the map $G_0 = G$ described above. Each row of surfaces illustrates G_s for a fixed s , beginning with G_0 in the top row of [Figure 13](#). We have chosen six representative values of s , hence six rows. ([Figure 15](#) enlarges two regions in [Figure 14](#) just to illustrate the detail there carefully.) In each row (corresponding to a fixed value of s), the surfaces illustrated are each of the form $M_{s,t} = G_s^{-1}(\{t\} \times [-1, 1])$, for nine representative values of t ranging from -1 to 1 . Each surface $M_{s,t}$ is drawn embedded in \mathbb{R}^3 ; each embedding is such that the function $G_s|_{M_{s,t}}: M_{s,t} \rightarrow \{t\} \times [-1, 1]$ is the height function, projection to the

z -axis. Thus the critical locus of each G_s can be seen as the trace of the critical points of each $G_s|_{M_{s,t}}$ as t ranges from -1 to 1 .

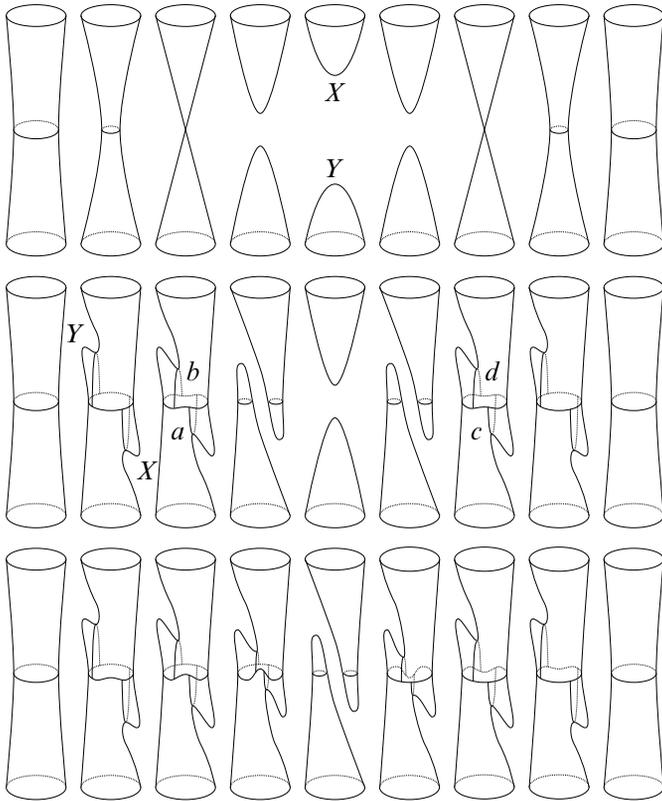


Figure 13: The first half of a 1-parameter family of Morse 2-functions on $S^1 \times B^2$: critical points are labeled for coordination with following figures. From the second row to the third row, we have broken the symmetry of the mid-level circle to prepare for the next move, going to the first row of Figure 14. Looking at the three middle surfaces in the bottom row, we see, from left to right, a 3-dimensional 2-handle attached, surgering the mid-level circle to two circles, followed by a 3-dimensional 1-handle which reattaches the two circles.

We can extract from these figures two related sequences of diagrams. The first, Figure 16, illustrates the images of the singular loci in the base $[-1, 1] \times [-1, 1]$ for G_s for each of the six values of s in Figures 13 and 14. The second, Figure 17, indicates how the respective ascending and descending manifolds of the critical points, for the vertical height functions $G_s|_{M_{s,t}}$, intersect the middle-level 1-manifold in each surface $M_{s,t}$

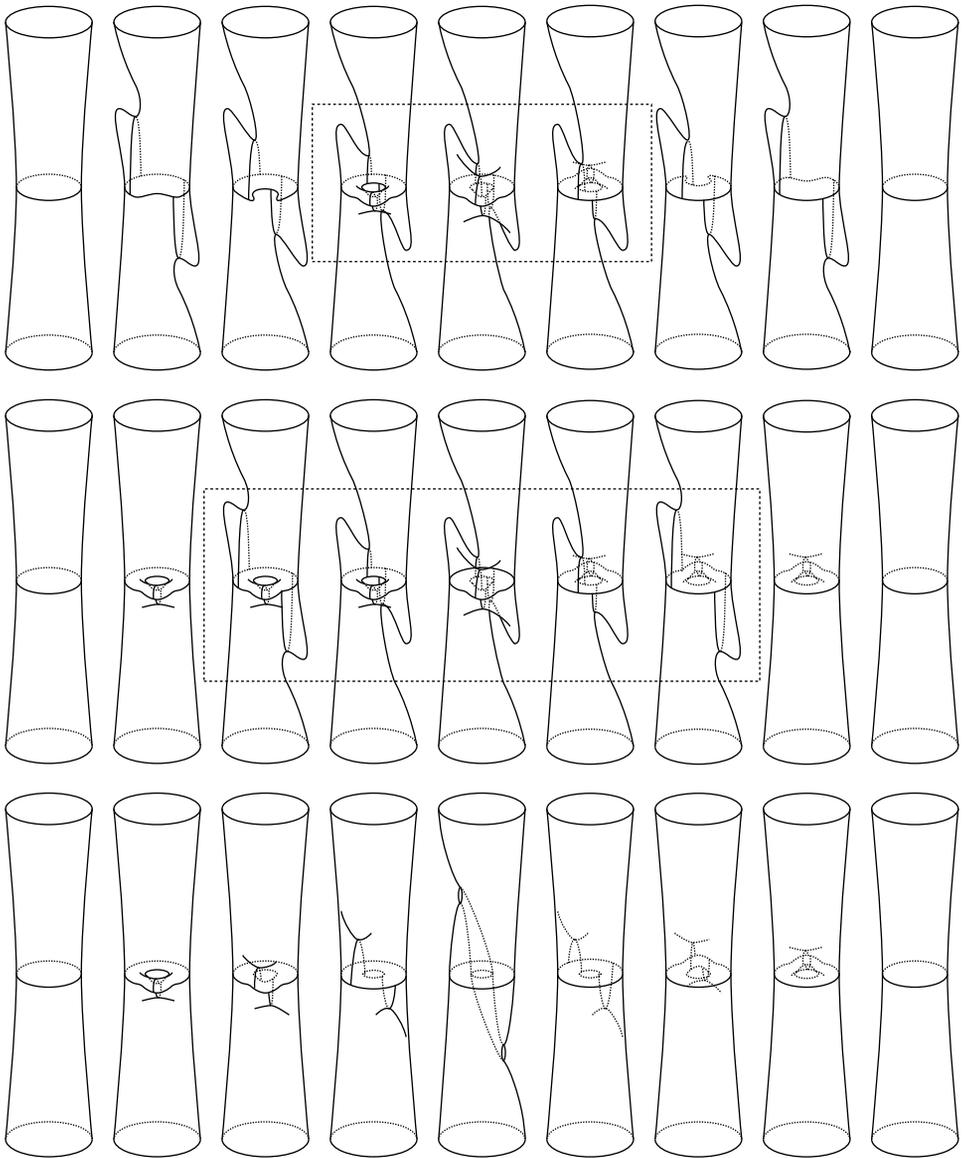


Figure 14: The second half of a 1–parameter family of Morse 2–functions on $S^1 \times B^2$, with definite folds at the beginning and indefinite folds at the end: the two regions enclosed in boxes are shown enlarged in Figure 15. The 3–dimensional 1–handle mentioned in the caption for Figure 13 is seen in the first row between the third and fourth surfaces, while the 2–handle is seen in the first row between the sixth and seventh surface.

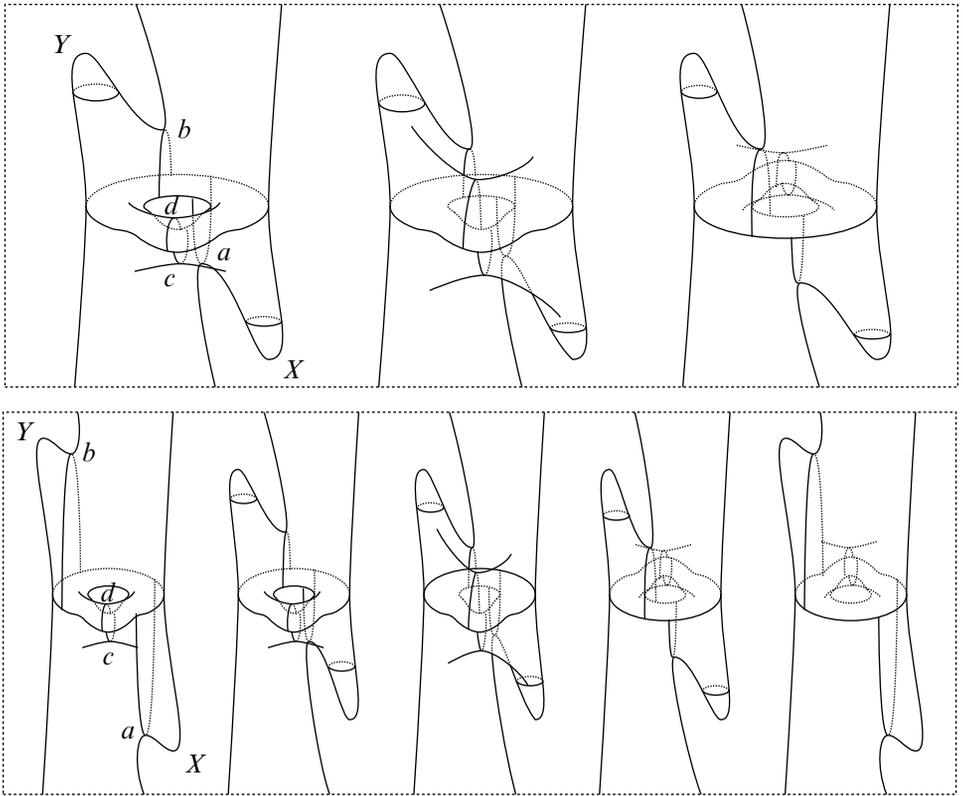


Figure 15: Zooming in on two regions of Figure 14: critical points are labeled corresponding to labels in Figure 13 and Figure 16. An important point to note here is that, in the second row, going from the first surface to the second, we see two handle slides: b slides over d and a slides under c . And, of course, we have the symmetric slides at the other end of that row.

from Figures 13 and 14. Here we have only drawn the diagrams for the final five values of s and for the middle five values of t .

There are several key features to note here.

- (1) For each Morse 2–function G_s , we can consider the function $t \circ G_s$, where $t: \mathbb{R}^2 \rightarrow \mathbb{R}$ is projection to the horizontal axis. In our examples, this “horizontal” function is in fact (locally) an ordinary Morse function, which we call the “horizontal Morse function” associated to G_s . The critical points of $t \circ G_s$ occur at precisely the vertical tangencies of the singular loci, as illustrated in Figure 16. These critical points should not be confused with the (2–dimensional) critical points of the vertical Morse function $z \circ G_s|_{M_{s,t}}$ on each surface $M_{s,t}$ in

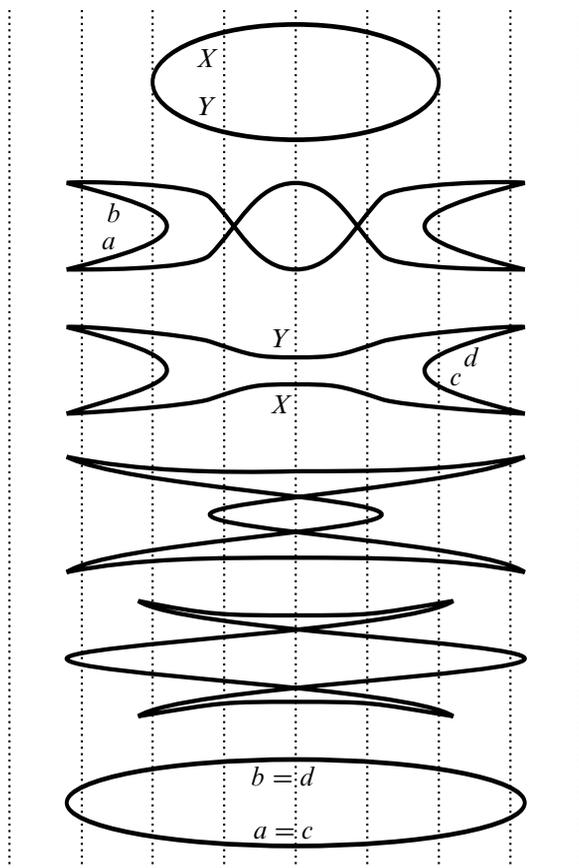


Figure 16: The singular loci of G_s for each of the six values of s corresponding to the rows of Figures 13 and 14: the nine values of t corresponding to the columns are indicated by vertical dotted lines. The correspondence with the critical points in Figures 13 and 14 is indicated by the letter labels. Capital letters X and Y indicate definite folds (2-dimensional 0- and 2-handles) while lowercase letters a , b , c and d indicate indefinite folds (2-dimensional 1-handles). Note that by the end of the homotopy, a and c have become the same fold and b and d have become the same fold; this arises due to the cancellation of the two “swallowtails” at the top and bottom of the preceding singular locus.

Figures 13 and 14. Looking at how $t \circ G_s$ varies with s , in the beginning, we have a (3-dimensional) critical point of index 2 on the left and a (3-dimensional) critical point of index 1 on the right. By the fourth row, the index-1 critical point has moved to the left and the index-2 critical point to the right.

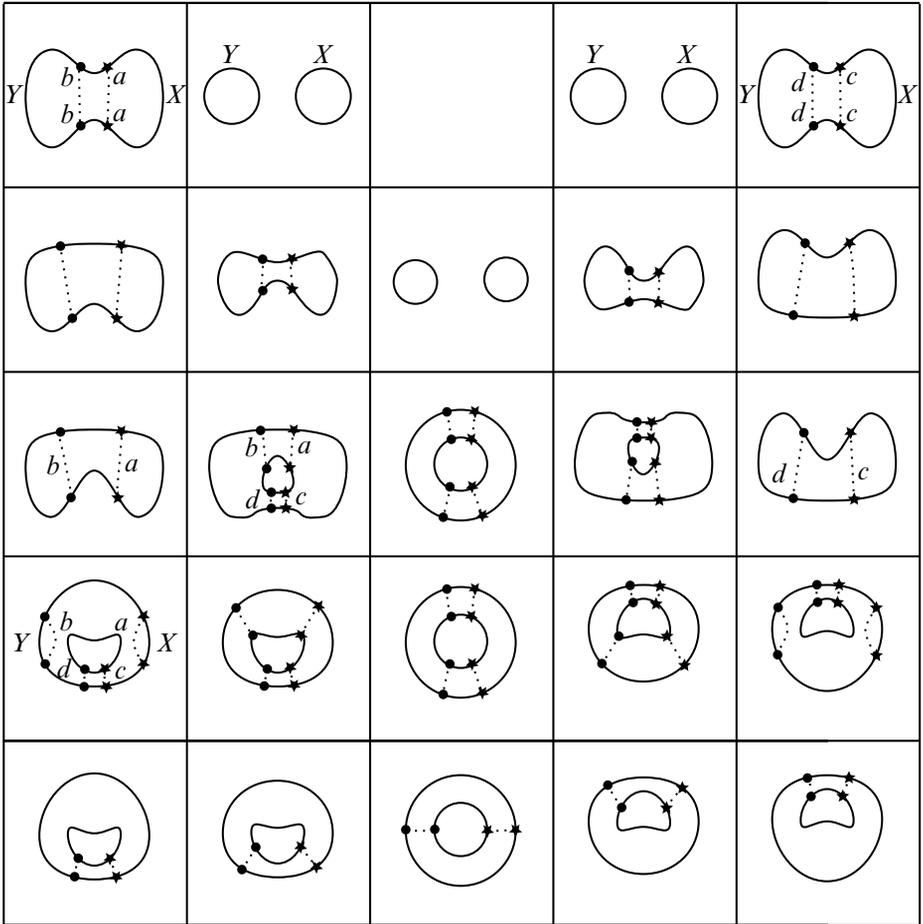


Figure 17: Handle attachment data for a 5×5 block from Figures 13 and 14: in each case we have drawn the middle-level 1–manifold for the surface, and shown where the ascending and descending manifolds of the index-1 critical points (for the vertical height function) intersect this 1–manifold. The circles indicate descending manifolds and the stars indicate ascending manifolds. We have also indicated the ascending manifold for the 0–handle X and the descending manifold for the 2–handle Y . Note again the handle slides in the fourth row. By the last row, the 0– and 2–handles have been canceled, the 1–handles b and d can no longer be distinguished, and the 1–handles a and c can no longer be distinguished.

- (2) The first Morse 2–function G_0 has only definite folds, the intermediate functions have both definite and indefinite folds, and the final function G_1 has only indefinite folds.

- (3) At G_1 , the fiber over points inside the circle of indefinite folds is $S^0 \times S^1$ and, considering rays going out from the center point of the circle, we see, over each such ray, a 2–dimensional 1–handle attached along $S^0 \times \{p\}$, where the point p moves once around the S^1 as the ray rotates once around the center point.
- (4) This example generalizes to a generic homotopy $G_s: X^n = S^1 \times B^{n-1} \rightarrow B^2$. The starting point G_0 is easy to describe in coordinates, exactly as we have done here for the case $n = 3$. The final map G_1 is harder to see in higher dimensions but this example makes it clear that, in the end, we get a circle of indefinite folds with the fiber over points inside the circle being $S^{n-3} \times S^1$. Furthermore, over rays going out from the center we see $(n - 1)$ –dimensional $(n - 2)$ –handles attached along $S^{n-3} \times \{p\}$, where p rotates once around S^1 as the ray rotates once around the center point. When $n = 4$, this example was already seen in [3] in their description of a broken fibration of S^4 over S^2 . This example also highlights a subtlety involved in reading off information about the total space of a Morse 2–function from data on the base; this subtlety is discussed in more detail in [8].
- (5) Such an example can be placed anywhere in a Morse 2–function by adding a cancelling 0–1 round handle pair along any loop in X^n which maps to an embedded circle bounding a disk in the base, so that the image of the 0–fold ends up on the inside of the 1–fold. After that, the round 0–handle can be traded for a round $(n - 2)$ –handle as we have seen here. Adding this cancelling 0–1 round handle pair and then trading the round 0–handle is a homotopy that starts and ends without definite folds but which passes through definite folds during the homotopy. A worthwhile example for the reader to consider, when following the proofs in this paper, is how to carry out this homotopy without definite folds. (The authors have not done this.)

4 Theorems about I –valued Morse functions on cobordisms

In this section we will prove Theorems 1.3 and 1.4 in the case where the base is the interval I , and we will prove Theorems 1.1 and 1.2 in the case where the total space is $X^n = I \times M^{n-1}$, the base is $I \times I$, and the Morse 2–functions are of the form $G(t, p) = (t, g_t(p))$. In preparation for general Morse 2–functions over $I \times I$ (see Section 5), we will need versions of Theorems 1.3 and 1.4 in which attaching maps for n –dimensional handles lie in level sets of I –valued Morse functions on $(n - 1)$ –manifolds.

Throughout this section, we are given the following data:

- (1) a connected m -dimensional cobordism M from $F_0 \neq \emptyset$ to $F_1 \neq \emptyset$, where F_0 and F_1 are compact $(m-1)$ -manifolds, possibly with boundary; we also assume $m \geq 2$ to avoid very-low-dimensional confusion;
- (2) a collection L_1, \dots, L_p (possibly empty) of closed manifolds with $\dim(L_i) = l_i < m/2$ and with mutually disjoint embeddings $\phi_i: [-\epsilon, \epsilon] \times B^{m-1-l_i} \times L_i \hookrightarrow (M \setminus \partial M)$, for some small $\epsilon > 0$; note that if $l_i < m/2$ then $l_i < m-1$: assume the order is such that $l_1 \leq \dots \leq l_p$;
- (3) a collection of values $z_1 < \dots < z_p \in (0, 1)$.

In the results that follow we will say that a Morse function $g: M \rightarrow I$ is *standard with respect to ϕ_i at height z_i* if $g \circ \phi_i: [-\epsilon, \epsilon] \times B^{m-1-l_i} \times L_i \rightarrow I$ is of the form $(z, x, p) \mapsto z + z_i$ on some neighborhood of $\{0\} \times \{0\} \times L_i$.

The most illuminating example to bear in mind is when $m = 3$ and each $l_i = 1$, and we think of $L_1 \cup \dots \cup L_p$ as a link in M and of each embedding ϕ_i as given by a framing of L_i . Then we are interested in Morse functions with L_i in the level set at level z_i , with framing coming from a framing in this level set. These can be constructed by starting with a given Morse function, projecting L_i into a level set, and then resolving crossings by stabilizing to increase the genus. More generally the ϕ_i 's are going to be attaching maps for handles, and we will see in the next section the importance of attaching handles along spheres lying in level sets of a Morse function, with tubular neighborhoods interacting well with the Morse function.

Remark 4.1 In the proofs of the following theorems, we will use generic gradient-like vector fields, and in particular the ascending and descending manifolds of critical points, as a tool to organize local modifications of Morse functions, such as cancellation of critical points. Recall that a gradient-like vector field is a vector field which is transverse to level sets and such that, near each critical point, there are local coordinates with respect to which the Morse function takes the usual form $-x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_m^2 = \mu_k^m(x)$ and the vector field is the usual Euclidean gradient of this function. For a fixed Morse function g , a generic gradient-like vector field is one for which the ascending and descending manifolds meet transversely in intermediate-level sets. For a generic homotopy g_t between Morse functions, a generic 1-parameter family of gradient-like vector fields V_t is one for which the 1-parameter families of ascending and descending manifolds intersected with intermediate level sets are transverse in the 1-parameter sense, and for a generic homotopy $g_{s,t}$ between generic homotopies we have the natural notion of a generic 2-parameter family of gradient-like vector fields $V_{s,t}$. (We should also require that, at the non-Morse singularities, the vector

field is the usual Euclidean gradient for the standard model of the singularity in local coordinates.) It is clear from the fact that the transversality properties of the ascending and descending manifolds are generic that the associated “genericity” properties of the vector fields are actually generic.

Theorem 4.2 *There exists an indefinite ordered Morse function $g: M \rightarrow I$, with critical values not in $\{z_1, \dots, z_p\}$, which is standard with respect to each of ϕ_1, \dots, ϕ_p , at heights z_1, \dots, z_p respectively. Furthermore, the indices of the critical values and the dimensions l_i of the submanifolds L_i are such that all critical values of index less than or equal to l_i are below z_i while all critical values of index greater than l_i are above z_i .*

In the following proof, we will make essential use of the standard lemma that, for a Morse function g with a generic gradient-like vector field, if there is a single gradient flow line from a critical point q of index $k + 1$ down to a critical point p of index k , then the two critical points can be canceled. More precisely, there exists a generic homotopy g_t , with $g_0 = g$, with exactly one death singularity at $g_{1/2}$ involving p and q , and no other birth or death singularities. Also note that no other critical values need to move if there is a regular level set between p and q such that the descending manifold for q and the ascending manifold for p avoid all other critical points on their way to this level set. If this is not the case then other critical points may need to move “out of the way” to facilitate the crossing. See Figure 18 for an illustration of these ideas.

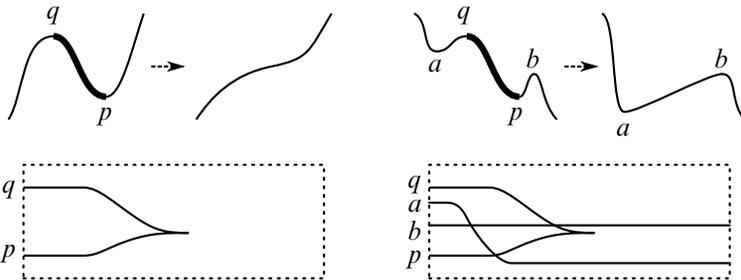


Figure 18: Cancellation of critical points q and p in one dimension, with accompanying Cerf graphics: in the first example, no other critical points need to move, but in the second example the critical point a needs to drop below p before q and p can cancel (or raise b above q).

Definition 4.3 Given a Morse function $g: M \rightarrow I$ with a gradient-like vector field and critical points q of index $k + 1$ and p of index k , with $g(q) > g(p)$, we say that q *cancels* p if there is a unique gradient flow line from q to p .

Then we can summarize the standard cancellation lemma (without proof) as follows:

Lemma 4.4 *Given $g: M \rightarrow I$ with a generic gradient-like vector field, if critical point q cancels critical point p , let C be the closure of the descending manifold for q inside $g^{-1}[g(p), g(q)]$. Note that $p \in C$. Then there is a generic homotopy g_t between Morse functions, with $g_0 = g$, which is independent of t outside an arbitrarily small neighborhood of C , passes through exactly one death singularity at $g_{1/2}$ in which q and p cancel, and has no other birth or death singularities. If C is actually just the descending disk, then no other critical values will change; otherwise all the critical points in C besides q and p will have to move below p before the death occurs.*

Proof of Theorem 4.2 First define $g: M \rightarrow I$ on the submanifolds $\phi_i([-\epsilon, \epsilon] \times B^{m-1-l_i} \times L_i)$ by ϕ_i^{-1} followed by projection to $[-\epsilon, \epsilon]$ followed by translation by z_i . This places $\phi_i(L_i)$ at height z_i as desired.

There is no obstruction to extending this map to a Morse function $g: M \rightarrow I$. (Issues of smoothness at the boundary of $\phi_i([-\epsilon, \epsilon] \times B^{m-1-l_i} \times L_i)$ are easily avoided either by use of tubular neighborhood theorems or by slightly shrinking the image of ϕ_i .) To make g indefinite and ordered, first choose a generic gradient-like vector field, so that we can construct arguments using gradient flow lines and ascending and descending manifolds. Indefiniteness is easily achieved because each critical point of index 0 (or m) must be canceled by a critical point of index 1 (or $m-1$), since M is connected. (If, for an index-0 critical point p , there was no such cancelling index-1 critical point, then there could not be a path from p to F_0 , and M would not be connected.) The set C for an index-1 critical point is 1-dimensional, and will thus miss a neighborhood of the $\phi_i(L_i)$'s by genericity of the gradient-like vector field. Here we use the fact that $\dim(L_i) = l_i < m/2 \leq m-1$, so that each $\phi_i(L_i)$ has positive codimension in the level set $g^{-1}(z_i)$. Thus we can cancel all the index-0 and index- m critical points without modifying g near the $\phi_i(L_i)$'s.

To arrange that g is ordered, suppose that p and q are critical points of index j and k , respectively, with $g(p) < g(q)$, and with no critical values in $(g(p), g(q))$. If the descending manifold D_q for q and the ascending manifold A_p for p are disjoint then there is a generic homotopy supported in a neighborhood of $D_q \cap g^{-1}[g(p), g(q)]$ which lowers $g(q)$ below $g(p)$ (without creating any new critical points). In this case there is also a generic homotopy supported in a neighborhood of $A_p \cap g^{-1}[g(p), g(q)]$ which raises $g(p)$ above $g(q)$. If $j \geq k$ this disjointness can always be arranged by a generic choice of gradient-like vector field, and thus we can get g to be ordered, using either raising or lowering homotopies each time we need to switch the relative order of two critical points. However, if we are not careful, we may mess up the behavior

of g near $\phi_i(L_i)$. To avoid this, we need to make sure that either D_q or A_p misses each $\phi_i(L_i)$, which can be done by generically choosing the gradient-like vector field, as long as $\dim(L_i) = l_i < m/2$. (To see this, count dimensions in the level set F containing $\phi_i(L_i)$ and note that we are asking for either $l_i < j$ or $l_i + k < m$; if $l_i \geq j \geq k$ then $l_i + k \leq 2l_i < m$.)

To arrange that the critical values are ordered nicely with respect to the values z_i as stated, we may need to further raise or lower some critical values, but the same dimension count argument works in this case. \square

The following theorem is about ordered, indefinite generic homotopies on M ; if F_0 and F_1 are connected, then ordered and indefinite implies fiber-connected. At the end of this section, in [Lemma 4.10](#), we will discuss fiber-connectedness without the ordered assumption.

Theorem 4.5 *Given two indefinite, ordered Morse functions $g_0, g_1: M \rightarrow I$ there exists an indefinite, ordered generic homotopy $g_t: M \rightarrow I$ from g_0 to g_1 . If $m \geq 3$ and both g_0 and g_1 are standard with respect to each ϕ_i at height z_i , then we can arrange that, for all t , g_t is standard with respect to each ϕ_i at height z_i . We can also arrange that all the births occur before the critical point crossings and that all the deaths occur after the critical point crossings.*

The proof of this theorem (and the one to follow about homotopies of homotopies) is in essentially the same spirit as the proof of the preceding theorem; we just need the right cancellation lemmas to get rid of definite critical points over time and we need to count dimensions to see that we can order critical points appropriately and avoid the submanifolds $\phi_i(L_i)$ as we modify the homotopy.

For the cancellation lemmas, we need to articulate conditions under which we can pass through eye, unmerge or swallowtail singularities to simplify the homotopy. When discussing a generic homotopy $g_t: M \rightarrow I$, we will frequently use the Cerf graphic to organize our argument; recall that this is the image in $I \times I$ of the critical points of g_t under the map $G: (t, p) \mapsto (t, g_t(p))$. We will use the term “ k -fold” to refer to an arc of index- k critical points in $I \times M$. If we label a k -fold P , then for a fixed time t , P_t will refer to the index- k critical point on P at time t . We will also fix a generic 1-parameter family of gradient-like vector fields so that we may refer to gradient flow lines for each g_t . (Here “generic” means that the 1-parameter families of descending manifolds intersect transversely in level sets, which means that handle slides occur at isolated times, with lower index critical points never sliding over higher index critical points.) We will say that a $(k+1)$ -fold Q “cancels” a k -fold P over a time interval A if Q_t cancels P_t for every time $t \in A$.

In the following three lemmas, suppose that $g_t: M \rightarrow I$ is a generic homotopy between Morse functions g_0 and g_1 with a generic 1-parameter family of gradient-like vector fields. We state conditions under which we can modify g_t so as to reverse the arrows in Figures 8, 9 and 10. We leave the proofs to the reader; the basic idea, as with the proof of the standard cancellation lemma (Lemma 4.4), is as follows. We first push extraneous critical points down far enough so that we only need to work with a single descending disk and ascending disk. Then we use the uniqueness of a gradient flow line to find coordinates on a neighborhood of the gradient flow line in which the gradient flow line itself lies in one coordinate axis, and in the other coordinates the Morse function has the usual Morse model. Then the cancellation occurs entirely in the coordinate chart containing the gradient flow line.

Lemma 4.6 (Unmerge) *Consider a k -fold P and a $(k + 1)$ -fold Q over a time interval $[t_0, t_1]$ such that Q cancels P over all of $[t_0, t_1]$. Then for some arbitrarily small $\delta > 0$ there is a generic homotopy between homotopies $g_{s,t}$, with $g_{0,t} = g_t$, passing through a single unmerge singularity (see Figure 9) at $s = \frac{1}{2}$ and no other 1-parameter singularities, independent of s for $t \in [0, t_0] \cup [t_1, 1]$, such that, with respect to $g_{1,t}$, the cancelling pair Q_t and P_t die at $t = t_0 + \delta$ and are reborn at $t = t_1 - \delta$. For each $t \in [t_0, t_1]$, $g_{s,t}$ is independent of s outside an arbitrarily small neighborhood of the descending manifold for Q_t . Also note that, with respect to $g_{1,t}$, Q_t still cancels P_t on $[t_0, t_0 + \delta)$ and on $(t_1 - \delta, t_1]$. Furthermore we can arrange that any other folds that canceled P on $[t_0, t_0 + \delta]$ or $[t_1 - \delta, t_1]$ will still cancel P there.*

Lemma 4.7 (Eye death) *Consider a k -fold P and a $(k + 1)$ -fold Q over a time interval $[t_0, t_1]$ such that Q cancels P over all of (t_0, t_1) . Also suppose that the critical points Q_t and P_t are born as a cancelling pair at time t_0 and die as a cancelling pair at time t_1 . Then for some small $\delta > 0$ there is a generic homotopy between homotopies $g_{s,t}$, with $g_{0,t} = g_t$, passing through a single eye death singularity (see Figure 8) at $s = \frac{1}{2}$ and no other 2-parameter singularities, independent of s for $t \in [0, t_0 - \delta] \cup [t_1 + \delta, 1]$, such that, in $g_{1,t}$, the cancelling pair Q_t and P_t have canceled for all $t \in [t_0, t_1]$. For each $t \in (t_0, t_1)$, $g_{s,t}$ is independent of s outside an arbitrarily small neighborhood of the descending manifold for Q_t . For $t \in \{t_0, t_1\}$, $g_{s,t}$ is independent of s outside a neighborhood of the birth/death point.*

Lemma 4.8 (Swallowtail death) *Consider a k -fold P and two $(k \pm 1)$ -folds Q and R over a time interval (t_0, t_1) . Suppose furthermore that Q and P are born as a cancelling pair at time t_0 , that R and P die as a cancelling pair at time t_1 , and that Q cancels P over $(t_0, (t_0 + t_1)/2 + \delta)$ while R cancels P over $((t_0 + t_1)/2 - \delta, t_1)$ for some small $\delta > 0$. Then there is a generic homotopy between homotopies, $g_{s,t}$, with*

$g_{0,t} = g_t$, passing through a single swallowtail death singularity (see [Figure 10](#)) at $s = \frac{1}{2}$ and no other 2–parameter singularities, independent of s for $t \in [0, t_0 - \delta] \cup [t_1 + \delta, 1]$ (again for some small $\delta > 0$) such that, in $g_{1,t}$, the k –fold P has disappeared and the $(k \pm 1)$ –folds Q and R have become the same fold. For each $t \in (t_0, (t_0 + t_1)/2 - \delta)$, $g_{s,t}$ is independent of s outside an arbitrarily small neighborhood of the descending/ascending (according to whether $\pm = +$ or $\pm = -$) manifold for Q_t , while for each $t \in ((t_0 + t_1)/2 + \delta, t_1)$ the independence is outside a neighborhood of the descending/ascending manifold for R_t , and for $t \in [(t_0 + t_1)/2 - \delta, (t_0 + t_1)/2 + \delta]$ we need a neighborhood of the union of the descending/ascending manifolds for Q_t and R_t . For $t \in \{t_0, t_1\}$ the independence is outside a neighborhood of the birth/death points.

Proof of Theorem 4.5 As in the proof of [Theorem 4.2](#), there is no difficulty in finding a generic homotopy $g_t: M \rightarrow I$, and if g_0 and g_1 are standard with respect to each ϕ_i at height z_i then we can make g_t independent of t on these neighborhoods. Arranging for births to happen first and deaths last is standard, using connectedness; g_t is modified via a generic homotopy between homotopies which passes through cusp-fold crossings, moving left-cusps (births) further left (earlier), and right-cusps (deaths) further right (later). However it requires some work to make g_t indefinite and then ordered for all $t \in (0, 1)$.

We will cancel the 0–folds one at a time. Consider a 0–fold P which is born at time a and dies at time b . At each time $t \in [a, b]$, there is some index-1 critical point q which cancels P_t . Thus there is a sequence $a = t_0 < t_1 < \dots < t_n = b$ and some $\delta > 0$, giving a covering of $[a, b]$ by intervals $I_1 = [a, t_1 + \delta]$, $I_2 = (t_1 - \delta, t_2 + \delta)$, \dots , $I_n = (t_{n-1} - \delta, b]$, and a sequence of 1–folds Q^1, \dots, Q^n such that each Q^i cancels P over the interval I_i . We do this so that Q^1 is the 1–fold born with P as a cancelling 0–1 pair at time $t_0 = a$ and so that Q^n is the 1–fold which dies with P as a cancelling pair at time $t_n = b$. Using the above three lemmas we can then cancel P with the Q^i 's. First cancel over the nonoverlapping parts of the open intervals I_2, \dots, I_{n-1} using [Lemma 4.6](#). Then cancel over the overlaps and I_1 and I_n using either [Lemma 4.7](#) or [Lemma 4.8](#), depending on whether the two cancelling 1–folds Q^i and Q^{i+1} in the overlap region $(t_i - \delta, t_i + \delta)$ are the same or different. Going from cancelling on the nonoverlapping regions to the overlapping regions requires the extra clauses in [Lemma 4.6](#) to the effect that whatever canceled P at the beginning still cancels P wherever it has not been killed with [Lemma 4.6](#).

The above argument ignored the issue of the submanifolds $\phi_i(L_i)$. If $m \geq 3$ and both g_0 and g_1 are standard with respect to each ϕ_i at height z_i , then we want to arrange that, for all t , g_t is standard with respect to each ϕ_i at height z_i . Since, before

cancelling the definite folds, we had arranged for this property to hold, we need to arrange that, in each application of Lemmas 4.6, 4.7 and 4.8, we avoid neighborhoods of each $\phi_i(L_i)$. In other words, all the descending manifolds for the cancelling 1–folds down to the level of the canceled 0–folds should avoid $\phi_i(L_i)$. Counting dimensions we see that this can generically be achieved at all but finitely many times t , which are distinct from the times t_0, t_1, \dots, t_n at which we switch from one cancelling 1–fold to another. If the descending manifold for a 1–fold Q^j intersects some $\phi_i(L_i)$ at time t_* , with $t_{j-1} < t_* < t_j$, we break Q^j into two 1–folds by introducing a 1–2 swallowtail (passing through a swallowtail singularity) at time t_* along Q^j . This is illustrated in Figure 19; we label the two new 1–folds Q^j_- and Q^j_+ as indicated in the figure, and observe that we can arrange for the descending manifold for Q^j_- to meet $\phi_i(L_i)$ at some time $t_- > t_*$ while the descending manifold for Q^j_+ meets $\phi_i(L_i)$ at some time $t_+ < t_*$. Then we break the interval $(t_{j-1} - \delta, t_j + \delta)$ into two overlapping intervals $(t_{j-1} - \delta, t_* + \delta)$ and $(t_* - \delta, t_j + \delta)$ and replace the single cancelling 1–fold Q^j with Q^j_- over $(t_{j-1} - \delta, t_* + \delta)$ and Q^j_+ over $(t_* - \delta, t_j + \delta)$. (We might, of course, need to decrease δ .)

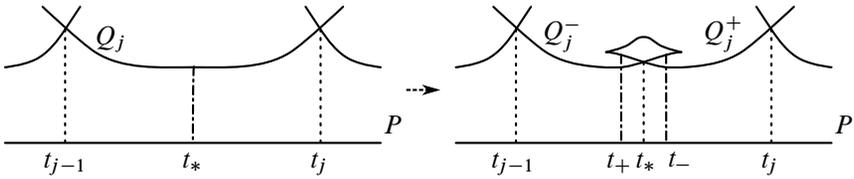


Figure 19: Splitting Q^j

Note that the above argument required $m \geq 3$ because otherwise the 2–fold in the 1–2 swallowtail is a definite fold.

To arrange that g_t is ordered when g_0 and g_1 are ordered, we first need to arrange that each birth or death of a cancelling k –($k + 1$) pair occurs above all the other k –folds and below all the other $(k + 1)$ –folds. This is straightforward because such a modification of g_t can be achieved through a generic homotopy supported in a neighborhood of an arc, which can be chosen to be disjoint from the $\phi_i(L_i)$ ’s. Now the only issue is pulling k –folds below j –folds when $k < j$ (or pushing j –folds above k –folds); this can be achieved if we ignore the $\phi_i(L_i)$ ’s for the same reason that it can be achieved for a fixed Morse function, as in Theorem 4.2, namely that, for a generic 1–parameter family of gradient-like vector fields, k –handles will not slide over j –handles if $k < j$. However, if we want to avoid modifying g_t near each $\phi_i(L_i)$ we need to be more careful, and to do this we count dimensions again.

Here we need to check that either the 1–parameter descending disk for the k –fold Q or the 1–parameter ascending disk for the j –fold P misses the l_i –dimensional sub-

manifold $\phi_i(L_i)$ in the level set $g^{-1}(z_i)$, which is presumed to be between Q and P . The level set is $(m-1)$ -dimensional, the descending sphere for Q in the level set is $(k-1)$ -dimensional, and the ascending sphere for P is $(m-j-1)$ -dimensional. However, because of the parameter t , we now want that either $(k-1)+1+l_i < m-1$ or that $(m-j-1)+1+l_i < m-1$, ie that $k+l_i < m-1$ or that $l_i < j-1$. If $l_i \geq j-1$, so that $k < j \leq l_i+1$, we have $k \leq l_i$ and thus $k+l_i \leq 2l_i$. Thus we are fine as long as $l_i < (m-1)/2$, but in our initial hypotheses we only assumed that $l_i < m/2$. The only potentially bad case is when m is odd, $k = l_i = (m-1)/2$ and $j = (m+1)/2$. In this case both the ascending sphere for P and the descending sphere for Q will intersect $\phi_i(L_i)$ at discrete times. However, now we simply note that, again by genericity, these times will be distinct for P and Q , and so at times when the ascending sphere for P intersects $\phi_i(L_i)$ we lower Q below P while at times when the descending sphere for Q intersects $\phi_i(L_i)$ we raise P above Q . \square

Theorem 4.9 *Suppose that $m \geq 3$. Given two indefinite, ordered generic homotopies $g_{0,t}, g_{1,t}: M \rightarrow I$ between indefinite Morse functions $g_{0,0} = g_{1,0}$ and $g_{0,1} = g_{1,1}$, there exists an indefinite, almost ordered generic homotopy of homotopies $g_{s,t}: M \rightarrow I$ from $g_{0,t}$ to $g_{1,t}$ with fixed endpoints. When $m \geq 4$ and F_0 and F_1 are both connected this guarantees that all level sets of each $g_{s,t}$ are connected. In the case where $m = 3$ and F_0 and F_1 are both connected, we can do a little extra work to arrange that all level sets of each $g_{s,t}$ are connected, even though the “almost ordered” condition is not sufficient to imply this.*

(Compare [9, Proposition 3.6], which deals with approximately the same issue, but is about cancelling critical points of arbitrary indices and requires high ambient dimensions. There are many striking similarities between that proof and our proof of Theorem 4.9.)

Note that we could also ask that $g_{s,t}$ behave well on neighborhoods of the L_i 's as in the preceding theorems, and presumably there are constraints in terms of the dimensions involved, but we have no need for such a result in this paper.

Although the statement of the theorem does not say this, we will actually be able to modify a given generic homotopy $g_{s,t}$ from $g_{0,t}$ to $g_{1,t}$ through a generic homotopy $g_{r,s,t}$ with yet one more parameter $r \in [0, 1]$, with $g_{0,s,t} = g_{s,t}$, so that $g_{1,s,t}$ satisfies the various conditions we want. In doing so we will pass through higher codimension singularities, and so we could have “cancellation” lemmas analogous to Lemmas 4.4, 4.7, 4.6 and 4.8. In our case they would involve the “butterfly singularity” and the “monkey saddle” (or “elliptic umbilic”). However, since each only occurs once in the proof, we just develop them in the course of the proof. After discovering the necessary homotopies corresponding to these singularities, we then realized that they also play a central role, for similar reasons, in the work of Hatcher and Wagoner [9].

Proof There is always a generic homotopy $g_{s,t}: M \rightarrow I$ rel boundaries, so the first issue is to make it indefinite, that is, to remove all 0–folds (m –folds are treated the same way using $1 - g_{s,t}$).

Instead of the traditional Cerf graphic, we consider a 1–parameter family of Cerf graphics, and in this case the 0–folds form a 2–dimensional immersed surface Σ , as in the example in Figure 20.

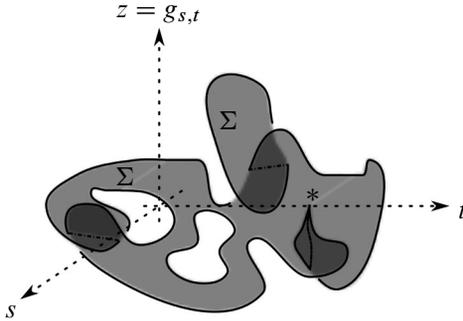


Figure 20: Example of a 2–dimensional surface Σ of 0–folds in a generic 2–parameter homotopy between Morse functions: note the swallowtail singularity at the point labeled $*$.

In Σ , with respect to the s direction, there are merges, unmerges, eyes and swallowtails; apart from the swallowtails, these appear as smooth curves with tangents parallel to the (t, z) plane.

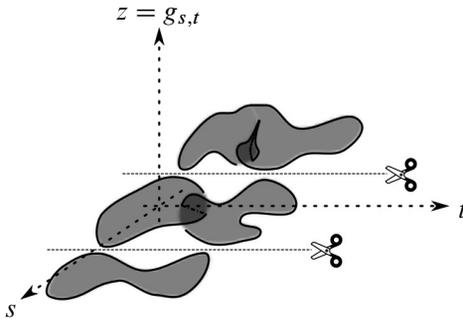


Figure 21: After the first cuts, along constant s slices: note that components are still not embedded, due to the bad swallowtail.

The first step is to cut Σ into pieces, each of which is embedded in $I \times I \times I$. We do this by first cutting Σ in the (t, z) direction at many fixed s values. Such a cut is done by first making $g_{s,t}$ independent of s in a small s –interval $[s_* - \delta, s_* + \delta]$, then

applying the technique in the proof of [Theorem 4.5](#) to modify the homotopy $g_{s_*,t}$ to get rid of definite folds at s_* , and then noting that the modification is through a generic homotopy between homotopies, which can then be run forward and backward in the s direction as s ranges from $s_* - \delta$ to s_* to $s_* + \delta$. A typical example of the result is illustrated in [Figure 21](#).

The components of Σ are not yet embedded because of the possibility that there are births of swallowtails in the middle of Σ . However, in [\[9, page 199 and item 1 on page 194\]](#), it is shown exactly how such a swallowtail birth can be extended past the 0–1 cusp and onto the surface of 1-folds by passing through a “butterfly singularity”, with the result that the swallowtail cuts Σ into two parts, which intersect each other but no longer have a self-intersection. The change in the movie of Cerf graphics is shown in [Figure 22](#), with accompanying graphs of the 1-dimensional Morse functions.

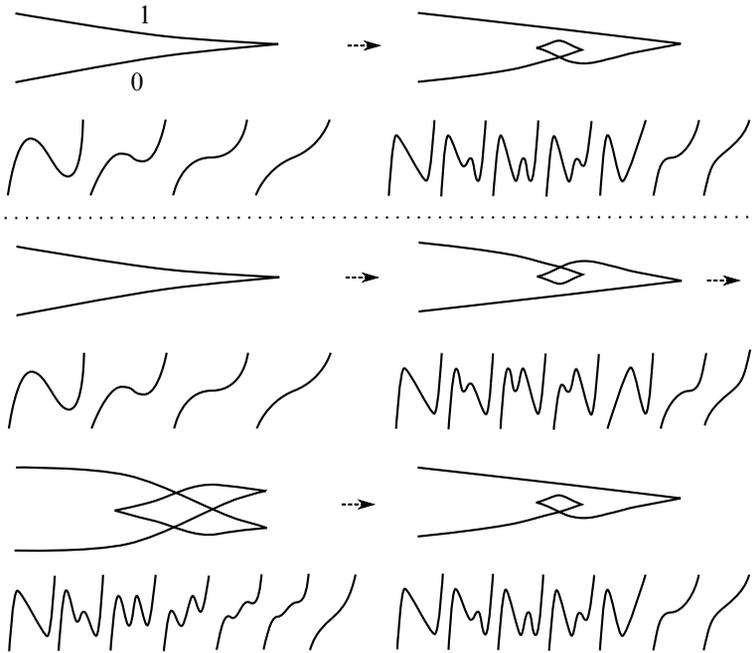


Figure 22: Cutting the bad swallowtail onto the surface of 1-folds using the butterfly singularity: above the dotted line, the swallowtail occurs in the middle of a 0-fold, leading to surface of 0-folds that is not embedded. Below the line, the swallowtail is born in the middle of a 1-fold, so that the two 0-folds which intersect are in distinct components of the surface of 0-folds.

Now each component \mathcal{P} of Σ is embedded in $I \times I \times I$ and we will eliminate these components one by one, in much the same way we eliminated individual 0-folds in

Theorem 4.5 and individual index-0 critical points in **Theorem 4.2**. Noting that, for each (s, t) , the index-0 critical point $\mathcal{P}_{s,t}$ coming from \mathcal{P} is canceled by some index-1 critical point, we can cover \mathcal{P} with open disks $\{\mathcal{P}^i\}$ over each of which we have chosen a particular disk \mathcal{Q}^i of cancelling index-1 critical points. We also arrange that every vertex in the nerve of this cover (including vertices on $\partial\mathcal{P}$) has valence 3 and that every edge of the nerve is transverse to constant s slices. Then we can use the ideas in the proof of **Theorem 4.5** to eliminate \mathcal{P} at all points in exactly one or two of the open sets of the cover, and we reduce to the case where \mathcal{P} is a union of disjoint triangles each canceled by three distinct surfaces of 1–folds.

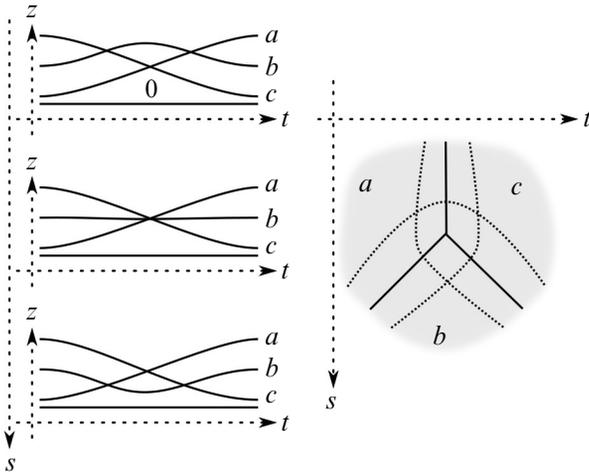


Figure 23: Three 1–folds a, b and c cancelling a 0–fold in two parameters

Figure 23 shows a sequence of Cerf graphics, representing the 2–parameter Cerf graphic where three open sets intersect; on the right we show the covering in parameter space, with the nerve and its trivalent vertex. The labels a, b and c indicate the 1–handles that cancel in each of the three open sets. At each point in parameter space, we adopt the convention that the closest index-1 critical point to the 0–fold is the cancelling one. This is not necessarily the case at first, but right before the cancellations this will be true. Next, **Figure 24** shows the result of cancelling the 0–fold with the appropriate 1–folds away from the overlaps in the cover, with the labels ab, bc and ac indicating the pairs of 1–folds which cancel in each region. Finally, **Figure 25** shows how this sequence of Cerf graphics is transformed by cancelling the swallowtails that remain when two of the open sets intersect, leaving the 0–fold uncanceled only in a cusped-triangular neighborhood of the trivalent vertex. (The boxed numbers indicate points in parameter space for reference in the next figure to come.)

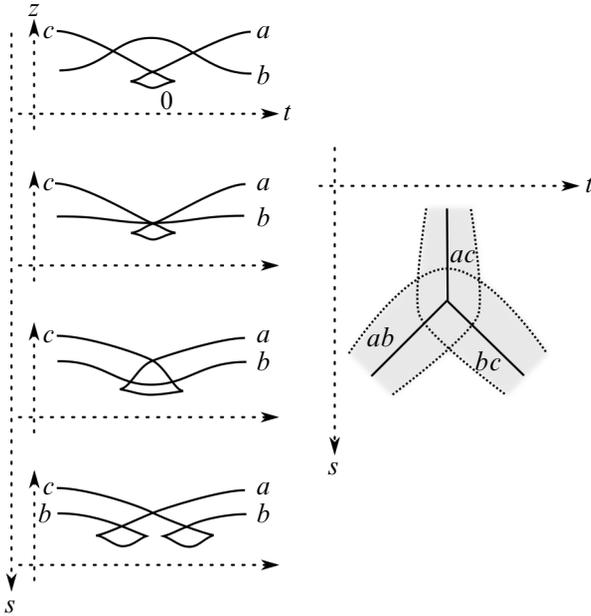


Figure 24: After cancelling the 0–1 pairs away from the overlapping regions

Because there are three 1–handles cancelling the 0–handle across this triangle, we can construct a local model in which we separate out two local coordinates in the domain in which the cancellations occur and keep the other coordinates as a sum of squares independent of the parameters. In other words, locally $g_{s,t}$ is given by $g_{s,t}(x_1, x_2, x_3, \dots, x_m) = h_{s,t}(x_1, x_2) + x_3^2 + \dots + x_m^2$. In Figure 26 we schematically illustrate the handle decomposition corresponding to $h_{s,t}$ at representative points in the (s, t) –parameter space, as labeled by boxed numbers in Figure 25. In [9, page 202], this is shown to be precisely the southern hemisphere of the S^2 –boundary of a B^3 space of deformations of the monkey saddle $h(x_1, x_2) = x_1^3 - 3x_1x_2^2$. The south pole is visualized as pushing down in the middle of the monkey saddle to create an index-0 critical point with three cancelling index-1 critical points, while the equator is a loop of Morse functions involving two index-1 critical points that slide over each other three times. To eliminate this 0–fold completely, we simply replace the southern hemisphere of this S^2 with the northern hemisphere, which involves pushing the middle of the monkey saddle upwards to create an index-2 critical point. We replace the triangular 0–fold surface with a triangular 2–fold surface, exactly as in [9, page 198, item (iv) and Lemma 4.1]. (This is where it is important that $m \geq 3$, so that index 2 is indefinite.)

Do this to each component of Σ and, upside down, do the same to the index- m critical points and we have an indefinite generic homotopy between homotopies.

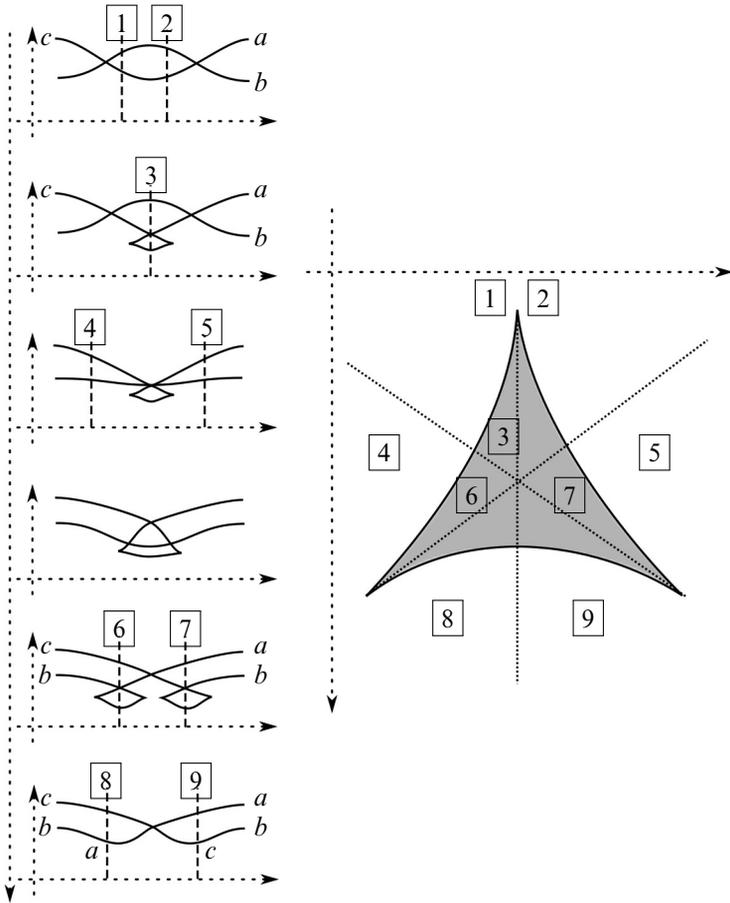


Figure 25: The 2–parameter Cerf graphic after cancelling along the double overlaps but not the triple overlaps: the dotted lines in the figure on the right indicate the points where two 1–folds intersect. Note that the labeling of the 1–folds a , b and c is no longer consistent because, as one moves across the bottom Cerf graphic, c becomes b and b becomes a , and as one moves across the top Cerf graphic, c becomes a .

Getting $g_{s,t}$ to be almost ordered when $g_{0,t}$ and $g_{1,t}$ are ordered follows the same argument as for a fixed Morse function or a homotopy between Morse functions, except that we now note that for a generic 2–parameter family of gradient-like vector fields the descending manifold for an index- j critical point may meet the ascending manifold for an index- $(j + 1)$ critical point at isolated points (s, t) in the 2–dimensional parameter space. This is why we can at most ask that, if an index- k critical point is above an index- j critical point, then $j \leq k + 1$. As noted earlier, when F_0 and F_1 are connected

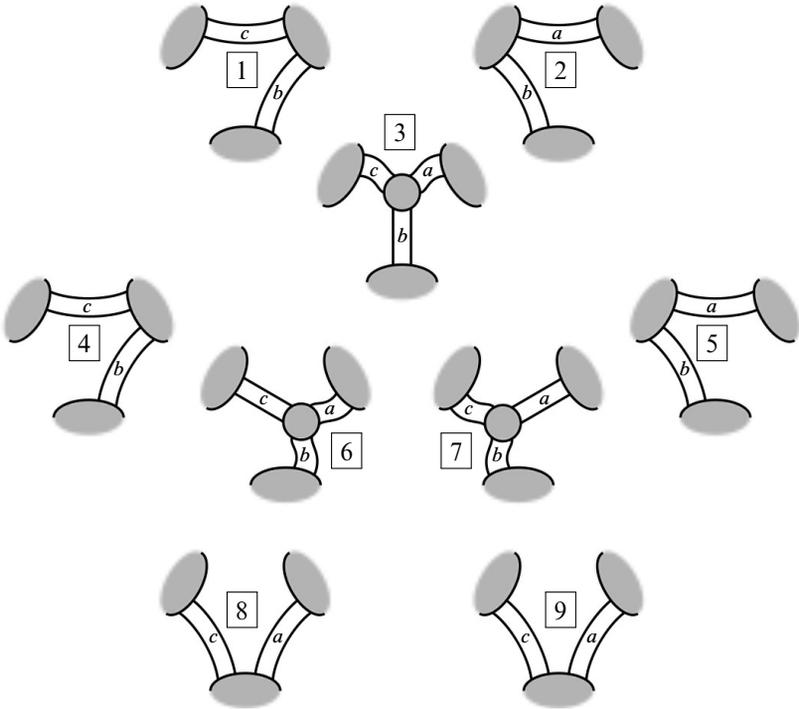


Figure 26: Handle decompositions at various points in the Cerf graphic in [Figure 25](#)

and $m \geq 3$ then ordered implies connected level sets. In fact, for connectedness of level sets we simply need that all index- $(m - 1)$ critical points are above all index-1 critical points, and thus almost ordered implies connected level sets when $m \geq 4$. So, if $m \geq 4$, the proof is complete.

For the remainder of the proof, we assume $m = 3$, in which case level sets are 2-dimensional.

There are no definite folds, so the only way for a level set to become disconnected is by adding a 2-handle H_2 to a separating circle C in a lower level set. Then this level set would remain disconnected all the way to the top of the Morse function unless a higher 1-handle H_1 is attached to the different components of the level set. In the 0- and 1-parameter cases, we can always arrange that the 1-handle is added below the 2-handle in which case the attaching circle C of the 2-handle does not separate. But in the 2-parameter case, the attaching 0-sphere of the 1-handle H_1 may go over the 2-handle H_2 and not be able to be pulled off. This is illustrated in [Figure 27](#), where one foot of the 1-handle moves around C_1 and over the 2-handle H_2 , as the

parameter runs over a 2-disk D . At the center of D , the foot is stuck at the critical point of H_2 .

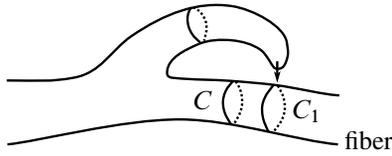


Figure 27: A 1-handle sliding over a 2-handle in a 2-parameter family

However, it is important to note that this problem occurs at isolated points, so we can pull the 1-handles below the 2-handles everywhere except at small disks exactly like this disk D .

To deal with the disconnectedness of the level set over the middle of D , above the 2-handle H_2 and below the 1-handle H_1 , we need a “helper” 1-handle H'_1 , so create a 1–2 cancelling pair over the entire disk D in which the helper 1-handle H'_1 is attached to a parallel copy C' of C and over H_2 , and H'_2 is a “tunnel” between H_1 and H'_1 . See Figure 28. Now perturb H'_1 so that the value of the parameter in D at which H'_1 hits the critical point of H_2 is different from $0 \in D$, say $d \in D$. Now the fibers are connected except for disjoint disk neighborhoods of 0 and d of radius less than $\|d\|/3$. However, in the disk around $0 \in D$, the helper 1-handle can be pulled below H_2 , so that H_2 does not disconnect and then H_1 is attached, and lastly H'_2 which obviously does not disconnect. By symmetry, the level sets can also be made connected in the disk around $d \in D$. □

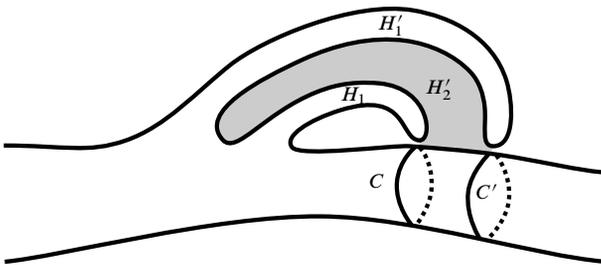


Figure 28: The helper 1-handle H'_1 , with cancelling 2-handle H'_2

We end this section with a lemma, of a slightly different flavor, that will be used several times in the following sections to relate connectedness of level sets to the property of being ordered. In particular, Theorem 4.5 requires, as a hypothesis, that the given Morse functions be ordered. The following lemma is used to prepare for this theorem.

We have already stated that ordered I –valued Morse functions have connected level sets, and the converse is obviously not true. However, we do have:

Lemma 4.10 *Let $g: M \rightarrow I$ be an indefinite Morse function which is standard with respect to each ϕ_i at height z_i , and suppose that all level sets of g are connected. Then there exists an indefinite generic homotopy $g_t: M \rightarrow I$ between Morse functions, with $g_0 = g$, such that g_1 is ordered, all level sets of g_t are connected for all t , and such that each g_t is standard with respect to each ϕ_i at height z_i , with the critical values of g_1 also ordered with respect to the heights z_i as in the statement of [Theorem 4.2](#).*

Proof In the process of ordering the critical points, we just need to check that we do not create disconnected level sets. Disconnected level sets only come from $(m-1)$ –handles attached along separating spheres. We will never be moving $(m-1)$ –handles below other handles in the ordering process. If the attaching sphere for an $(m-1)$ –handle H is nonseparating, it can only be made separating by attaching another $(m-1)$ –handle below H . Thus we never need to create disconnected level sets while ordering if all level sets were connected to begin with. \square

For the sake of completeness, we could also state a parameterized version, in which one homotopes a homotopy with connected level sets to an ordered homotopy of homotopies. This would be used as preparation for applying [Theorem 4.9](#). However, we will only need such a lemma at two points (in the proofs of [Theorem 5.2](#) and [Lemma 5.7](#)) so we will prove it when we need it.

5 Theorems about I^2 –valued Morse 2–functions on cobordisms between cobordisms

Throughout this section let X be an oriented connected n –dimensional cobordism from M_0 to M_1 , where each M_i is a nonempty oriented $(n-1)$ –dimensional cobordism from F_{i0} to F_{i1} , with $F_{00} \cong F_{10}$ and $F_{01} \cong F_{11}$, with each F_{ij} oriented, closed and nonempty. Recall that this means that ∂X is equipped with a fixed identification with $-M_0 \cup (I \times F_{00}) \cup (I \times (-F_{01})) \cup M_1$, with $F_{0j} \subset M_0$ identified with $\{0\} \times F_{0j}$ and $F_{1j} \subset M_1$ identified with $\{1\} \times F_{0j}$ (see [Figure 1](#)). Also suppose that we are given indefinite, ordered Morse functions $\zeta_0: M_0 \rightarrow I$ and $\zeta_1: M_1 \rightarrow I$. On I^2 we use coordinates (t, z) , thinking of t as horizontal and z as vertical.

Note that we are not at this point assuming that the M_i ’s or the F_{ij} ’s are connected. However, the assumption that each ζ_i is indefinite implies that each component of M_i is a connected cobordism between nonempty components of the F_{ij} ’s.

Our goal in this section is to prove the following two theorems, an existence theorem and a uniqueness theorem for square Morse 2–functions:

Theorem 5.1 *Suppose that $n \geq 3$. Given any indefinite, ordered Morse function $\tau: X \rightarrow I$ which is projection to I on $I \times F_{00}$ and $I \times (-F_{01})$ in ∂X , there exists an indefinite square Morse 2–function $G: X \rightarrow I^2$ such that $z \circ G|_{M_0} = \zeta_0$, $z \circ G|_{M_1} = \zeta_1$ and $t \circ G = \tau$. If $n \geq 4$ and each M_i and F_{ij} is connected, then we can arrange that G is fiber-connected.*

Theorem 5.2 *Suppose that $n \geq 4$. Given two indefinite square Morse 2–functions $G_0, G_1: X \rightarrow I^2$ with $z \circ G_0|_{M_0} = z \circ G_1|_{M_0} = \zeta_0$ and $z \circ G_0|_{M_1} = z \circ G_1|_{M_1} = \zeta_1$, and such that both $\tau_0 = t \circ G_0$ and $\tau_1 = t \circ G_1$ are ordered and indefinite, there exists an indefinite generic homotopy $G_s: X \rightarrow I^2$ between G_0 and G_1 such that $\tau_s = t \circ G_s$ is ordered (and τ_s restricted to $I \times F_{0i}$ is projection onto I , as in Definition 2.10). If in addition G_0 and G_1 are fiber-connected then we can arrange that G_s is fiber-connected.*

Note that in the preceding section we have already proved these two theorems in the special case that $X = [0, 1] \times M$, $M_0 = \{0\} \times M$, $M_1 = \{1\} \times M$, and $\tau = \tau_0 = \tau_1$ is projection to $[0, 1]$. This is because a generic homotopy $g_t: M \rightarrow I$ between Morse functions $g_0, g_1: M \rightarrow I$ gives a square Morse 2–function $G: X \rightarrow I^2$ defined by $G(t, p) = (t, g_t(p))$, with $z \circ G|_{M_i} = g_i$. The key difference between a general square Morse 2–function and one coming from a generic homotopy between Morse functions is that the “Cerf graphic” for a general square Morse 2–function, ie the image $G(Z_G) \subset I^2$ of the critical point set, may have vertical tangencies. These vertical tangencies correspond precisely to critical points of the horizontal Morse function $t \circ G: X \rightarrow I$. In the absence of such vertical tangencies, the horizontal Morse function is trivial and hence X is a product. We exploit these ideas repeatedly in the following proofs.

Before working on the proofs, we spend some time understanding neighborhoods of these critical points. In other words, when $t \circ G: X \rightarrow I$ has a critical point at $p \in X$, what can we say about G near p ? We first construct two local models which are illustrated in Figure 29 (recall that $\mu_k^n(x) = \mu_k^n(x_1, \dots, x_n) = -x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$):

- (1) We call the following map a *forward index- k critical point*, or a *forward k –handle*: $\Gamma_k^+(x_1, \dots, x_n) = (\mu_k^n(x), x_{k+1})$. Note that this only makes sense for $0 \leq k \leq n - 1$. By construction $t \circ \Gamma_k^+$ has a critical point of index k at 0. Reparametrizing the range by $(t, z) \mapsto (t - z^2, z)$ transforms Γ_k^+ to $(x_1, \dots, x_n) \mapsto (\mu_k^{n-1}(x_1, \dots, \hat{x}_{k+1}, \dots, x_n), x_{k+1})$, showing that Γ_k^+ really

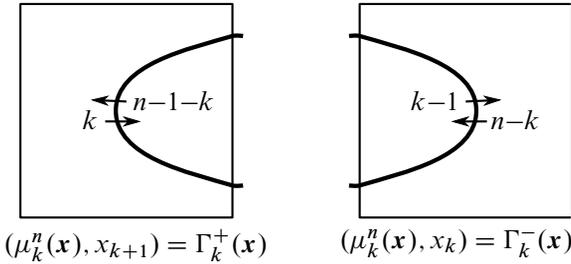


Figure 29: Local models for Morse critical points of index k realized via Morse 2–functions: on the left is a “forward k –handle”, on the right is a “backward k –handle”. Again, the arrows indicate the index of the fold in the given direction, not the index of the Morse critical point.

is a (local) Morse 2–function with a single fold $Z_{\Gamma_k^+}$ along the x_{k+1} axis and with the image $\Gamma_k^+(Z_{\Gamma_k^+})$ of this fold equal to the rightward-opening parabola $t = z^2$. The fold is indefinite when $0 < k < n - 1$.

- (2) This map is a *backward index- k critical point*, or a *backward k –handle*:

$$\Gamma_k^-(x_1, \dots, x_n) = (\mu_k^n(\mathbf{x}), x_k).$$

This only makes sense for $1 \leq k \leq n$. Again, by construction $t \circ \Gamma_k^-$ has a critical point of index k at 0. In this case, reparametrizing the range by $(t, z) \mapsto (t + z^2, z)$ transforms Γ_k^- to

$$(x_1, \dots, x_n) \mapsto (\mu_{k-1}^{n-1}(x_1, \dots, \hat{x}_k, \dots, x_n), x_k),$$

showing that Γ_k^- really is a (local) Morse 2–function with a single fold $Z_{\Gamma_k^-}$ along the x_k axis and with the image $\Gamma_k^-(Z_{\Gamma_k^-})$ of this fold equal to the leftward-opening parabola $t = -z^2$. This fold is indefinite when $1 < k < n$.

Note that we could have defined Γ_k^+ by $\Gamma_k^+(\mathbf{x}) = (\mu_k^n(\mathbf{x}), \pm x_j)$ for any $j \in \{k + 1, \dots, n\}$ and it would still have all the properties listed, and in fact such a definition is equivalent to the one given up to a change of coordinates in the domain. Similarly we could define $\Gamma_k^-(\mathbf{x}) = (\mu_k^n(\mathbf{x}), \pm x_j)$ for any $j \in \{1, \dots, k\}$. However, Γ_k^+ and Γ_k^- are not in general related by a change of coordinates, even allowing a change of coordinates in the range, because the indices of the folds are different, as can be seen in Figure 29. If we turn a forward k –handle backwards, ie postcompose Γ_k^+ with the time-reversal $(t, z) \mapsto (-t, z)$, it becomes a backward $(n - k)$ –handle.

Here are some further observations about forward k –handles, illustrated below in Figure 30; the reader can figure out the parallel statements for backward handles.

- (1) When $0 < k \leq n - 1$, the descending disk $\{x_{k+1} = \dots = x_n = 0\}$ has image equal to the horizontal line $\{t \leq 0, z = 0\}$. When $k = 0$ there is, of course, no descending disk.
- (2) When $0 \leq k < n - 1$, the ascending disk $\{x_1 = \dots = x_k = 0\}$ has image equal to the “interior” of the parabola $\{t \geq z^2\}$. (Of course, when $k = 0$ the ascending disk is the whole domain of the function.) When $k = n - 1$, the image of the ascending disk is just the parabola $\{t = z^2\}$.
- (3) For $0 < k \leq n - 1$, the descending $(k - 1)$ -sphere $\{x_{k+1} = \dots = x_n = 0, x_1^2 + \dots + x_k^2 = R^2\}$ has image equal to the point $(-R^2, 0)$.
- (4) For $0 < k \leq n - 1$, the attaching region for the k -handle, which we identify as the set

$$\begin{aligned} \{\mu_k^n(\mathbf{x}) = -R^2, -\epsilon \leq x_{k+1} \leq \epsilon, x_{k+2}^2 + \dots + x_n^2 \leq \epsilon^2\} \\ \cong S^{k-1} \times [-\epsilon, \epsilon] \times B^{n-1-k}, \end{aligned}$$

has image equal to the line segment $\{t = -R^2, -\epsilon \leq z \leq \epsilon\}$, and the map to this line segment is simply projection onto $[-\epsilon, \epsilon]$. This is where we first see the relevance of the conditions in the preceding section regarding constructing Morse functions which are standard with respect to embeddings of $[-\epsilon, \epsilon] \times B^{n-1-k} \times S^{k-1}$.

- (5) For $k < n - 1$, the ascending $(n - k - 1)$ -sphere $\{x_1 = \dots = x_k = 0, x_{k+1}^2 + \dots + x_n^2 = R^2\}$ maps to the line $\{t = R^2\}$ via the standard Morse function on a sphere, with image equal to the interval $\{-R \leq z \leq R\}$. For $k = n - 1$ the ascending sphere is two points mapping to $(R^2, \pm R)$.

First we prove existence using these local models:

Proof of Theorem 5.1 If τ has no critical points then we use a gradient flow to identify X with $I \times M_0$ such that $\tau(t, p) = t$, and then we see ζ_0 and ζ_1 as two indefinite Morse functions on M_0 . Then our result follows from Theorem 4.5; we get an indefinite generic homotopy ζ_t and we let $G(t, p) = (t, \zeta_t(p))$.

Thus if we can now prove the theorem in the case where τ has exactly one critical point $p \in X$, then we are done. Suppose that $\tau(p) = \frac{1}{2}$ and that p has index $k \leq n/2$. (If $k > n/2$ then replace τ with $1 - \tau$ and switch M_0 and M_1 .) Choose a gradient-like vector field V for τ , and use this to find an embedding $\phi: [-\epsilon, \epsilon] \times B^{n-1-k} \times S^{k-1} \hookrightarrow M_0$ which gives the gluing map for the associated handle with appropriate framing. (We have split the normal direction into a product of $[-\epsilon, \epsilon]$ and B^{n-1-k} as preparation for the use of Theorem 4.2. Note that $k - 1$ is the dimension referred to as l_i in the statement

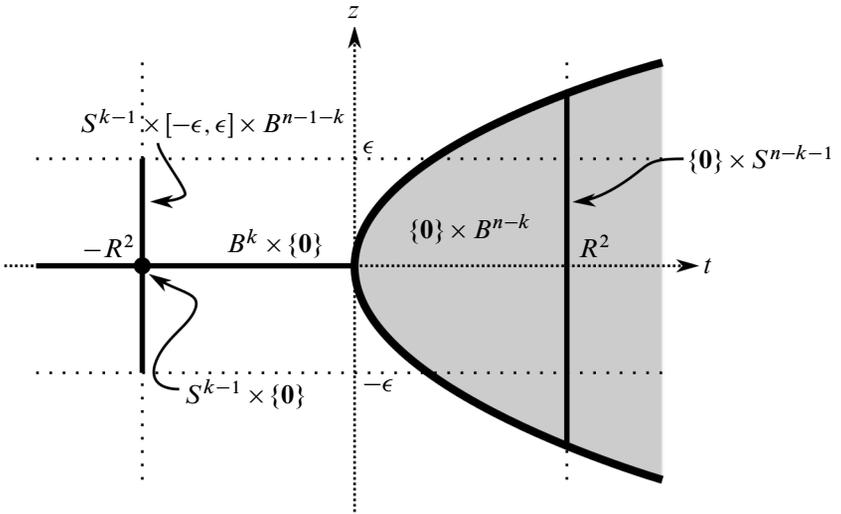


Figure 30: Images of ascending and descending disks and spheres, and attaching region, for a forward k -handle $B^k \times B^{n-k}$: the parabola $t = z^2$ is the image of the fold.

of that theorem, and that the dimension of M_0 is $m = n - 1$. We need to verify that $k - 1 < (n - 1)/2$, which we do have, because $k - 1 \leq (n/2) - 1 = (n - 2)/2 < (n - 1)/2$.) Let $T \subset \tau^{-1}(\frac{1}{2} - \delta)$ be the image of ϕ . For some small $\delta > 0$ we can then decompose X into a union of four parts, $X = X_0 \cup X_c \cup H \cup X_1$ with the following properties. (Figure 31 shows where these four parts will sit in I^2 and shows what the singular locus $G(Z_G)$ will look like in each part.)

- (1) $X_0 = \tau^{-1}[0, \frac{1}{2} - \delta]$ and is identified, via V , with $[0, \frac{1}{2} - \delta] \times M_0$ in such a way that $\tau|_{X_0}(t, p) = t$.
- (2) $X_1 = \tau^{-1}[\frac{1}{2} + \delta, 1]$ and is identified, via V , with $[\frac{1}{2} + \delta, 1] \times M_1$ in such a way that $\tau|_{X_1}(t, p) = t$.
- (3) $X_c \subset \tau^{-1}[\frac{1}{2} - \delta, \frac{1}{2} + \delta]$ and is identified, via V , with $[\frac{1}{2} - \delta, \frac{1}{2} + \delta] \times (M_0 \setminus (T \setminus \partial T))$, in such a way that $\tau|_{X_c}(t, p) = t$.
- (4) H is the k -handle, the union of the forward flow lines for V starting at T , together with the ascending manifold of p (using V), intersected with $\tau^{-1}[\frac{1}{2} - \delta, \frac{1}{2} + \delta]$. On H we have coordinates (x_1, \dots, x_n) with respect to which $\tau(\mathbf{x}) = \frac{1}{2} + \mu_k^n(\mathbf{x})$ and $V = -2x_1 \partial_{x_1} - \dots - 2x_k \partial_{x_k} + 2x_{k+1} \partial_{x_{k+1}} + \dots + 2x_n \partial_{x_n}$. We choose these coordinates so that the x_{k+1} direction is the $[-\epsilon, \epsilon]$ direction in the attaching region $[-\epsilon, \epsilon] \times B^{n-1-k} \times S^{k-1}$ (and (x_{k+2}, \dots, x_n) give the B^{n-1-k} directions while the sphere in the (x_1, \dots, x_k) coordinates gives the S^{k-1} factor).

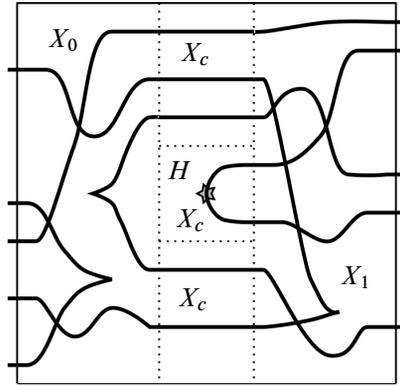


Figure 31: Constructing a Morse 2–function with a single critical point in the horizontal Morse function: this diagram shows the images of the four parts of X , X_0 , X_1 , X_c and the handle H ; the image of the critical point is at the star. Note that the images of X_0 and X_1 are disjoint, while the image of X_c contains the image of H , even though X_c and H intersect in X only along their boundaries.

In order to construct G , we first use [Theorem 4.2](#) to each component of M_0 to construct an indefinite, ordered Morse function $\zeta_{1/2-\delta}: M_0 \rightarrow I$ which is standard with respect to ϕ at height $\frac{1}{2}$, so that $\zeta_{1/2-\delta}$ on the attaching region T is of the form $(t, x, p) \mapsto t + \frac{1}{2}$ (identifying T with $[-\epsilon, \epsilon] \times B^{n-1-k} \times S^{k-1}$ via ϕ), with critical values of index less than or equal to $k - 1$ at heights less than $\frac{1}{2}$ and critical values of index greater than or equal to k at heights greater than $\frac{1}{2}$. Now use [Theorem 4.5](#) to construct an indefinite, ordered generic homotopy ζ_t (for $t \in [0, \frac{1}{2} - \delta]$) connecting ζ_0 to $\zeta_{1/2-\delta}$. Then we let $G: X_0 \rightarrow [0, \frac{1}{2} - \delta] \times I$ be $G(t, p) = (t, \zeta_t(p))$, after identifying X_0 with $[0, \frac{1}{2} - \delta] \times M_0$ as above. On H , at first just let $G(x) = \Gamma_k^+(x) + (\frac{1}{2}, \frac{1}{2})$, a forward k -handle. This gives the single vertical tangency as part of a horizontal parabola seen in the middle in [Figure 31](#). This fits together smoothly with the definition of G on X_0 . Now we postcompose with an isotopy of $[\frac{1}{2} - \delta, \frac{1}{2} + \delta] \times [0, 1]$ to make the image of H exactly equal to the square $[\frac{1}{2} - \delta, \frac{1}{2} + \delta] \times [\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]$, so that G as defined on X_0 and H extends smoothly to

$$X_c \cong [\frac{1}{2} - \delta, \frac{1}{2} + \delta] \times (M_0 \setminus (T \setminus \partial T))$$

via $G(t, p) = (t, \zeta_{1/2-\delta}(p))$. One sees then that these definitions fit together smoothly to define G over $\tau^{-1}[0, \frac{1}{2} + \delta]$, and that $z \circ G$ then defines an indefinite Morse function $\zeta_{1/2+\delta}$ on $\tau^{-1}\{\frac{1}{2} + \delta\}$, which is identified with M_1 via V . Finally, we use [Theorem 4.5](#) to construct an indefinite generic homotopy ζ_t (for $t \in [\frac{1}{2} + \delta, 1]$) connecting $\zeta_{1/2+\delta}$ to ζ_1 , and we define G on $X_1 \cong [\frac{1}{2} + \delta, 1] \times M_1$ by $G(t, p) = (t, \zeta_t(p))$.

If we arranged that ζ_t is ordered for $t \in [\frac{1}{2} + \delta, 1]$, we would have completed the proof of the connectedness assertion. This is fine if $\zeta_{1/2+\delta}$ (handed to us by the construction on $X_0 \cup X_c \cup H$) is ordered. There are two different cases when $\zeta_{1/2+\delta}$ will not be ordered. The first is when $k = n/2$. Here we have a critical point of index $n/2$ below a critical point of index $n - 1 - n/2 = n/2 - 1$. The relevant indices are indicated on the left in Figure 32. However, as long as the level sets of $\zeta_{1/2+\delta}$ are connected, we can start off the homotopy ζ_t , $t \in [\frac{1}{2} + \delta, 1]$, by switching the heights of the two offending critical points as indicated on the right in Figure 32. Because $\zeta_{1/2-\delta}$ has critical values of index less than or equal to $k - 1$ below $\frac{1}{2}$ and greater than or equal to k above $\frac{1}{2}$, $\zeta_{1/2+\delta}$ is now ordered. The only case in which the level sets might be disconnected in the short space when ζ_t is not ordered is when $n/2 = n - 2$ and $n/2 - 1 = 1$, ie when $n = 4$ and $k = 2$. In this case we should make sure, in $\zeta_{1/2-\delta}$, the attaching S^1 for the 2–handle H does not separate the level set in which it lies. A moment’s thought about the proof of Theorem 4.2 shows this is easy to achieve.

The second case when $\zeta_{1/2+\delta}$ will not be ordered is when $n - 1 - k > k$, in which case the upper of the two new critical points, which has index $n - 1 - k$, may have critical points of lower index above it. In this case start off the homotopy ζ_t , $t \in [\frac{1}{2} + \delta, 1]$, by lifting this critical point above those of lower index, and then proceed as above. The only case when this could conceivably cause any connectedness problems is when $k = 1$ and $n - k - 1 = n - 2$, but adding a 1–handle and its dual $(n - 2)$ –handle to a fiber that is already connected cannot disconnect the fiber. \square

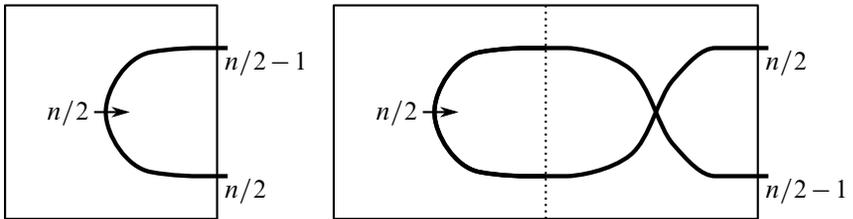


Figure 32: After attaching a handle of index $n/2$, the vertical Morse function will not be ordered, as on the left. The arrow labeled $n/2$ indicates that this fold is of index $n/2$ when looked at in the direction of the arrow. The numbers to the right of each box are the index of the critical points of the vertical Morse function there. In the box on the right, we have switched the two critical points to restore order.

The rest of this section is devoted to proving uniqueness, ie the proof of Theorem 5.2. To do this, we first show that our two models, forward and backward handles, are a complete list of local models, in the following sense:

Lemma 5.3 Consider a square Morse 2–function $G: X \rightarrow I^2$ and a critical point $p \in X$ of $t \circ G: X \rightarrow I$, of index k , with $G(p) = (t_p, z_p)$. Suppose we are given standard coordinates (x_1, \dots, x_n) on a neighborhood ν of p such that $\tau = t \circ G(x_1, \dots, x_n) = \mu_k^n(\mathbf{x}) + t_p$. Then there exists an arc G_s of Morse 2–functions (ie G_s is Morse for all s) supported inside ν , with $G_0 = G$ and $t \circ G_s$ independent of s , such that, inside a smaller neighborhood $\nu' \subset \nu$, $G_1(\mathbf{x}) = \Gamma_k^\pm(\mathbf{x}) + (t_p, z_p)$. It will be Γ_k^+ , ie a forward k –handle, exactly when the point (t_p, z_p) is a local minimum for $t|_{G(Z_G)}$, and it will be Γ_k^- , a backward k –handle, exactly when (t_p, z_p) is a local maximum.

Proof Let $\tau = t \circ G$ and $\zeta = z \circ G$, ie $G(\mathbf{x}) = (\tau(\mathbf{x}), \zeta(\mathbf{x}))$. We know $\tau = \mu_k^n(\mathbf{x}) + t_p$. Because the rank of DG at $p = 0$ must be 1, we know that ζ is nonsingular at 0 and so, after a small perturbation of ζ (keeping G Morse) supported in a neighborhood of 0 we can assume that ζ is linear of the form $\zeta(x_1, \dots, x_n) = a_1x_1 + \dots + a_nx_n + z_p$. Now note that $-a_1^2 - \dots - a_k^2 + a_{k+1}^2 + \dots + a_n^2 \neq 0$, because otherwise we would have a 1–dimensional subset of the singular locus of G on which G was constant, which does not happen in any of the local models for a Morse 2–function. Thus there is a linear transformation of \mathbb{R}^n preserving the quadratic form μ_k^n and taking (a_1, \dots, a_n) to $(0, \dots, 0, c, 0, \dots, 0)$ for some positive c , with the c in either the k^{th} or $(k + 1)^{\text{st}}$ coordinate. Smoothly interpolate from the identity to this linear transformation while going in radially towards the origin in our neighborhood, and this creates an ambient isotopy of the domain. Precomposing with this ambient isotopy and postcomposing with a rescaling isotopy gives the arc of Morse 2–functions G_s with the desired properties. Because we preserved the quadratic form μ_k^n , we didn’t change τ in this isotopy, and thus $t \circ G_s$ is independent of s . □

At some point one might wish that a certain forward index- k critical point was actually a backward index- k critical point, or vice versa. The next lemma addresses this:

Lemma 5.4 Suppose that $G: X \rightarrow I^2$ is a square Morse 2–function and that $p \in G$ is a critical point of $t \circ G$ of index k with local coordinates with respect to which $G(\mathbf{x}) = \Gamma_k^\pm(\mathbf{x}) + (t_p, z_p)$. If $2 \leq k \leq n - 2$ there exists an indefinite generic homotopy G_s of Morse 2–functions supported inside this coordinate neighborhood, with $G_0 = G$, such that, inside a smaller neighborhood of p , we have $G_1(\mathbf{x}) = \Gamma_k^\mp(\mathbf{x}) + (t_p, z_p)$. If all fibers of G are connected then we can arrange that all fibers of G_s are connected for all s . We can further arrange that $t \circ G_s$ is independent of s .

Proof Figure 33 shows how to do this (without the indefinite condition) when $n = 2$ and $k = 1$. For this we should in principle be able to write down an explicit formula,

but the illustration probably explains what is going on better. We know that a forward handle has become a backward handle simply because the vertical tangency in the fold locus has changed from being a rightward-opening parabola to a leftward-opening parabola. The homotopy has passed through a swallowtail singularity. To get the higher-dimensional version, and indefiniteness, consider Figure 33 to be a picture of what is happening in the (x_k, x_{k+1}) -plane, and keep the homotopy independent of s in the other coordinates (x_1, \dots, x_{k-1}) and (x_{k+2}, \dots, x_n) . It is easy to see that, for $n \geq 4$ and $2 \leq k \leq n - 2$, this is an indefinite homotopy and does not disconnect fibers. The way we have drawn it, the critical point moves slightly to the left, but after modifying by a small isotopy we can keep $t \circ G_s$ independent of s throughout the homotopy. □

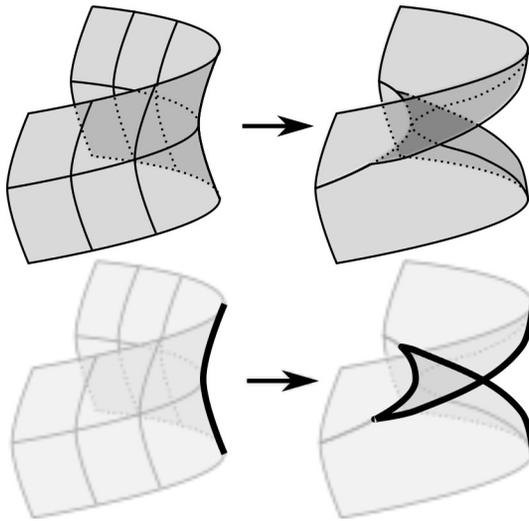


Figure 33: Bending a forward handle to a backward handle: the point is to see that the only change to the projection to the horizontal axis in this homotopy is that the critical point moves a little to the left, but that when we look at the map to 2 dimensions, ie projection to the page exactly as in this figure, the fold at the critical point changes from opening to the right to opening to the left. This figure is in fact nothing more than the standard homotopy that introduces a swallowtail.

We will want to accompany the lemmas above with a lemma stating that, outside the standard neighborhood of a critical point of $t \circ G$, G can be taken to be “constant in the t direction”. We state this precisely as follows:

Lemma 5.5 Consider a square Morse 2-function $G: X \rightarrow I^2$ and an index- k critical point $p \in X$ of $\tau = t \circ G$, with local coordinates near p with respect to which

$G = \Gamma_k^+(\mathbf{x}) + (t_p, z_p)$. In these local coordinates, let V be the standard gradient vector field for $\tau = t \circ G = \mu_k^n(\mathbf{x}) + t_p$ and, for any small $\epsilon > 0$, let H_ϵ be the k -handle neighborhood of p obtained by flowing forward along V starting from the attaching region $\{(x_1, \dots, x_n) \mid \tau(x_1, \dots, x_n) = -\epsilon + t_p, x_{k+1}^2 \leq \epsilon^2, x_{k+2}^2 + \dots + x_n^2 \leq \epsilon^2\} \cong S^{k-1} \times [-\epsilon, \epsilon] \times B^{n-k-1}$, together with the ascending disk, and stopping at $\tau = t_p + \epsilon$. Note that $H_\epsilon \subset \tau^{-1}[t_p - \epsilon, t_p + \epsilon]$; let H_ϵ^c be the closure of $\tau^{-1}[t_p - \epsilon, t_p + \epsilon] \setminus H_\epsilon$ and $M_{t_p - \epsilon}^c = H_\epsilon^c \cap \tau^{-1}(t_p - \epsilon)$. Using V , H_ϵ^c is naturally identified with $[t_p - \epsilon, t_p + \epsilon] \times M_{t_p - \epsilon}^c$. Then there exists an $\epsilon > 0$ and an arc of Morse 2-functions G_s , with $G_0 = G$, which is independent of s inside H_ϵ and independent of s outside $\tau^{-1}(t_p - 2\epsilon, t_p + 2\epsilon)$, such that, on H_ϵ^c , identified with $[t_p - \epsilon, t_p + \epsilon] \times M_{t_p - \epsilon}^c$, G_1 is of the form $(t, x) \mapsto (t, g(x))$ for a fixed Morse function $g = \zeta|_{\tau^{-1}(t_p - \epsilon)}$. In addition, we can arrange that there are no critical values of $g|_{M_{t_p - \epsilon}^c}$ in $[z_p - \epsilon, z_p + \epsilon]$.

Proof Given the standard model on the handle, the complement of the handle in $\tau^{-1}([t_p - 2\epsilon, t_p + 2\epsilon])$ is a product cobordism and G on this product is identified with an arc of Morse functions g_t . A standard homotopy can make g_t independent of t for $t \in [t_p - \epsilon, t_p + \epsilon]$. □

We now present the proof of uniqueness for indefinite square Morse 2-functions as a sequence of steps forward-referencing two further lemmas which will be stated and proved afterwards.

Proof of Theorem 5.2 We are given two indefinite square Morse 2-functions

$$G_0, G_1: X \rightarrow I^2$$

which agree on M_0 and M_1 , ie $z \circ G_0|_{M_i} = \zeta_i = z \circ G_1|_{M_i}$, for $i = 0, 1$. Then we need to construct an indefinite generic homotopy $G_s: X \rightarrow I^2$ between G_0 and G_1 , and we need to address the issue of connected fibers. The steps are as follows.

- (1) Let $\tau_0 = t \circ G_0$ and $\tau_1 = t \circ G_1$. These are indefinite I -valued Morse functions, and we are given that they are ordered. Let $\tau_s: X \rightarrow I$ be an indefinite, ordered generic homotopy from τ_0 to τ_1 such that all the births of cancelling pairs of critical points occur for $s \in [0, \frac{1}{4}]$ and all the deaths occur for $s \in [\frac{3}{4}, 1]$, and such that τ_s is independent of s for all $s \in [\frac{1}{4}, \frac{3}{4}]$. We then construct the desired generic homotopy G_s for $s \in [0, \frac{1}{4}]$ and $s \in [\frac{3}{4}, 1]$ such that $t \circ G_s = \tau_s$. (We need to slightly modify τ_s to achieve this.) This step is carried out in [Lemma 5.6](#). The key outcome of this step is $t \circ G_{1/4} = t \circ G_{3/4}$, so when we construct G_s for $s \in [\frac{1}{4}, \frac{3}{4}]$, we can leave $t \circ G_s$ independent of s and work on $z \circ G_s$.

- (2) Now we need to connect $G_{1/4}$ to $G_{3/4}$. Let $\tau = t \circ G_{1/4} = t \circ G_{3/4}$. Our next step is to extend the homotopy G_s to $s \in [\frac{1}{4}, \frac{1}{2}]$, keeping $t \circ G_s = \tau$ for all $s \in [\frac{1}{4}, \frac{1}{2}]$ so that, for some $\epsilon > 0$ and for each critical value t_* of τ , $G_{1/2}$ and $G_{3/4}$ agree on $\tau^{-1}([t_* - \epsilon, t_* + \epsilon]) = G_{1/2}^{-1}([t_* - \epsilon, t_* + \epsilon] \times I) = G_{3/4}^{-1}([t_* - \epsilon, t_* + \epsilon] \times I)$. This step is carried out in [Lemma 5.8](#).
- (3) Finally, we extend G_s to $s \in [\frac{1}{2}, \frac{3}{4}]$ to connect $G_{1/2}$ to $G_{3/4}$ as follows. The parts of X where $G_{1/2}$ and $G_{3/4}$ do not yet agree are of the form $X_* = \tau^{-1}([t_* + \epsilon, t'_* - \epsilon])$, for two consecutive critical values $t_* < t'_*$ of τ . But then X_* can be identified with a product $[t_* + \epsilon, t'_* - \epsilon] \times M_*$, where $M_* = \tau^{-1}(t_* + \epsilon)$. Furthermore, with this identification, for $s = \frac{1}{2}$ and $s = \frac{3}{4}$, we see that $G_s|_{X_*}$ is of the form $(t, p) \mapsto (t, g_{s,t}(p))$, precisely because $t \circ G_{1/2} = t \circ G_{3/4} = \tau$. Thus we can use [Theorem 4.9](#) from the preceding section to construct a homotopy $g_{s,t}$ from $g_{1/2,t}$ to $g_{3/4,t}$, and then define G_s for $s \in [\frac{1}{2}, \frac{3}{4}]$ and $p \in X_* = [t_* + \epsilon, t'_* - \epsilon] \times M_*$ by $G_s(t, p) = (t, g_{s,t}(p))$. (Here we actually need a parameterized version of [Lemma 4.10](#), which is discussed in a similar context in the proof of [Lemma 5.7](#).) Finally, since $G_{1/2}$ and $G_{3/4}$ already agree on $\tau^{-1}([t_* - \epsilon, t_* + \epsilon])$ for critical points t_* , we can define $G_s = G_{1/2} = G_{3/4}$ on these subsets, for all $s \in [\frac{1}{2}, \frac{3}{4}]$, and we are done.

In the above steps we did not address the issue of keeping fibers connected when G_0 and G_1 are fiber-connected. We have already arranged for τ_s to be ordered for all s . In this case, [Lemma 5.6](#) also states that G_s will have all fibers connected for all $s \in [0, \frac{1}{4}]$ and for all $s \in [\frac{3}{4}, 1]$. [Lemma 5.8](#) then explicitly asserts that, if the fibers of $G_{1/4}$ and $G_{3/4}$ are connected in each $\tau^{-1}([t_* - \epsilon, t_* + \epsilon])$, for t_* a critical value of τ , then we can keep the fibers of G_s connected there when we construct G_s for $s \in [\frac{1}{4}, \frac{1}{2}]$. Finally, when we use [Theorem 4.9](#) to construct G_s for $s \in [\frac{1}{2}, \frac{3}{4}]$, we should first use [Lemma 4.10](#) to get each of the Morse functions $g_{1/2,t_*+\epsilon} = g_{3/4,t_*+\epsilon}$ and $g_{1/2,t'_*-\epsilon} = g_{3/4,t'_*-\epsilon}$ to be ordered. □

We now state and prove the two lemmas referenced in the proof above.

Lemma 5.6 *Given any indefinite Morse 2–function $G: X \rightarrow I^2$ and an indefinite, ordered generic homotopy $\tau_s: X \rightarrow I$ between Morse functions, with $t \circ G = \tau_0$ and with no deaths of cancelling pairs of critical points, there exists an indefinite, ordered generic homotopy $\tau'_s: X \rightarrow I$, with $\tau'_0 = \tau_0$ and $\tau'_1 = \tau_1$, which is connected to τ_s by an arc of generic homotopies, and there exists an indefinite generic homotopy of Morse 2–functions $G_s: X \rightarrow I^2$ with $G_0 = G$ and with $t \circ G_s = \tau'_s$. If G is fiber-connected then we can arrange that G_s is fiber-connected for all s .*

Proof of Lemma 5.6 We will show how to construct generic indefinite homotopies G_s such that $t \circ G_s$ is a generic homotopy between Morse functions which realizes either (1) a desired birth of a cancelling pair or (2) a desired crossing of two critical points. The given τ_s will then tell us where the births should be and which critical points should cross when. Making the births occur at these points and the right critical points cross at the right time, and pre- and postcomposing G with ambient isotopies, we can construct G_s so that $\tau'_s = t \circ G_s$ is connected to τ_s by an arc of generic homotopies. (To see this, first note that an arc of Morse functions can always be realized by pre- and postcomposing with ambient isotopies. This is because we can postcompose with an isotopy so that the critical values are constant, then precompose so that the critical points are constant, then precompose again to arrange that the homotopy is constant on neighborhoods of the critical points, and finally integrate a time-like vector field to get the full isotopy. Then note that births and critical point crossings can be localized by using bump functions to keep given homotopies stationary for short time periods outside standard neighborhoods.)

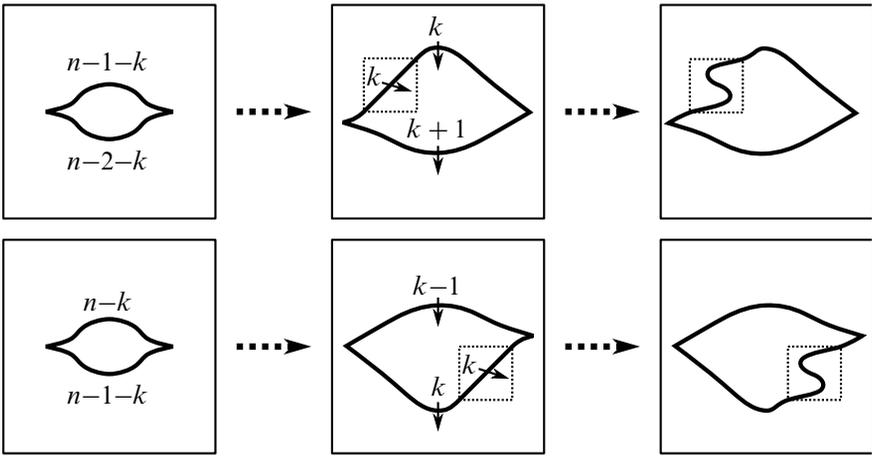


Figure 34: Two ways to realize a birth of a cancelling $k-(k+1)$ pair in τ_s via a homotopy of G : the birth occurs inside the dotted square.

We deal with these two moves as follows.

- (1) The easiest way to arrange a birth is to arrange that, inside the ball in which the birth should occur, G has the form $G(x_1, \dots, x_n) = (-x_1^2 - \dots - x_k^2 + x_{k+1} + x_{k+2}^2 + \dots + x_n^2, x_{k+1})$. This is a fold which is index k looked at from left to right and the image of the fold set is the line $z = t$. We can arrange this via, for example, an eye birth as illustrated in Figure 34; there are two cases, one which is indefinite for $1 \leq k \leq n-3$ and one which is indefinite for $2 \leq k \leq n-2$.

Now let $f_s(x)$ be a function which equals $x^3 - sx$ for x in a neighborhood of 0, has no critical points outside that neighborhood for any $s \in [-\epsilon, \epsilon]$, and is linear in x and independent of s outside a slightly larger neighborhood. Finally let $G_s(x_1, \dots, x_n) = (-x_1^2 - \dots - x_k^2 + f_s(x_{k+1}) + x_{k+2}^2 + \dots + x_n^2, x_{k+1})$, the result of which is also illustrated in Figure 34. As long as this eye birth is indefinite, it will not disconnect fibers.

- (2) Since τ_s is ordered, we only need to switch critical values of the same index. Lemma 5.7 below deals with more general critical values switches, but applies in particular to this case. (We have separated that lemma from the proof of this lemma because we will need the more general case in the following section.)

This completes the proof. □

Lemma 5.7 *Given an indefinite Morse 2–function $G: X \rightarrow I^2$, let $\tau = t \circ G$. Consider two critical points p and q of τ , with p having index $k \leq n/2$, with q having index $l \geq k$, with $\tau(q) < \tau(p)$ and with no other critical values between $\tau(q)$ and $\tau(p)$. Fix a generic gradient-like vector field for τ . Then there exists an indefinite generic homotopy G_s with $G_0 = G$ such that $\tau_s = t \circ G_s$ agrees with τ outside a neighborhood of the descending manifold for p and, inside this neighborhood, no new critical points are born and $\tau_s(p)$ decreases monotonically from $\tau(p)$ to $\tau(q) - \epsilon$ for some small $\epsilon > 0$. Furthermore, if G is fiber-connected then we can arrange that G_s is fiber-connected for all s .*

Proof Suppose that $\tau(p) = \frac{2}{3}$ and $\tau(q) = \frac{1}{3}$ and that there are no other critical values between $\frac{1}{3}$ and $\frac{2}{3}$. Using Lemmas 5.3 and 5.4, we arrange for local coordinates near p with respect to which G has the form $G = \Gamma_k^+(\mathbf{x}) + (\frac{2}{3}, z_p)$, a forward k –handle, and we arrange for a disjoint coordinate system near q with respect to which $G = \Gamma_l^\pm(\mathbf{x}) + (\frac{1}{3}, z_q)$, a forward or backward l –handle. We further use Lemma 5.5 to arrange that G is constant in the t direction outside a neighborhood of q for $t \in [\frac{1}{3} - \epsilon, \frac{1}{3} + \epsilon]$. First we consider the case that q is a backward l –handle.

Since there are no critical points between p and q , $\tau^{-1}([\frac{1}{3} + \epsilon, \frac{2}{3} - \epsilon])$ is diffeomorphic to $[\frac{1}{3} + \epsilon, \frac{2}{3} - \epsilon] \times M$ for an $(n - 1)$ –manifold M (each component of which is a connected cobordism between nonempty manifolds), and via this diffeomorphism G has the form $(t, p) \mapsto (t, g_t(p))$ where g_t is an indefinite generic homotopy between Morse functions on M . Furthermore, because of the local models we have found for G near p and q , we have an embedding ϕ_q of a neighborhood of the ascending sphere for q and an embedding ϕ_p of a neighborhood of the descending sphere for p in M such that $g_{1/3+\epsilon}$ is standard with respect to ϕ_q at height z_q while $g_{2/3-\epsilon}$ is standard with respect to ϕ_p at height z_p .

We will now sequentially improve g_t in preparation for lowering p past q . This is illustrated in Figure 35, where “lowering” p really means moving p to the left. At each stage we produce an improved g_t , which must be connected to the preceding g_t by a homotopy $g_{s,t}$. The homotopy $g_{s,t}$ is produced by appealing to Theorem 4.9. The one hitch here, as pointed out helpfully by our anonymous referee, is that Theorem 4.9 requires ordered Morse homotopies as input, and at certain stages our two Morse homotopies may have connected level sets without necessarily being ordered. For this, we need a parameterized version of Lemma 4.10. Now we have a given homotopy g_t which has connected level sets and must be made ordered, respecting standardization with respect to ϕ_p and/or ϕ_q (see below). Here we simply pull the arcs of critical points down or up, as appropriate, using exactly the same argument as in last two paragraphs of the proof of Theorem 4.5.

Now here is the sequence of improvements of g_t . First we use Lemmas 4.10 and 5.5 and Theorems 4.2 and 4.5 to arrange that g_t is ordered and standard with respect to ϕ_p at height z_p for t near $\frac{1}{3}$. Similarly we arrange for g_t to be ordered near $\frac{2}{3}$, keeping it standard with respect to ϕ_p at height z_p near $\frac{2}{3}$. (Note the importance of the t -independence of G outside a neighborhood of q , as given by Lemma 5.5, so that we can smoothly connect the behavior of $g_{1/3+\epsilon}$ to the behavior of $g_{1/3-\epsilon}$, and similarly for the $g_{2/3-\epsilon}$ and $g_{2/3+\epsilon}$.) Then we use Theorem 4.5 to arrange that g_t is standard with respect to ϕ_p at height z_p (and ordered) for all intermediate values of t . The argument from the preceding paragraph connects these improved g_t 's by the appropriate homotopies of homotopies of Morse functions, which turn into homotopies of Morse 2-functions. Finally, having arranged the standardness of g_t for the point p on the entire interval $[\frac{1}{3} - \epsilon, \frac{2}{3} + \epsilon]$, we can easily lower p past q .

In the case that q is a forward l -handle, we use the same argument as above but look first at $M' = \tau^{-1}(\frac{1}{3} - \epsilon)$, with a vertical Morse function $g_{1/3-\epsilon}: M' \rightarrow I$ which is standard with respect to the descending manifold for q at height z_q . We modify this to be standard with respect to the descending manifolds for both p and q , using the t -independence of G outside a neighborhood of q for $t \in [\frac{1}{3} - \epsilon, \frac{1}{3} + \epsilon]$, so that again we get $g_{1/3+\epsilon}: M \rightarrow I$ to be standard with respect to the descending manifold for p and proceed as above.

Regarding fiber-connectedness, note that the only potential problem arises when the attaching sphere for p has codimension 1 in the fiber, and then we need to make sure that it never separates the fibers as we lower p past q . Thus we only need to worry when $k = n - 2$, but since $k \leq n/2$ and $n \geq 4$, the only case of concern is $n = 4$ and $k = 2$. (To a 4-manifold topologist this is, of course, the most interesting case.) Thus, when we apply Theorems 4.2 and 4.5 and Lemma 4.10 to arrange that g_t is standard with respect to ϕ_p at height z_p for all values of t between $\frac{1}{3}$ and $\frac{2}{3}$, we

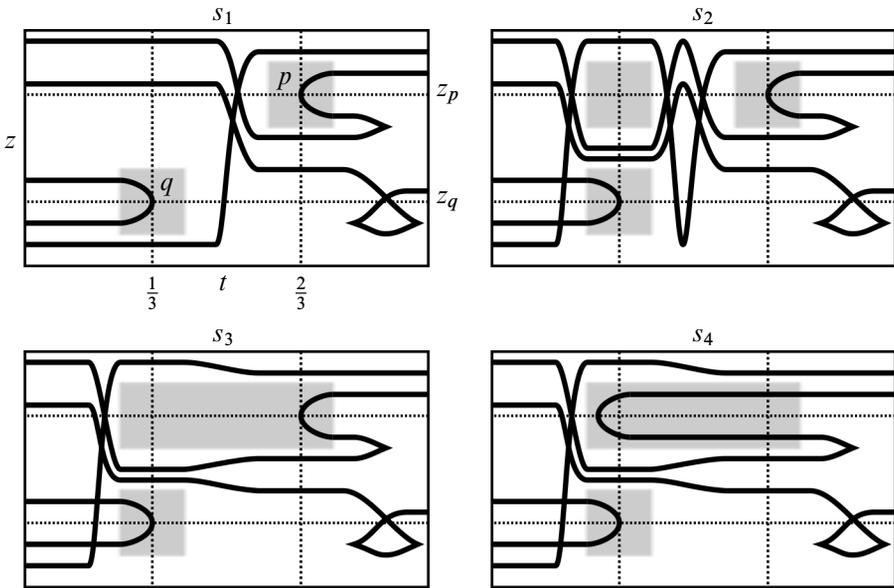


Figure 35: Passing two critical points past each other; the homotopy is illustrated at four successive values of s . Shaded squares are regions where g_t is standard with respect to ϕ_p or ϕ_q . To pass from s_1 to s_2 , we modify g_t in a neighborhood of $t = \frac{1}{3}$ so that the framed attaching sphere for p lies in a level set at $t = \frac{1}{3}$. Going from s_2 to s_3 , we modify g_t between $t = \frac{1}{3}$ and $t = \frac{2}{3}$ so that this attaching sphere lies in a level set on the whole interval $[\frac{1}{3}, \frac{2}{3}]$.

actually need to do a little more. We need to ensure that the descending sphere for p remains nonseparating in its fiber for all t . Thus we need a slight improvement on [Theorem 4.5](#) which says that, in this special case $m = n - 1 = 3$ and $l_i = k - 1 = 1$, we can maintain the nonseparating property throughout a generic homotopy between Morse functions. The easiest way to do this is to arrange that a dual circle to the attaching circle also lies in the fiber throughout the homotopy, ie that the union of two circles intersecting transversely at one point stays in the fiber. Since the proof of the relevant part of [Theorem 4.5](#) simply involves counting dimensions and appealing to transversality, this slight improvement is straightforward. \square

The other lemma needed in the proof of [Theorem 5.2](#) is:

Lemma 5.8 *Given two indefinite Morse 2-functions $G, G': X \rightarrow I^2$ such that $z \circ G|_{M_0} = z \circ G'|_{M_0} = \zeta_0$, $z \circ G|_{M_1} = z \circ G'|_{M_1} = \zeta_1$ and $t \circ G = t \circ G' = \tau: X \rightarrow I$, there exist an $\epsilon > 0$ and a generic indefinite homotopy of Morse 2-functions $G_s: X \rightarrow I^2$ such that:*

- (1) $G_0 = G$.
- (2) $t \circ G_s = \tau$ for all s .
- (3) For each critical value t_* of τ , letting $X_* = \tau^{-1}[t_* - \epsilon, t_* + \epsilon]$, $G_1|_{X_*} = G'|_{X_*}$.

If G and G' have all fibers connected then we can arrange for G_s to have all fibers connected as well.

Proof Again, we use [Lemma 5.3](#) to standardize $G = (\tau, \zeta)$ and $G' = (\tau, \zeta')$ near each critical point of τ , so that ζ and ζ' are equal in a neighborhood of each critical point. Then use [Lemma 5.5](#) to make G and G' “constant in the t direction” inside each $\tau^{-1}[t_* - \epsilon, t_* + \epsilon]$ but away from the critical point. Thus if we homotope $\zeta|_{\tau^{-1}(t_* - \epsilon)}$ to $\zeta'|_{\tau^{-1}(t_* - \epsilon)}$ without changing ζ on the attaching region for the handle associated to this critical point, this homotopy can be spread out over $[t_* - \epsilon, t_* + \epsilon]$ to give the desired homotopy of G . The homotopy from $\zeta|_{\tau^{-1}(t_* - \epsilon)}$ to $\zeta'|_{\tau^{-1}(t_* - \epsilon)}$ is given by [Lemma 4.10](#) followed by [Theorem 4.5](#). □

6 The main results

In this section we prove the theorems stated in the introduction. We will apply Thom–Pontrjagin type arguments to reduce to the case of maps to disks, so first we show quickly how the results of the preceding two sections immediately give our main theorems in the case when we are mapping to B^1 or B^2 .

Proof of Theorem 1.3 for maps to B^1 This is exactly [Theorem 4.2](#), with ordered implying fiber-connected when $m > 2$. □

Proof of Theorem 1.4 for maps to B^1 Given two indefinite Morse functions

$$g_0, g_1: M^m \rightarrow B^1,$$

we need to construct an indefinite generic homotopy g_t connecting them, and if $m \geq 3$ and g_0 and g_1 are fiber-connected we need to arrange that g_t is fiber-connected. Apply [Lemma 4.10](#) to homotope g_0 and g_1 to be ordered, without destroying fiber-connectedness if they are given as fiber-connected. Then apply [Theorem 4.5](#). □

Proof of Theorem 1.1 for maps to B^2 Here we are given an indefinite Morse function $g: \partial X^n \rightarrow S^1$ and we need to construct an indefinite Morse 2–function $G: X \rightarrow B^2$ with $G|_{\partial X} = g$. When $n \geq 4$ and g is fiber-connected we need to arrange that G is fiber-connected. Arbitrarily identify S^1 with ∂I^2 so that g has no critical

values in $I \times \{0, 1\} \subset \partial I^2$, and then use this identification to realize X as a cobordism from $M_0 = g^{-1}(\{0\} \times I)$ to $M_1 = g^{-1}(\{1\} \times I)$. Apply [Lemma 4.10](#) to begin the construction of G on a collar neighborhood of ∂X so as to reduce to the case where g is ordered on M_0 and M_1 , without destroying fiber-connectedness if we had it to begin with. This gives us ζ_0 and ζ_1 as in the preceding section, in preparation for [Theorem 5.1](#). Also apply [Theorem 4.2](#) to X to produce the desired indefinite, ordered Morse function τ . Finally apply [Theorem 5.1](#) to produce G . \square

Proof of [Theorem 1.2](#) for maps to B^2 Now we are given two indefinite Morse 2–functions $G_0, G_1: X^n \rightarrow B^2$ which agree on ∂X , and, assuming that $n \geq 4$, we need to construct an indefinite generic homotopy G_s connecting them. When G_0 and G_1 are fiber-connected we need to arrange that G_s is fiber-connected. In order to reduce to [Theorem 5.2](#), we first proceed as in the previous proof, identifying B^2 with I^2 and applying [Lemma 4.10](#) to get ζ_0 and ζ_1 ordered as in [Theorem 5.2](#). Now we need to arrange that $\tau_0 = t \circ G_0$ and $\tau_1 = t \circ G_1$ are ordered. We do this with [Lemma 5.7](#), switching critical values so as to order them by index, making sure to always move critical points of index less than or equal to $n/2$ to the left and critical points of index greater than $n/2$ to the right. Now we can apply [Theorem 5.2](#). \square

Now we can consider more interesting topology in our target spaces. For existence of indefinite, fiber-connected S^1 –valued Morse functions, for example, we need to arrange for one fiber to be connected, and then we can cut open along that fiber and reduce to the B^1 –valued case. These proofs are essentially a refinement of the Thom–Pontrjagin construction, in which maps to S^n and homotopies between them are constructed and characterized by specifying the fiber over a point. However, we need to be more careful than in the basic Thom–Pontrjagin construction because of the fact that we need to arrange for connected fibers, and because we want to construct homotopies which connect fibers without introducing extraneous components along the way.

In the traditional Thom–Pontrjagin construction for maps from S^n to S^k , a framed codimension- k submanifold of S^n determines a map by sending the submanifold to the north pole and the framed normal k –disk bundle to S^k , with boundary and the complement of the normal disk bundles all mapping to the south pole. A framed cobordism in $I \times S^n$ between two such framed submanifolds determines a homotopy between their maps in the same way.

In our setting, the framed cobordisms are easy to see but we require more control on the associated homotopies than we would get by applying the standard Thom–Pontrjagin arguments. In particular, if we want to construct a homotopy which connects two components of a disconnected fiber, we will need to choose an arc connecting the

components and then construct a homotopy with support in a neighborhood of the arc, such that, during the homotopy, extraneous components are not introduced.

We begin with the existence and uniqueness results for indefinite, fiber-connected, Morse functions. As mentioned earlier, we already have these results when mapping to I , so we focus now on maps to S^1 .

Proposition 6.1 *Given a Morse function $g: M^m \rightarrow S^1$ which is surjective on π_1 , with M connected, and given a regular value $q \in S^1$, if $g^{-1}(q)$ is disconnected then we can find two components of $g^{-1}(q)$ connected by an arc $a \subset M$ which intersects $g^{-1}(q)$ only at its endpoints and which is disjoint from either $g^{-1}[q - \epsilon, q)$ or $g^{-1}(q, q + \epsilon]$, for some small $\epsilon > 0$.*

Proof Begin with an arc connecting any two components of $g^{-1}(q)$, meeting $g^{-1}(q)$ transversely, intersecting $g^{-1}(q)$ with opposite signs at its endpoints, and projecting to a homotopically trivial loop in S^1 . (Here we use π_1 -surjectivity.) If this arc hits $g^{-1}(q)$ anywhere in its interior, either take an innermost arc intersecting distinct components with opposite signs at its endpoints, or, if an innermost arc starts and ends in the same component, shortcut in the obvious way, reducing the number of intersections. □

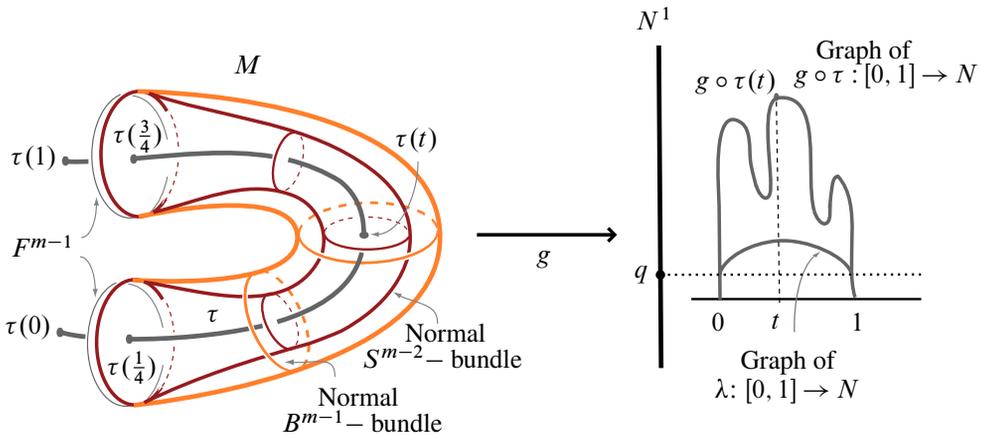


Figure 36: Connecting components of $g^{-1}(q)$

We will use such arcs to construct homotopies that connect components of $g^{-1}(q)$. For the basic idea, referring to Figure 36, consider a given Morse function $g: M^m \rightarrow N^1$, with regular value q and suppose the fiber $F = g^{-1}(q)$ is disconnected. Choose an arc $\tau: [0, 1] \rightarrow M$ transverse to F , intersecting F only at $\tau(\frac{1}{4})$ and $\tau(\frac{3}{4})$, connecting two components of F . (If τ intersects F at more points, choose a shorter arc.) Note that τ has a normal B^{m-1} -bundle with boundary S^{m-2} -bundle. We wish to construct

a homotopy g_t starting at $g_0 = g$, supported inside this B^{m-1} -bundle, such that, watching the level sets $g_t^{-1}(q)$ as t ranges from 0 to 1, we see two fingers stick out from F following τ , meeting at $\tau(\frac{1}{2})$, and merging to achieve a surgery on the 0-sphere $\{\tau(\frac{1}{4}), \tau(\frac{3}{4})\}$. In other words, we connect the two components of F with a tube along τ , without introducing any extraneous components during the homotopy.

To visualize the homotopy g_t , choose coordinates for the normal B^{m-1} -bundle to the arc $\tau([0, 1])$ so that g is projection on the t coordinate, that is, each normal B^{m-1} maps to a point of N . Then g_t should equal g_0 on the boundary S^{m-2} of each normal B^{m-1} and should push the interior of this B^{m-1} down N until the center of B^{m-1} has gone past q . If the center is below q , then a normal S^{m-2} will map to q , and these spheres will make up the cylinder $I \times S^{m-2}$ which connects the two components of the fiber.

In order not to create extra components during the homotopy g_t , the interiors of the normal B^{m-1} 's need to be pushed across q in turn, starting monotonically at $t = \frac{1}{4}$ and $t = \frac{3}{4}$, and finally at $t = \frac{1}{2}$. To organize this, choose a parabolic arc λ as drawn in Figure 36 and first push all the centers of the normal B^{m-1} 's, ie $\tau(t)$, down to λ , and then push the centers down as though λ was being translated down.

The details are as follows:

Lemma 6.2 *Suppose that $g: M^m \rightarrow N^1$ is a Morse function and that $\tau: I \rightarrow M$ is a smooth embedded path avoiding the critical points of g such that $\gamma = g \circ \tau$ is homotopically trivial (rel endpoints) in N . Let $q \in N$ be a regular value for both $g: M \rightarrow N$ and $\gamma: [0, 1] \rightarrow N$, with $\gamma^{-1}(q) = \{\frac{1}{4}, \frac{3}{4}\}$. Suppose that $\Gamma: I \times I \rightarrow N$ is a homotopy, with $\Gamma(0, x) = \gamma(x)$, such that q is regular for Γ and such that $\Gamma^{-1}(q)$ is an arc $a \subset I \times I$ from $(0, \frac{1}{4})$ to $(0, \frac{3}{4})$ and such that projection onto the first factor of $I \times I$ restricts to a as a Morse function with a single index-1 critical point at $(\frac{1}{2}, \frac{1}{2})$. Then, inside any neighborhood of $\tau(I)$ there is an embedding $\bar{\tau}: B^{m-1} \times I \rightarrow M$, with $\bar{\tau}(\mathbf{0}, x) = \tau(x)$ such that, letting $\bar{\gamma} = g \circ \bar{\tau}: B^{m-1} \times I \rightarrow N$, we have a homotopy $\bar{\Gamma}: I \times B^{m-1} \times I \rightarrow N$ satisfying the following properties.*

- (1) For any $(0, \mathbf{y}, x) \in I \times B^{m-1} \times I$, $\bar{\Gamma}(0, \mathbf{y}, x) = \bar{\gamma}(\mathbf{y}, x)$.
- (2) For any $(t, \mathbf{0}, x) \in I \times B^{m-1} \times I$, $\bar{\Gamma}(t, \mathbf{0}, x) = \Gamma(t, x)$.
- (3) For any $(t, \mathbf{y}, x) \in I \times ((S^{m-2} \times I) \cup (B^{m-1} \times \{0, 1\}))$, $\bar{\Gamma}(t, \mathbf{y}, x) = \bar{\Gamma}(0, \mathbf{y}, x) = \bar{\gamma}(\mathbf{y}, x)$.
- (4) The point $q \in N$ is a regular value for $\bar{\Gamma}$.
- (5) $\bar{\Gamma}^{-1}(q)$ is an m -dimensional submanifold of $I \times B^{m-1} \times I$ on which the projection onto the first factor of $I \times B^{m-1} \times I$ has a single Morse critical point of index 1 at $(\frac{1}{2}, \mathbf{0}, \frac{1}{2})$. This is illustrated in Figure 37.

(6) This submanifold $\bar{\Gamma}^{-1}(q)$ intersects the boundary of $I \times B^{m-1} \times I$ as

$$\begin{aligned} (\{0\} \times B^{m-1} \times I) \cap \bar{\Gamma}^{-1}(q) &= \{0\} \times B^{m-1} \times \{\frac{1}{4}, \frac{3}{4}\}, \\ (I \times S^{m-2} \times I) \cap \bar{\Gamma}^{-1}(q) &= I \times S^{m-2} \times \{\frac{1}{4}, \frac{3}{4}\}. \end{aligned}$$

(7) In particular, this implies that $(\{1\} \times B^{m-1} \times I) \cap \bar{\Gamma}^{-1}(q)$ is properly embedded in $\{1\} \times B^{m-1} \times I$ and diffeomorphic to $S^{m-2} \times [\frac{1}{4}, \frac{3}{4}]$, only meeting the boundary of $\{1\} \times B^{m-1} \times I$ at $S^{m-2} \times \{\frac{1}{4}, \frac{3}{4}\}$.

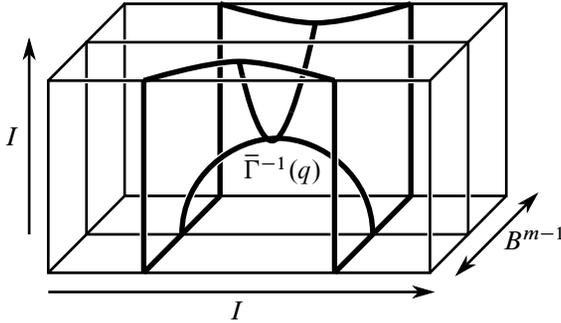


Figure 37: The model for $\bar{\Gamma}^{-1}(q)$ in the statement of Lemma 6.2

Proof Let $\bar{\tau}$ be a parametrization of a small neighborhood of $\tau([0, 1])$ such that $\bar{\tau}^{-1}(q) = B^{m-1} \times \{\frac{1}{4}, \frac{3}{4}\}$. Let H be a model saddle hypersurface in $I \times B^{m-1} \times I$ satisfying the behavior given in the statement of the lemma for $\bar{\Gamma}^{-1}(q)$. Then, the constraints given for $\bar{\Gamma}$, where we ask that $\bar{\Gamma}^{-1}(q) = H$, completely determine $\bar{\Gamma}$ on the following subset C of $I \times B^{m-1} \times I$:

$$C = (\{0\} \times B^{m-1} \times I) \cup (I \times S^{m-2} \times I) \cup H \cup (I \times \{0\} \times I) \cup (I \times B^{m-1} \times \{0, 1\}).$$

There is a natural extension of this prescribed behavior of $\bar{\Gamma}$ on C to a smooth function on a regular neighborhood ν of C so that q becomes a regular value and such that $\bar{\Gamma}(\partial\nu)$ avoids q . Then note that $I \times B^{m-1} \times I$ deformation retracts onto C , so there is a continuous extension of $\bar{\Gamma}$ to all of $I \times B^{m-1} \times I$, which can be smoothed and made generic without changing the behavior on ν , and without introducing any components of $\bar{\Gamma}^{-1}(q)$ outside ν . □

Existence: Proof of Theorem 1.3 We have already proved this for the case of maps to B^1 . Now we consider the case of maps to S^1 . We are given a homotopically nontrivial map $g': M \rightarrow S^1$ and we want to produce an indefinite Morse function $g: M \rightarrow S^1$ which is homotopic to g' . To simplify notation, we will simply refer to all our maps to S^1 as g , modifying g in stages by homotopies. So we begin with $g = g'$.

Because g is homotopically nontrivial, we can lift to a finite cover of S^1 so that the induced map on π_1 is surjective, and thus reduce to the case $g_*(\pi_1(M)) = \pi_1(S^1)$. Fiber-connectedness is not preserved under postcomposition with a covering space map, but we are not asking for fiber-connectedness unless g is π_1 -surjective to begin with.

So now assume g is π_1 -surjective and Morse. Identify S^1 with \mathbb{R}/\mathbb{Z} so that 0 is a regular value. We want to homotope g to arrange that $g^{-1}(0)$ is connected. (The new g should still be Morse and 0 should still be regular.) Then our theorem will reduce to the cobordism case, [Theorem 4.2](#), by cutting M open along $g^{-1}(0)$.

Choose some $\epsilon > 0$ such that there are no critical values of g in $[-\epsilon, 0]$. Choose two components of $g^{-1}(0)$ as in [Proposition 6.1](#), and, extending the arc produced in the proposition, choose a path $\tau: [0, 1] \rightarrow M$ connecting them in M , but starting and ending in $g^{-1}(-\epsilon)$ and passing through the two components at $\tau(\frac{1}{4})$ and $\tau(\frac{3}{4})$, respectively, such that $\gamma = g \circ \tau$ is homotopic to $0 \in \pi_1(S^1, -\epsilon)$. We can then choose a homotopy $\Gamma: I \times I \rightarrow N$ as in the hypotheses of [Lemma 6.2](#), with 0 being the regular value q in the lemma. Then the embedding $\bar{\tau}$ and the homotopy $\bar{\Gamma}$ gives us a homotopy g_t defined as the identity outside the image of $\bar{\tau}$ and as $g_t(p) = \bar{\Gamma}(t, \bar{\tau}^{-1}(p))$ for p in the image of $\bar{\tau}$. The effect on $g^{-1}(0)$ is to replace two B^{m-1} neighborhoods in $g^{-1}(0)$ of the two points $\tau(\frac{1}{4})$ and $\tau(\frac{3}{4})$ with a tube diffeomorphic to $S^{m-2} \times I$, and thus the two components get connected. Repeating, we connect all the components of $g^{-1}(0)$.

Note that the final homotopy is a concatenation of homotopies each supported in a neighborhood of an arc in M . Thus we could just as well have performed all the homotopies at the same time, as long as we can choose all the relevant arcs to be disjoint from the beginning, which we can do if $m \geq 3$. This point of view is more useful in the next proof. \square

Uniqueness: Proof of [Theorem 1.4](#) Again, we have already proved this in the case of homotopies between maps to B^1 , so now we consider the case of homotopies between maps to S^1 . We are given two homotopic indefinite Morse functions $g_0, g_1: M \rightarrow S^1$ and we need to produce an indefinite homotopy g_t between them; when g_0 and g_1 are fiber-connected then g_t should also be fiber connected for all t . Throughout the rest of this proof suppose we are trying to prove both indefiniteness and fiber-connectedness. Assuming that g_0 and g_1 are indefinite and fiber-connected implies that both maps are π_1 -surjective. The proof without asking for fiber-connectedness follows by lifting to an appropriate cover.

Again, identify S^1 with \mathbb{R}/\mathbb{Z} and assume that 0 and $\frac{1}{2}$ are regular values of both g_0 and g_1 . We first choose any generic homotopy g_t , and we will show how to modify g_t

so as to arrange the following connectedness of level sets property: for a sequence of values $0 = t_0 < t_1 < \cdots < t_{2n} = 1$, we have that $g_t^{-1}(0)$ is connected and regular on $[t_0, t_1] \cup [t_2, t_3] \cup \cdots \cup [t_{2n-2}, t_{2n-1}]$ and that $g_t^{-1}(\frac{1}{2})$ is connected and regular on $[t_1, t_2] \cup [t_3, t_4] \cup \cdots \cup [t_{2n-1}, t_{2n}]$.

As in the proof of [Theorem 1.3](#) we can make either of the level sets $g_t^{-1}(0)$ or $g_t^{-1}(\frac{1}{2})$ connected for a fixed t , as long as 0 or $\frac{1}{2}$ is a regular value. The construction depends on a choice of arcs in M based at the level set in question, and then produces a homotopy supported in a neighborhood of those arcs and loops. Even if one of the level sets contains a single Morse critical point, we can keep the arcs and loops away from that critical point, either connecting the entire singular level set if the critical point is indefinite or connecting all the components other than the single point component when the critical point is a minimum or maximum.

Now note that the arcs and the guiding homotopies in S^1 which are used for a fixed value $t = t_0$ will work for all t in some short interval $[t_0 - \epsilon, t_0 + \epsilon]$, modulo modifying the arcs by small ambient isotopies near their endpoints. Also note that, in the middle of one of these homotopies, we do not introduce new components of the level set in question. This is because, in [Lemma 6.2](#), $\bar{\Gamma}^{-1}(q)$ has a single critical point of index 1 with respect to projection onto the first factor of $I \times B^{m-1} \times I$, so for each t , $(\{t\} \times B^{m-1} \times I) \cap \bar{\Gamma}^{-1}(q)$ either has two components (for $t < \frac{1}{2}$) or one component (for $t \geq \frac{1}{2}$), but never more.

Thus, for a fixed level set $g_t^{-1}(z_0)$, as long as z_0 is regular or a Morse critical value for each t , we can cover $[0, 1]$ by a finite collection of such short intervals, using disjoint arcs and loops where the intervals overlap (recall that $m \geq 3$, where m is the dimension of M), and use bump functions to patch together the homotopies over the whole of $[0, 1]$. The result will be that $g_t^{-1}(z_0)$ is connected for all t outside a short interval around each time t_* at which $g_t^{-1}(z_0)$ contains a definite critical point. In these short intervals, $g_t^{-1}(z_0)$ will be connected, say, for all $t < t_*$, will be the disjoint union of a connected manifold and a point for $t = t_*$, will have an isolated sphere component for t slightly larger than t_* , and then that component will get connected back to the rest of the level set immediately thereafter. (The phenomenon described in the preceding sentence could also occur with time reversed, of course.)

Do this once for $g_t^{-1}(0)$. Now do this for $g_t^{-1}(\frac{1}{2})$, noting that generically $\frac{1}{2}$ and 0 will never be critical values at the same time, and also noting that the arcs used to connect components of $g_t^{-1}(\frac{1}{2})$ can be made to avoid $g_t^{-1}(0)$ because $g_t^{-1}(0)$ is now connected or has a single isolated sphere component that dead-ends immediately above or below 0. Because these arcs are disjoint from $g_t^{-1}(0)$, the homotopy used to connect components of $g_t^{-1}(\frac{1}{2})$ does not destroy the connectedness properties of $g_t^{-1}(0)$. We

have also assumed here that non-Morse singularities, ie births and deaths, never occur at 0 or $\frac{1}{2}$.

The zigzag argument We have now achieved the connectedness property advertised: for $0 = t_0 < t_1 < \dots < t_{2n} = 1$, we have that $g_t^{-1}(0)$ is connected and regular on $[t_0, t_1] \cup [t_2, t_3] \cup \dots \cup [t_{2n-2}, t_{2n-1}]$ and $g_t^{-1}(\frac{1}{2})$ is connected and regular on $[t_1, t_2] \cup [t_3, t_4] \cup \dots \cup [t_{2n-1}, t_{2n}]$, as in Figure 38. If g_0 and g_1 agreed on, say, $g_0^{-1}(0) = g_1^{-1}(0)$, then we could cut open M along L and obtain two indefinite Morse functions to I , thus reducing, with the help of Lemma 4.10, to the already proven Theorem 4.5. The goal now is to arrange for this to be true for g_{t_i} and $g_{t_{i+1}}$ for $0 = t_0 < t_1 < \dots < t_{2n} = 1$.

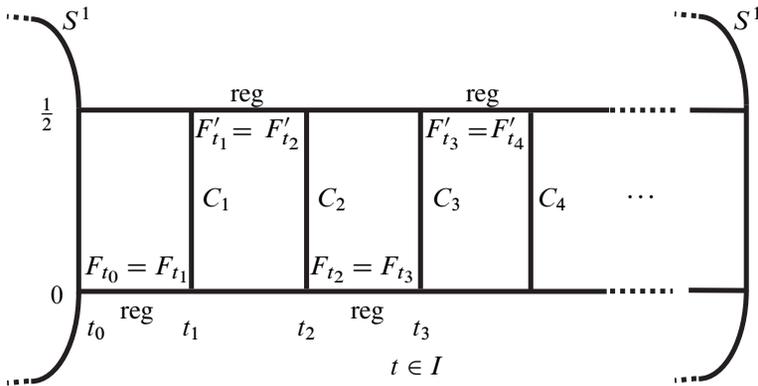


Figure 38: The zigzag argument

Let $F_{t_i} = g_{t_i}^{-1}(0)$ and let $F'_{t_i} = g_{t_i}^{-1}(\frac{1}{2})$. Note that F_{t_0} is isotopic to F_{t_1} because 0 is regular for each g_t , $t \in [t_0, t_1]$. Similarly, $F_{t_{2i}}$ is isotopic to $F_{t_{2i+1}}$. We can assume they are, in fact, equal. The same works for $F'_{t_{2i+1}}$ and $F'_{t_{2i+2}}$.

The vertical cobordism $C_i = g_{t_i}^{-1}[0, \frac{1}{2}]$ between F_{t_i} and F'_{t_i} is connected because the top, $g_{t_i}^{-1}(\frac{1}{2})$, is connected, as is the bottom, and because $g_{t_i}: M \rightarrow S^1$ is surjective on π_1 . (For, given $p, q \in C_i$, there is an arc joining them in M which intersects F_{t_i} and F'_{t_i} algebraically 0 each, and now connectedness of F_{t_i} and F'_{t_i} allows the arc to be replaced by an arc in C_i .)

Now for each $i = 1, \dots, 2n - 1$ we construct, by Theorem 4.2, intermediate functions $h_{t_i}: M \rightarrow S^1$ homotopic to g_{t_i} such that $h_{t_i}^{-1}[0, \frac{1}{2}] = C_i$ and h_{t_i} is indefinite and fiber-connected.

Then g_0 and h_{t_1} agree on the level set $F_{t_0} = g_0^{-1}(0)$ and are homotopic rel this level set and thus are joined by an indefinite fiber-connected homotopy (using Lemma 4.10 and Theorem 4.5). Also h_{t_1} and h_{t_2} agree on the level set $F'_{t_2} = h_{t_2}^{-1}(\frac{1}{2})$ so they are

also joined by an indefinite fiber-connected homotopy. Repeat this up and down zigzag construction up to $h_{t_{2n-1}}$, and then finally join $h_{t_{2n-1}}$ to $g_{t_{2n}} = g_1$. This ends the proof. \square

We have proved [Theorem 1.1](#) when Σ^2 is B^2 , and we could prove the case $\Sigma = S^2$ by homotoping G so that a fiber F is connected and then removing a B^2 -bundle neighborhood of F to reduce to the case $\Sigma = B^2$. However when Σ is closed and not S^2 , there can be problems keeping G an epimorphism on π_1 after removing F . To resolve this issue, we need some constructions which use the following lemma. This is essentially the Morse 2-function version of [Lemma 6.2](#). However, when written as a direct analog of [Lemma 6.2](#), the statement of this lemma becomes unwieldy.

Lemma 6.3 *Let $G: X \rightarrow \Sigma$ be a Morse 2-function. Let a be a nonseparating simple closed curve or properly embedded arc in Σ which meets all folds transversely so that $G^{-1}(a) = M^{n-1}$ is a smooth manifold. Let $\beta: [-1, 2] \rightarrow X$ be a smoothly embedded arc in X which intersects M transversely in two points, $\beta(1)$ and $\beta(0)$, of opposite sign. Let the arc $b: [-1, 2] \rightarrow \Sigma$ be defined as $b = G \circ \beta$. Now suppose that $b|_{[0,1]}$ is homotopic by $h_s, s \in [0, 1]$ to an arc joining $G(\beta(1))$ and $G(\beta(0))$, and call this arc a' .*

Then there exists a homotopy $G_s, s \in [0, 2]$ satisfying:

- (1) $G_0 = G$.
- (2) $G_s = G$ outside of an arbitrarily thin tubular neighborhood of $\beta([-1, 2])$.
- (3) If the homotopy h_s of b never takes b across a except at time $s = 1$, then $G_2^{-1}(a)$ equals M surgered along the 0-sphere, $\beta(1) \cup \beta(0)$, thus connecting the components containing $\beta(1)$ and $\beta(0)$ by a tube $[0, 1] \times S^{n-2}$.
- (4) If h_s does take b across a , then $G_2^{-1}(a)$ is the result of 0-surgery on M as described above together with the possible addition of new closed components.

Proof The argument is a version of the well-known Thom–Pontrjagin method for calculating $\pi_n(S^2)$ by simplifying the preimage of the north pole of S^2 by first connecting its components.

First it is easy to alter h_s so that the end h_1 of the homotopy taking $b([0, 1])$ to a' is a diffeomorphism.

We have that M in X has a trivial normal line bundle which locally is compatible with the normal lines to a in Σ , meaning that the lines map to lines. Then, using this product structure around $\beta(1)$ and $\beta(0)$ and $a \subset \Sigma$, it is easy to extend the homotopy h_s to a homotopy of the full arc $b: [-1, 2] \rightarrow \Sigma$, with the homotopy time parameter s extended from $[0, 1]$ to $[0, 2]$, so that points under $h_s, s \in [0, 2]$ follow the curved lines beginning at b and ending up on a parallel copy of a' called a'' as drawn in [Figure 39](#).

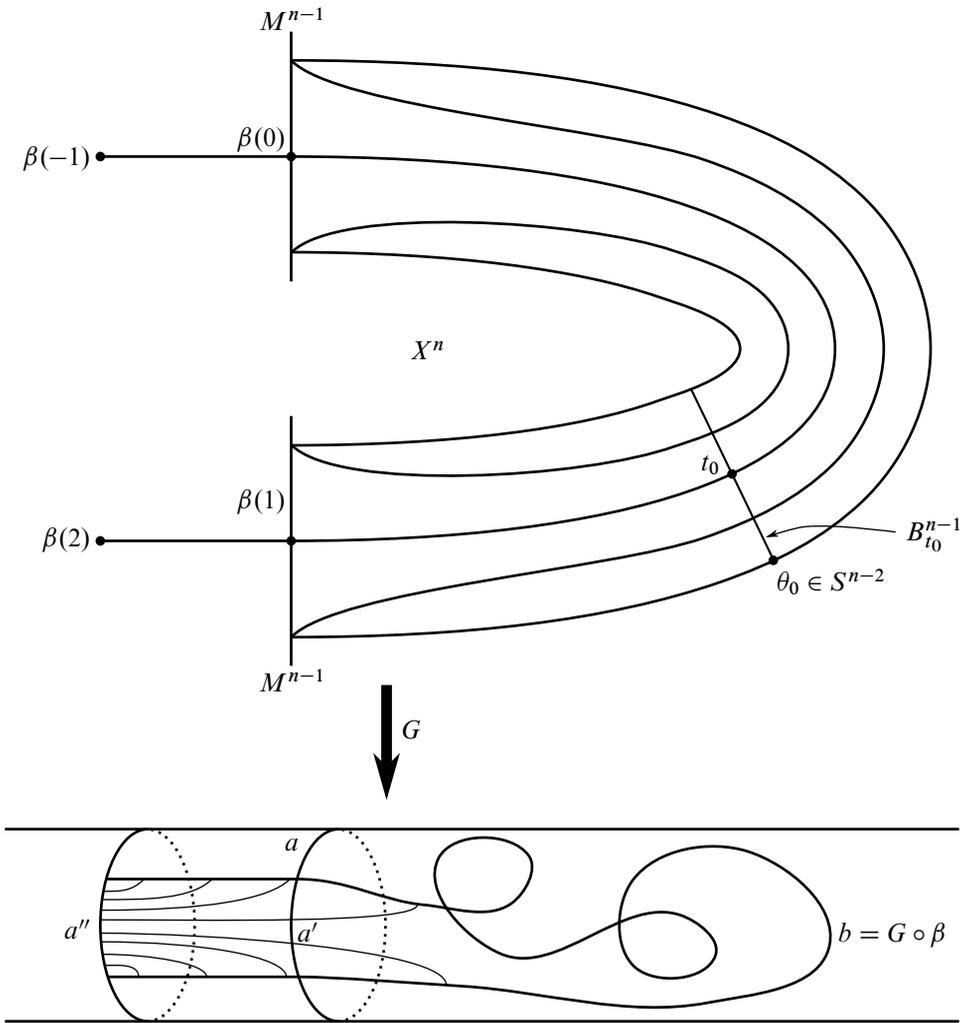


Figure 39: Connecting components of M

The arc β has a tubular neighborhood, $\beta \times B^{n-1}$, and each B^{n-1} has polar coordinates (t, r, θ) where $t \in [-1, 2]$, $r \in [0, 1]$, $\theta \in S^{n-2}$ (see Figure 39). A ray (t_0, r, θ_0) , $r \in [0, 1]$ determined by the pair (t_0, θ_0) is mapped by G to a path $\rho = \rho_{t_0, \theta_0} \subset \Sigma$. The endpoint of ρ at b is moved by the homotopy h_s , $s \in [0, 2]$, along a path ending on a'' ; extend ρ to this longer path $\bar{\rho}$.

The homotopy of G now maps the ray (t_0, r, θ_0) to ρ and as time progresses stretches the ray over more and more of $\bar{\rho}$ until it is onto. It is easy to see that this homotopy is constant on β 's normal sphere bundle, and it is also constant on the top and bottom

of this cylinder because these points in $G^{-1}(a'')$ are also not moved. At the end G_2 is a Morse 2–function on a neighborhood of M (because we chose endpoints of b disjoint from folds and then M and nearby copies change only by surgery on small neighborhoods of the endpoints of β) but may need to be perturbed to make it a Morse 2–function again elsewhere.

The difference between the last two parts of the lemma is fairly evident. The surgery statement follows because neighborhoods of $\beta(1)$ and $\beta(0)$ are pushed off of a' and replaced by a horn shaped cylinder which is mapped to a' , as can be seen in [Figure 39](#). And if the original homotopy h_s does move points across a' , then closed components can be added to M . □

Existence: Proof of Theorem 1.1 Here we are given a compact, connected, oriented n –manifold X and a 2–manifold Σ and an indefinite, surjective Morse function $g: \partial X \rightarrow \partial \Sigma$ which extends to a map $G': X \rightarrow \Sigma$. We wish to homotope G' rel boundary to an indefinite Morse 2–function $G: X \rightarrow \Sigma$, and perhaps also arrange fiber-connectedness. Again, we will drop the primes and simply refer to all our maps to Σ as G , constructing or modifying G in stages.

The base case is when Σ is the disk B^2 , which we have already addressed. We postpone the case $\Sigma = S^2$ briefly until [Remark 6.4](#) below, and now we reduce all other cases to the case $\Sigma = B^2$. The given map G can be homotoped to a Morse 2–function; we need to make it indefinite if $[G_*(\pi_1(X)) : \pi_1(\Sigma)] < \infty$ and fiber-connected if $G_*(\pi_1(X)) = \pi_1(\Sigma)$. If $[G_*(\pi_1(X)) : \pi_1(\Sigma)] < \infty$ we can lift to a finite cover of Σ and reduce to the case that G is π_1 –surjective.

First suppose that Σ and X are closed, in which case there is no boundary condition g . We first arrange that a nice basis for $\pi_1(\Sigma, \sigma_0)$ lifts to X (we only know that the basis is homotopic to one that lifts). We do this as follows:

Describe Σ as a 0–handle, $2g$ 1–handles and a 2–handle in the standard way, with the 1–handles coming in dual pairs. Let a_1, a_2, \dots, a_{2g} be the cores of the 1–handles and $\bar{a}_1, \dots, \bar{a}_{2g}$ be the extensions of these arcs to smooth loops by “coning” their endpoints to the core, σ_0 , of the 0–handle in a smooth way. We can assume that $G^{-1}(\bar{a}_i) = M^{n-1}$ is a manifold.

Focus on a_1 and \bar{a}_1 and drop the subscript for simplicity. \bar{a} is homotopic to a loop \bar{b} which lifts to X , and the homotopy fixes σ_0 and can be made to fix the arc, $\bar{a} - a$, also. This gives an arc b with $\partial a = \partial b$. Let β be the lift of b to X .

We can assume that β meets $M = G^{-1}(\bar{a})$ transversely at $\partial\beta$ and that these two intersection points have opposite signs. Now apply [Lemma 6.3](#) to homotope G so

that a , hence \bar{a} , has a lift α to $M \subset X$. Do the same process for each a_i , noting that the arcs β_i need not intersect, nor their thin neighborhoods. Also note that all the lifts α_i can contain the basepoint of $x_0 \in X$.

Thus we have homotoped G so that each a_i and \bar{a}_i have lifts which we call α_i and $\bar{\alpha}_i$. Now we want to apply [Theorem 1.1](#) to an $M_i = G^{-1}(\bar{a}_i)$, where we know G is an epimorphism on π_1 , but $M = M_i$ may not yet be connected.

We will make $M = M_1$ connected by again using [Lemma 6.3](#). It suffices to show how to homotope G so as to connect two points, p and q in $M = G^{-1}(\bar{a})$, where p belongs to the component of X containing x_0 , without introducing any new components in the process.

For this we need some notation. The two dual curves \bar{a}_1, \bar{a}_2 , define a punctured torus T_0 and its one-point compactification T , and there is a projection from X to Σ to T to $\bar{a}_i, i = 1, 2$, and we name the composition $p_i, i = 1, 2$.

In X , p and q are connected by an arc β which intersects M transversely in some points including p and q . We form a loop γ in Σ by joining the endpoints of $G(\beta)$ by the subarc a' of a which does not contain the basepoint σ_0 . We want to arrange that γ is homotopically trivial in Σ .

First, it may be that $p_1(\gamma)$ is not homotopically trivial, so we connect sum β in M with multiples of the lift of \bar{a}_1 , α_1 , so that it is now homotopically trivial. Recall that p belongs to the component containing $x_0 \in \alpha_1$ so the connect sum is taken near x_0 . And we push the copies of α_1 slightly to one side of M so as to avoid unnecessary intersections.

Next consider whether the new γ projected by p_2 to \bar{a}_2 is homotopically trivial. If not, we connect sum parallel copies of α_2 (which lie in M_2) to arrange triviality. Furthermore we choose these parallel copies so that all their projections to Σ all lie on the same side of the basepoint σ_0 .

Continue in this way with the other α_i 's so as to arrange that γ is homotopically trivial in Σ . Note that the other α_i 's do not intersect $M_1 = M$.

Next we want to arrange that γ not intersect \bar{a}_1 except along a' . For this, consider the universal cover of Σ in which we see "parallel" copies of the lift of \bar{a}_1 , and also a copy of the lift of γ which contains a given lift of the basepoint; see [Figure 40](#). Look for the last (rightmost in [Figure 40](#)) lift of \bar{a}_1 which intersects the lift of γ . Pick a subarc λ of the lift of γ which intersects this lift of \bar{a}_1 in its two endpoints, necessarily of opposite sign. If that subarc λ has endpoints in the same component of M , then connect them in M , changing β and γ by removing these two points of intersection. Proceed until

we get two points in different components. Then λ , projected back to Σ , is extended to a contractible loop, which we again call γ , by adding a segment of a_1 . The subarc λ corresponds to a subarc of β which intersects M only in its endpoints, with opposite sign. Now apply Lemma 6.3 using this subarc of β to connect two distinct components of M , without introducing any new components because the loop γ now lies to one side of a_1 . These steps are iterated to make M connected.

At this point we pause in the proof for a useful remark, needed for the case $\Sigma = S^2$ and for the proof of Theorem 1.2 below.

Remark 6.4 Notice that if the two points $b(0)$ and $b(1)$ which belong to M , actually belong to a single fiber of G , then the construction we have just described connects the two components of that fiber. And if the hypothesis of part 3 of the lemma holds, namely that the homotopy h_t does not move b across a except at time $t = 1$, then no components have been added to the fiber. Thus an iteration of these steps can be used to make a single fiber connected.

When $\Sigma = S^2$, this remark shows us how to homotope G so that a regular fiber is connected, and then removing the fiber cross disk reduces to the case $\Sigma = B^2$.

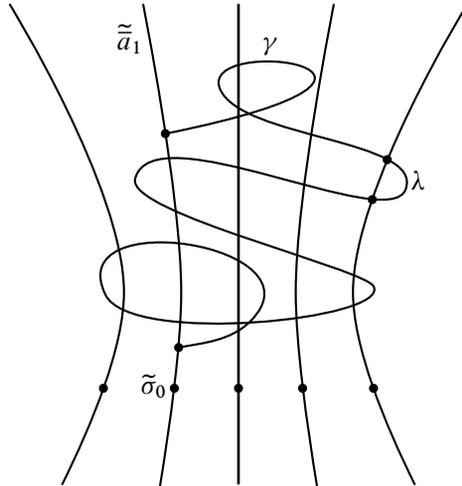


Figure 40: Shortening γ in the universal cover of Σ (the hyperbolic plane or \mathbb{R}^2 if $\Sigma = T^2$)

We now return to the general case $\Sigma \neq S^2$. Now $G|_M: M \rightarrow \bar{a} = \bar{a}_1 = S^1$ is π_1 -surjective and M is connected, so by Theorem 1.3, $G|_M$ can be homotoped to be indefinite with connected fibers. This homotopy extends to X , moving points only in a

thin product neighborhood, $M \times [-1, 1]$ in X . Now we remove $M \times (-1, 1)$ from X and $\bar{a} \times (-1, 1)$ from Σ and reduce to the case $\partial X \neq \emptyset$.

Note that after removing $M \times (-1, 1)$, G is still surjective on π_1 , because we still have the lifts, α_i , of the remaining generators $\bar{a}_i, i \geq 2$ of the fundamental group of $\Sigma \setminus \bar{a}_1$.

The proof of the case in which $\partial\Sigma \neq \emptyset$, assuming only that π_1 is onto and that $G|_{\partial X}$ is indefinite with connected fibers, proceeds nearly identically to the case just done. Again we describe Σ as a 0–handle and some 1–handles, and arrange that the loops defined by the cores of the 1–handles lift to loops in X , in order to make sure that G remains π_1 –surjective during the remaining steps.

We now apply the technique we just used to make M connected to make the inverse image of a nonseparating arc from boundary to boundary connected, eg a cocore of a 1–handle. The universal cover argument follows in the same way, and we apply [Theorem 1.3](#) in the interval-valued Morse function case rather than the circle-valued case. Then we cut Σ along this arc and X along its preimage and proceed inductively. After cutting, G is still π_1 –surjective because the remaining generators of π_1 still have their lifts. □

Uniqueness: Proof of [Theorem 1.2](#) Now we are given a compact, connected, oriented n –manifold X and a 2–manifold Σ and two indefinite Morse 2–functions $G_0, G_1: X \rightarrow \Sigma$ which agree on ∂X and are homotopic rel ∂ . We need to construct an indefinite generic homotopy G_s between them, which is fiber-connected when G_0 and G_1 are fiber-connected. Begin with an arbitrary generic homotopy G_s and we will modify this in stages, referring to it as G_s before and after each modification. We also reduce to the fiber-connected case by lifting to a cover, as in preceding proofs.

As in the preceding proof, the base case is when $\Sigma = B^2$, which we have addressed. We also postpone the case $\Sigma = S^2$ and now reduce by cutting along closed curves and/or arcs to the case $\Sigma = B^2$, using induction on $-\chi(\Sigma)$.

Choose a nonseparating simple closed curve (if Σ is closed) or properly embedded arc (if $\partial\Sigma \neq \emptyset$) a which is transverse to the folds of both G_0 and G_1 , and let a' be a parallel copy of a with the same transversality property. [Figure 41](#) illustrates the cases where a and a' are arcs, cobounding a rectangle R with two sides in $\partial\Sigma$; for the closed case, R would be a cylinder. We will parallel the zigzag argument in the proof of the S^1 –valued case of [Theorem 1.4](#), with a and a' playing the role that 0 and $\frac{1}{2}$ played in that proof. We use [Lemma 6.3](#) to arrange the following property: for a sequence $0 = s_0 < s_1 < \dots < s_{2n} = 1$, $G_s^{-1}(a)$ is connected and G_s is π_1 –surjective on the complement of $G_s^{-1}(a)$, for all $s \in [s_0, s_1] \cup [s_2, s_3] \cup \dots \cup [s_{2n-2}, s_{2n-1}]$, while

the same is true for a' on the other intervals $[s_1, s_2] \cup [s_3, s_4] \cup [s_{2n-1}, s_{2n}]$. Again, this works because all the homotopies involved are supported in neighborhoods of arcs which can be taken to be disjoint. By further subdivisions if necessary, we can also arrange that, on the s -intervals where $G_s^{-1}(a)$ is connected, the restriction of G_s to $G_s^{-1}(a)$ is a Morse function to a , and similarly for a' . Recall that, in this paper, Morse functions have distinct critical points mapped to distinct critical values. By [Theorem 1.3](#) we can also arrange that this Morse function $G_s: G_s^{-1}(a) \rightarrow a$ is indefinite with connected fibers.

The zigzag argument Now, in [Figure 41](#), we set up the usual zigzag argument where the vertical interval $[0, \frac{1}{2}]$ in [Figure 38](#) is replaced by R , with 0 replaced by a and $\frac{1}{2}$ replaced by a' .

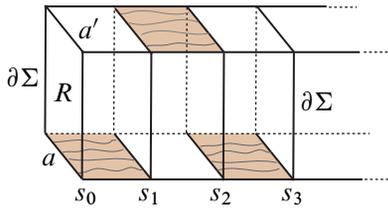


Figure 41: The zigzag argument for Morse 2–functions

Since the restriction of G_s to a is a Morse function for all $s \in [s_0, s_1]$, it follows that $G_{s_0}^{-1}(a)$ is isotopic, and hence can be taken to be equal, to $G_{s_1}^{-1}(a)$, and in fact G_s can be taken to be independent of s on $G_s^{-1}(a)$ for all $s \in [s_0, s_1]$. Similarly, G_s is constant along $G_{s_1}^{-1}(a') = G_{s_2}^{-1}(a')$ for all $s \in [s_1, s_2]$, etc. Note that $G_{s_1}^{-1}(R)$ is a connected cobordism between the connected submanifolds $G_{s_1}^{-1}(a)$ and $G_{s_1}^{-1}(a')$, on each of which G_{s_1} is indefinite and fiber-connected. Thus we can construct an intermediate indefinite, fiber-connected Morse 2–function H_{s_1} on X which agrees with G_{s_1} over a and a' , and is homotopic rel boundary to G_{s_1} over both R and the closure of the complement of R . Then G_0 agrees with H_{s_1} over a , so there exists an indefinite fiber-connected homotopy between G_0 and H_{s_1} (because $-\chi(\Sigma \setminus a) < -\chi(\Sigma)$ and the inductive hypothesis holds). Similarly we construct an appropriate H_{s_2} , so that H_{s_1} and H_{s_2} agree over a' , so they are homotopic via an indefinite fiber-connected homotopy. This argument can be iterated over each interval $[s_{2i}, s_{2i+2}]$, to finish the proof when $\Sigma \neq S^2$.

When $\Sigma = S^2$, there are no π_1 –surjectivity issues, and we can adapt the above zigzag argument to use [Remark 6.4](#) to zigzag back and forth between two fibers $G_s^{-1}(p)$ and $G_s^{-1}(q)$ as opposed to inverse images of arcs or closed curves $G_s^{-1}(a)$ and $G_s^{-1}(a')$. The rectangle R is replaced by an arc between p and q . □

References

- [1] **S Akbulut, Ç Karakurt**, *Every 4–manifold is BLF*, J. Gökova Geom. Topol. 2 (2008) 83–106 [MR2466002](#)
- [2] **D Asimov**, *Round handles and nonsingular Morse–Smale flows*, Ann. of Math. 102 (1975) 41–54 [MR0380883](#)
- [3] **D Auroux, S K Donaldson, L Katzarkov**, *Singular Lefschetz pencils*, Geom. Topol. 9 (2005) 1043–1114 [MR2140998](#)
- [4] **R İ Baykur**, *Existence of broken Lefschetz fibrations*, Int. Math. Res. Not. 2008 (2008) [MR2439543](#)
- [5] **R İ Baykur, N Sunukjian**, *Round handles, logarithmic transforms and smooth 4–manifolds*, J. Topol. 6 (2013) 49–63 [MR3029421](#)
- [6] **J Cerf**, *La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudoisotopie*, Inst. Hautes Études Sci. Publ. Math. (1970) 5–173 [MR0292089](#)
- [7] **D T Gay, R C Kirby**, *Constructing Lefschetz-type fibrations on four-manifolds*, Geom. Topol. 11 (2007) 2075–2115 [MR2350472](#)
- [8] **D T Gay, R C Kirby**, *Fiber-connected, indefinite Morse 2–functions on connected n –manifolds*, Proc. Natl. Acad. Sci. USA 108 (2011) 8122–8125 [MR2806648](#)
- [9] **A Hatcher, J Wagoner**, *Pseudoisotopies of compact manifolds*, Astérisque 6, Soc. Math. France, Paris (1973) [MR0353337](#)
- [10] **K Honda**, *Local properties of self-dual harmonic 2–forms on a 4–manifold*, J. Reine Angew. Math. 577 (2004) 105–116 [MR2108214](#)
- [11] **K Honda**, *Transversality theorems for harmonic forms*, Rocky Mountain J. Math. 34 (2004) 629–664 [MR2072799](#)
- [12] **R C Kirby**, *A calculus for framed links in S^3* , Invent. Math. 45 (1978) 35–56 [MR0467753](#)
- [13] **R C Kirby, P Melvin, P Teichner**, *Cohomotopy sets of 4–manifolds*, from: “Proceedings of the Freedman Fest”, (R Kirby, V Krushkal, Z Wang, editors), Geom. Topol. Monogr. 18 (2012) 161–190 [MR3084237](#)
- [14] **C LeBrun**, *Yamabe constants and the perturbed Seiberg–Witten equations*, Comm. Anal. Geom. 5 (1997) 535–553 [MR1487727](#)
- [15] **Y Lekili**, *Wrinkled fibrations on near-symplectic manifolds*, Geom. Topol. 13 (2009) 277–318 [MR2469519](#)
- [16] **T Perutz**, *Lagrangian matching invariants for fibred four-manifolds, I*, Geom. Topol. 11 (2007) 759–828 [MR2302502](#)
- [17] **T Perutz**, *Lagrangian matching invariants for fibred four-manifolds, II*, Geom. Topol. 12 (2008) 1461–1542 [MR2421133](#)

- [18] **O Saeki**, *Elimination of definite fold*, Kyushu J. Math. 60 (2006) 363–382 [MR2268242](#)
- [19] **CH Taubes**, *Seiberg–Witten and Gromov invariants for symplectic 4–manifolds*, First Int. Press Lecture Series 2, International Press, Somerville, MA (2000) [MR1798809](#)
- [20] **LR Taylor**, *The principal fibration sequence and the second cohomotopy set*, from: “Proceedings of the Freedman Fest”, (R Kirby, V Krushkal, Z Wang, editors), Geom. Topol. Monogr. 18 (2012) 235–251 [MR3084240](#)
- [21] **M Usher**, *The Gromov invariant and the Donaldson–Smith standard surface count*, Geom. Topol. 8 (2004) 565–610 [MR2057774](#)
- [22] **G Wassermann**, *Stability of unfoldings in space and time*, Acta Math. 135 (1975) 57–128 [MR0433497](#)
- [23] **J Williams**, *The h –principle for broken Lefschetz fibrations*, Geom. Topol. 14 (2010) 1015–1061 [MR2629899](#)

Euclid Lab

160 Milledge Terrace, Athens, GA 30606, USA

Department of Mathematics, University of California, Berkeley

Berkeley, CA 94720-3840, USA

d.gay@euclidlab.org, rckirby@berkeley.edu

Proposed: Simon Donaldson

Received: 3 February 2011

Seconded: David Gabai, Cameron Gordon

Revised: 3 February 2011