TSUKUBA J. MATH.
Vol. 41 No. 2 (2017), 215–233

ON T-COERCIVE INTERIOR TRANSMISSION EIGENVALUE PROBLEMS ON COMPACT MANIFOLDS WITH SMOOTH BOUNDARY

By

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Abstract. We consider an *interior transmission eigenvalue problem* on two compact Riemannian manifolds with common smooth boundary. We assume that this problem is *locally anisotropic type*. Then we prove that the set of interior transmission eigenvalues forms a discrete subset of complex plane. Moreover, we also mention the interior transmission eigenvalue free region. In order to prove our results, we employ the so-called T-coercivity method.

1. Introduction

In the present paper, we study the interior transmission eigenvalue problem on two compact Riemannian manifolds with common smooth boundary. As we explain in §2 and §4, the interior transmission eigenvalue problem (the **ITE** problem for short) is a boundary value problem for a system of Helmholtz equations on the support of the scattering media. The ITE problem arises from scattering theory, in particular, from non-scattering phenomena (see e.g., Vesalainen [14], [15] for quantum and acoustic scattering). As is pointed out in [14], [15], the ITE problem is closely related to the problem of non-scattering energy or non-scattering wave number. The ITE problem was first studied by Colton and Monk [8], in which they dealt with the case of isotropic media. Since [8], the ITE problem in the isotropic case has been studied by a lot of people. (For more details, see the survey of Cakoni and Haddar [6].) On the other hand, a similar ITE problem with anisotropic media was studied by Bonnet-Ben Dhia,

2010 *Mathematics Subject Classification*: Primary 81U40; Secondary 47A40.
Key words and phrases: interior transmission eigenvalue, non-scattering energy, non-scattering wave number, T-coercivity, eigenvalue free-region, scattering theory, conductive boundary condition.
Received February 17, 2017.
Revised December 15, 2017.

Chesnel and Haddar [4]. Currently, there are only a few results in the anisotropic case. In addition, it should be also noted that all of those papers deal with the ITE problems on Euclidean spaces. In recent years, scattering theory in non-compact manifolds with ends, in particular hyperbolic manifolds, has been widely studied. Therefore, it is natural to consider an ITE problem on Riemannian manifolds.

Our main purpose in this article is to study the distribution of these eigenvalues.

2. Background

Let us recall some basic notions of scattering theory in Euclidean case. We now consider the case of time harmonic acoustic scattering problem on d-dimensional Euclidean space $\mathbf{R}^d$ for $d \geq 2$ with compactly supported and bounded inhomogeneity n. We assume that there exists a bounded domain $D \subset \mathbf{R}^d$ with smooth boundary ∂D such that $n(x) = 1$ outside D.

We deal with a stationary acoustic total wave u satisfying the perturbed Helmholtz equation

$$(-\Delta - k^2 n)u = 0 \quad \text{in } \mathbf{R}^d,\ k > 0 \tag{2.1}$$

where Δ is the Laplacian on $\mathbf{R}^d$. Then we find that a solution to (2.1) is written in the form

$$u = u^i + u^s.$$

Here, u^i is an incident wave satisfying the free Helmholtz equation

$$(-\Delta - k^2)u^i = 0 \quad \text{in } \mathbf{R}^d$$

and u^s is the corresponding scattered wave satisfying some asymptotic behavior near infinity. Now let $u^i = u^i(x)$ be a plane wave $e^{ikx\cdot\omega}$ with an incident direction ω satisfying $|\omega| = 1$ and a fixed positive wave number k (or a fixed positive energy k^2). If u^s satisfies

$$u^s(x) = C(k)|x|^{-(d-1)/2} e^{ik|x|} a(k; \omega, \hat{x}) + o(|x|^{-(d-1)/2}) \quad \text{as } |x| \to \infty$$

for some positive constant $C(k)$ depending on k, there exists a unique solution $u = u^i + u^s$ of (2.1) (see e.g., [7]). Here, $\hat{x} = x/|x|$ is the scattered direction of u^s and the function $a(k; \omega, \hat{x})$ is called the scattering amplitude. Let $\hat{F}(k)$ be the integral operator on the space of square integrable functions on the $(d-1)$-dimensional sphere with integral kernel $a(k; \omega, \hat{x})$. Then the S-matrix is given by

$\hat{S}(k) = 1 - 2\pi i \hat{F}(k)$. If one is an eigenvalue of $\hat{S}(k)$ for $k > 0$, then k is called a *non-scattering wave number* (or k^2 is called a *non-scattering energy*). We denote the set of all non-scattering wave numbers by σ_N. For $k \in \sigma_N$, the corresponding scattered wave $u^s = u^s(k; x)$ vanishes outside D from the Rellich type uniqueness theorem (see e.g., [12], [13]). Hence, if k is in σ_N, there exists a non-trivial solution of the boundary value problem for a system of Helmholtz equations for u^i and u of the form

$$(-\Delta - k^2)u^i = 0 \quad \text{in } \mathbf{R}^d; \tag{2.2}$$

$$(-\Delta - k^2 n)u = 0 \quad \text{in } D; \tag{2.3}$$

$$u^i - u = 0 \quad \text{on } \partial D; \tag{2.4}$$

$$\partial_\nu u^i - \partial_\nu u = 0 \quad \text{on } \partial D; \tag{2.5}$$

where ∂_ν is the outward normal derivative on ∂D. Conversely, we suppose that (2.2)–(2.5) depending on a positive constant k has a non-trivial solution. Putting $u = u^i$ outside D, we can extend u as a solution of (2.1). Letting $u^s = u - u^i$, we can show that the scattering amplitude corresponding to u^s identically vanishes. Hence, k is in σ_N. Therefore, k is in σ_N if and only if there exists a nontrivial solution of the boundary value problem (2.2)–(2.5).

In order to study the spectral properties of non-scattering wave numbers, we consider the boundary value problem for a system of Helmholtz equations for unknown functions v and w of the form

$$(-\Delta - k^2)v = 0 \quad \text{in } D; \tag{2.6}$$

$$(-\Delta - k^2 n)w = 0 \quad \text{in } D; \tag{2.7}$$

$$v - w = 0 \quad \text{on } \partial D; \tag{2.8}$$

$$\partial_\nu v - \partial_\nu w = 0 \quad \text{on } \partial D. \tag{2.9}$$

The above boundary value problem is called an *interior transmission eigenvalue problem*. If there exists a non-trivial solution of the ITE problem (2.6)–(2.9) for some $k \in \mathbf{C}$, we call such a complex number k an *interior transmission eigenvalue*. We denote the set of all interior transmission eigenvalues by σ_I. We note that the ITE problem (2.6)–(2.9) is an eigenvalue problem for a non-selfadjoint operator. Therefore, interior transmission eigenvalues do not necessarily belong to $\mathbf{R}$. Also note that from the definition of σ_N and σ_I, the inclusion relation $\sigma_N \subset \sigma_I$ holds. Hence, as the first step to study detailed properties of the non-scattering wave numbers, in this paper, we will focus on the distribution of the interior transmission eigenvalues.

3. Notation

For $d \geq 2$, let M be a d-dimensional connected and compact oriented Riemannian manifold endowed with a smooth Riemannian metric g and with a smooth boundary ∂M. We fix local coordinates $x = (x_1, \dots, x_d)$ of M. We regard $g = g(x)$ as a positive-definite symmetric matrix valued function and we write $g(x) = (g_{ij}(x))_{i,j=1}^d$. We denote the inverse matrix of $g(x)$ by $g(x)^{-1} = (g^{ij}(x))_{i,j=1}^d$. The determinant of $g(x)$ and the volume element on M are denoted by $G(x)$ and $dV_g := \sqrt{G}\,dx = \sqrt{G}\,dx_1 \wedge \cdots \wedge dx_d$, respectively. Here, $dx_1, \dots, dx_d$ are the 1-forms providing an oriented basis for the cotangent bundle of M and the symbol $\wedge$ means the wedge product, respectively. A symbol dS denotes the surface element on ∂M induced by dx. For $x \in M$, T_xM and TM denote the tangent space of M at $x \in M$ and the tangent bundle of M, respectively. We write tangent vectors X_x, Y_x on T_xM as $X_x = \sum_{i=1}^d X_i(x)(\partial_i)_x$, $Y_x = \sum_{i=1}^d Y_i(x)(\partial_i)_x \in T_xM$, respectively. Here, X_i and Y_i are smooth functions on M and $\{(\partial_i)_x\}_{i=1}^d$ is a basis of T_xM. We denote the inner product and the norm on T_xM by

$$(X_x, Y_x)_g = \sum_{i,j=1}^d g_{ij}(x) X_i(x) \overline{Y_j(x)}, \quad |X_x|_g = \sqrt{(X_x, X_x)_g},$$

respectively. The space of all smooth vector fields on M is denoted by $\Gamma(TM)$. Let $\Delta_g : C^\infty(M) \to C^\infty(M)$ and $\nabla_g : C^\infty(M) \to \Gamma(TM)$ be the Laplace-Beltrami operator and the gradient operator on M, respectively. In local coordinates on M, those operators are written in the form

$$\Delta_g u = G^{-1/2} \sum_{i,j=1}^d \partial_i (g^{ij} G^{1/2} \partial_j u), \quad (\nabla_g u)_x = \sum_{i,j=1}^d g^{ij} (\partial_i u)(\partial_j)_x$$

for all $u \in C^\infty(M)$, respectively. Here, $(\nabla_g u)_x$ denotes the corresponding tangent vector in T_xM.

For measurable functions u on M and f on ∂M, we define

$$\|u\|_{L^\infty(M)} = \inf\{C_1 \geq 0 \mid |u(x)| \leq C_1 \text{ a.e., } x \in M\},$$

$$\|f\|_{L^\infty(\partial M)} = \inf\{C_2 \geq 0 \mid |f(x)| \leq C_2 \text{ a.e., } x \in \partial M\},$$

respectively. Next, we define $L^\infty(M)$ and $L^\infty(\partial M)$ by the space of all measurable functions u on M such that $\|u\|_{L^\infty(M)} < \infty$ and the space of all measurable functions f on ∂M such that $\|f\|_{L^\infty(\partial M)} < \infty$, respectively. We denote the $L^2(M)$-inner product and the $L^2(M)$-norm on $C^\infty(M)$ and the $L^2(\partial M)$-inner product and the $L^2(\partial M)$-norm on $C^\infty(\partial M)$ by

$$(u, v)_M = \int_M u\bar{v}\, dV_g, \quad \|u\|_M = \sqrt{(u, u)_M}, \quad u, v \in C^\infty(M),$$

$$(f, g)_{\partial M} = \int_{\partial M} f\bar{g}\, dS, \quad \|f\|_{\partial M} = \sqrt{(f, f)_{\partial M}}, \quad f, g \in C^\infty(\partial M),$$

respectively. Then the completion of $C^\infty(M)$ by $\|\cdot\|_M$ and the completion of $C^\infty(\partial M)$ by $\|\cdot\|_{\partial M}$ are denoted by $L^2(M)$ and $L^2(\partial M)$, respectively. We denote the $L^2(TM)$-inner product and the $L^2(TM)$-norm on $\Gamma(TM)$ by

$$\left.\begin{aligned} (X, Y)_{TM} &= \int_M (X_x, Y_x)_g\, dV_g, \\ \|X\|_{TM} &= \sqrt{(X, X)_{TM}}, \end{aligned}\right. \qquad X, Y \in \Gamma(TM),$$

respectively. Then the completion of $\Gamma(TM)$ by $\|\cdot\|_{TM}$ is denoted by $L^2(TM)$. We denote the $H^1(M)$-inner product and the $H^1(M)$-norm on $C^\infty(M)$ by

$$\left.\begin{aligned} (u, v)_{H^1(M)} &= (\nabla_g u, \nabla_g v)_{TM} + (u, v)_M, \\ \|u\|_{H^1(M)} &= \sqrt{(u, u)_{H^1(M)}}, \end{aligned}\right. \qquad u, v \in C^\infty(M),$$

respectively. Then the completion of $C^\infty(M)$ by $\|\cdot\|_{H^1(M)}$ is denoted by $H^1(M)$.

4. Main Theorem

To begin with, let us explain our setting. For $d \geq 2$, let M_1 and M_2 be d-dimensional connected and compact smooth oriented Riemannian manifolds endowed with Riemannian metrics g_1 and g_2 and with smooth boundaries ∂M_1 and ∂M_2, respectively. Throughout the paper, we assume that

(A-1)
- M_1 and M_2 have a common boundary $\Gamma := \partial M_1 = \partial M_2$.
- Γ is a disjoint union of a finite number of connected and closed components $\Gamma_1, \ldots, \Gamma_N$, namely $\Gamma = \amalg_{j=1}^N \Gamma_j$.
- Let $\Sigma := M_1 \cap M_2$. Then there exist connected neighborhoods Σ_j of Γ_j $(1 \leq j \leq N)$ such that Σ is written as the disjoint union of $\Sigma_1, \ldots, \Sigma_N$, namely, $\Sigma = \amalg_{j=1}^N \Sigma_j$ (see Figure 1).

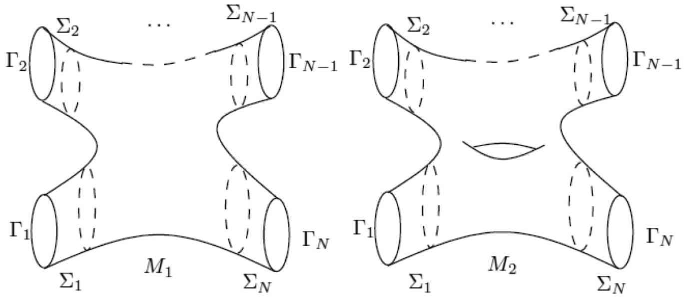

Figure 1

Here, we note that we do not necessarily assume that M_1 and M_2 are diffeomorphic.

Now, for functions $n_l \in L^\infty(M_l)$ $(l = 1, 2)$ and $\zeta \in L^\infty(\Gamma)$ and for $k \in \mathbf{C}$, we consider a boundary value problem for a system of Helmholtz equations for unknown functions u_1 and u_2 of the form

$$(-\Delta_{g_1} - k^2 n_1)u_1 = 0 \quad \text{in } M_1; \tag{4.1}$$

$$(-\Delta_{g_2} - k^2 n_2)u_2 = 0 \quad \text{in } M_2; \tag{4.2}$$

$$u_1 - u_2 = 0 \quad \text{on } \Gamma; \tag{4.3}$$

$$\sqrt{G_1}\partial_{\nu,1}u_1 - \sqrt{G_2}\partial_{\nu,2}u_2 = \zeta u_1 \quad \text{on } \Gamma. \tag{4.4}$$

Here, in the above, $\partial_{\nu,1}$ and $\partial_{\nu,2}$ denote the outward normal derivatives on Γ with respect to g_1 and g_2, respectively. Similarly as in (2.6)–(2.9), we also call the above boundary value problem an *interior transmission eigenvalue problem*.

Remark 4.1. In scattering theory, the above functions n_l $(l = 1, 2)$ and ζ are called a *refractive index* and a *conductive boundary parameter*, respectively. Usually, we assume that n_1 and n_2 are real valued functions and that ζ is a purely imaginary valued function. For the details, see [3]. However, in this article, we allow n_1, n_2 and ζ to be complex valued functions.

We put

$$\mathbf{H} := H^1(M_1) \times H^1(M_2).$$

Then $\mathbf{H}$ is a Hilbert space equipped with the inner product $(\cdot\,,\cdot)_{\mathbf{H}} := (\cdot\,,\cdot)_{H^1(M_1)} + (\cdot\,,\cdot)_{H^1(M_2)}$ and the norm $\|\cdot\|_{\mathbf{H}} := (\cdot\,,\cdot)_{\mathbf{H}}^{1/2}$. Now let us go into the definition of an interior transmission eigenvalue.

Definition 4.2. If there exists a non-trivial solution $(u_1, u_2) \in \mathbf{H}$ of the ITE problem (4.1)–(4.4) for some $k \in \mathbf{C}$, we call such a complex number k an *interior transmission eigenvalue*.

Definition 4.3.
· We denote the set of interior transmission eigenvalues by σ_I.
· A pair of functions $(u_1, u_2) \in \mathbf{H}$ is called an *interior transmission eigenfunction* associated with $k \in \sigma_I$, if (u_1, u_2) satisfies the ITE problem (4.1)–(4.4) corresponding to k.
· The dimension of the space spanned by all interior transmission eigenfunctions (u_1, u_2) associated with $k \in \sigma_I$ is called the multiplicity of k.

Our first main result is stated as follows.

Theorem 4.4. *Let $n_l \in L^\infty(M_l)$ $(l = 1, 2)$ and $\zeta \in L^\infty(\Gamma)$ be complex valued functions. We assume that $g_2/\sqrt{G_2} \le c g_1/\sqrt{G_1}$ on Σ for some constant $0 < c < 1$. Then there exists a constant $\zeta_0 > 0$ such that for ζ with $\operatorname{Re} \zeta \ge -\zeta_0$, the set σ_I of interior transmission eigenvalues is a discrete subset of $\mathbf{C}$. The point at infinity is the only possible accumulation point of σ_I. Furthermore, the multiplicity of each interior transmission eigenvalue is finite.*

Remark 4.5. The ITE problem (4.1)–(4.4) is said to be *locally anisotropic type* on Σ, if $g_1(x) \ne g_2(x)$ for some $x \in \Sigma$. The condition on g_1 and g_2 in Theorem 4.4 implies that the ITE problem (4.1)–(4.4) is locally anisotropic type on Σ.

For $r, \theta > 0$, we put

$$N(r, \theta) := \{k \in \mathbf{C} \mid |k| > r \text{ and } |\operatorname{Im} k| > (\tan \theta)|\operatorname{Re} k|\}$$

(see Figure 2). Then our second main result is given by the following.

Theorem 4.6. *Let $n_l \in L^\infty(M_l)$ $(l = 1, 2)$ and $\zeta \in L^\infty(\Gamma)$ be complex valued functions. We assume that $\operatorname{Re} n_1$ and $\operatorname{Re} n_2$ are strictly positive functions. We also assume that n_1, n_2, g_1 and g_2 satisfy*

$$\sup_\Sigma(\sqrt{G_1}(\operatorname{Re} n_1)) < \inf_\Sigma(\sqrt{G_2}(\operatorname{Re} n_2)), \quad \frac{g_2}{\sqrt{G_2}} \le c\frac{g_1}{\sqrt{G_1}} \quad on\ \Sigma \tag{4.5}$$

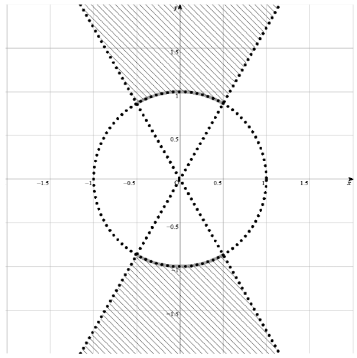

Figure 2: An example of $N(r, \theta)$ $(r = 1,\ \theta = \pi/3)$.

for some constant $0 < c < 1$. *Then there exist positive constants* r, θ, ε_0 *and* ζ_0 *such that there are no interior transmission eigenvalues in the region* $N(r, \theta)$ *for* n_1 *with* $|\mathrm{Im}\, n_1| < \varepsilon_0$ *in* Σ *and for* ζ *with* $\mathrm{Re}\, \zeta \geq -\zeta_0$ *on* Γ.

In [4], by using analytic Fredholm theorem (see e.g., [2, Theorem 1]), Bonnet-Ben Dhia, Chesnel and Haddar proved the discreteness of σ_I. In our setting, instead of analytic Fredholm theorem, we use the theory of compact operators to simplify their argument. As a result, we are able to remove their assumption which is essential to use analytic Fredholm theorem. Furthermore, we note that in this paper, we introduce a new function ζ called a boundary conductive parameter in the ITE problem (4.1)–(4.4). This parameter ζ plays an important role in scattering problem with conductive transmission condition. In this sense, we can say that our problem is a slightly more generalized version of the original ITE problem.

5. T-coercivity Method

In order to prove the discreteness of σ_I, we employ the T-coercivity method (see for example [4], [5]). Let

$$\mathbf{H}_0 := \{(u_1, u_2) \in \mathbf{H} \mid u_1 = u_2 \text{ on } \Gamma\}.$$

Let ∇_{g_1} and ∇_{g_2} be the gradient operators on (M_1, g_1) and on (M_2, g_2), respectively. We define a sesquilinear form $A_k[\cdot\,,\cdot]$ on $\mathbf{H}_0 \times \mathbf{H}_0$ by

$$\begin{aligned} A_k[(u_1, u_2), (v_1, v_2)] := {} & (\nabla_{g_1} u_1, \nabla_{g_1} v_1)_{TM_1} - (\nabla_{g_2} u_2, \nabla_{g_2} v_2)_{TM_2} \\ & - k^2((n_1 u_1, v_1)_{M_1} - (n_2 u_2, v_2)_{M_2}) - (\zeta u_1, v_1)_\Gamma \end{aligned}$$

for all $(u_1, u_2), (v_1, v_2) \in \mathbf{H}_0$. We can easily show that the ITE problem (4.1)–(4.4) has a non-trivial solution $(u_1, u_2) \in \mathbf{H}$ if and only if the variational problem of the form

$$A_k[(u_1, u_2), (v_1, v_2)] = 0 \quad \text{for all } (v_1, v_2) \in \mathbf{H}_0$$

has a non-trivial solution $(u_1, u_2) \in \mathbf{H}_0$. We define an operator T on $\mathbf{H}_0$ by

$$T(u_1, u_2) = (u_1 - 2\chi u_2, -u_2) \tag{5.1}$$

for $(u_1, u_2) \in \mathbf{H}_0$. Here, χ is a smooth cut-off function on M_2 such that $\chi = 1$ in a small neighborhood of Γ with support in $\Sigma \cap M_2$ and $0 \le \chi(x) \le 1$ for all $x \in M_2$. Let $I_{\mathbf{H}}$ be the identity operator on $\mathbf{H}$. Since $T^2 = I_{\mathbf{H}}$, T is an isomorphism on $\mathbf{H}_0$. By using this isomorphism, we define a sesquilinear form $A_k^T[\cdot\,,\cdot]$ on $\mathbf{H}_0 \times \mathbf{H}_0$ by

$$A_k^T[(u_1, u_2), (v_1, v_2)] := A_k[(u_1, u_2), T(v_1, v_2)]$$

for all $(u_1, u_2), (v_1, v_2) \in \mathbf{H}_0$. We can easily show that the above sesquilinear form $A_k^T[\cdot\,,\cdot]$ is non-degenerate and bounded on $\mathbf{H}_0 \times \mathbf{H}_0$. Hence, applying the first representation theorem (see e.g., [9, Page 322, Theorem 2.1]) or the Riesz representation theorem to the sesquilinear form $A_k^T[\cdot\,,\cdot]$, we find that there exists a bounded linear operator $\mathscr{A}^T(k)$ on $\mathbf{H}_0$ such that

$$A_k^T[(u_1, u_2), (v_1, v_2)] = (\mathscr{A}^T(k)(u_1, u_2), (v_1, v_2))_{\mathbf{H}}$$

for all $(u_1, u_2), (v_1, v_2) \in \mathbf{H}_0$. Summarizing the above argument, we obtain the following proposition.

Proposition 5.1. *A point $k \in \mathbf{C}$ is in σ_I if and only if the operator $\mathscr{A}^T(k)$ on $\mathbf{H}_0$ has a non-trivial kernel. In this case, each element of the kernel of $\mathscr{A}^T(k)$ is interior transmission eigenfunction associated with $k \in \sigma_I$. The multiplicity of $k \in \sigma_I$ coincides with the dimension of the kernel of $\mathscr{A}^T(k)$.*

Now, let us introduce the notion of *strictly coercive*.

Definition 5.2. Let H be a Hilbert space equipped with inner product $(\cdot\,,\cdot)_H$ and norm $\|\cdot\|_H = \sqrt{(\cdot\,,\cdot)_H}$. A bounded linear operator $B : H \to H$ is said to be *strictly coercive* if there exists a constant $C > 0$ such that

$$\mathrm{Re}(B\varphi, \varphi)_H \ge C\|\varphi\|_H^2$$

for all $\varphi \in H$.

The following theorem is well-known as the Lax-Milgram theorem.

Theorem 5.3 (see e.g., [10, Page 201, Theorem 13.23]). *In a Hilbert space H, a strictly coercive bounded linear operator $B : H \to H$ has a bounded inverse.*

Let $\kappa \in \mathbf{R}\setminus\{0\}$ and $\varepsilon, \delta > 0$ be constants such that

$$\varepsilon^* := \sup_{\Sigma}(\sqrt{G_1})\varepsilon < \inf_{\Sigma}(\sqrt{G_2})\delta =: \delta_*.$$

We define a sesquilinear form $A_{i\kappa,\varepsilon,\delta}[\cdot\,,\cdot]$ on $\mathbf{H}_0 \times \mathbf{H}_0$ by

$$A_{i\kappa,\varepsilon,\delta}[(u_1,u_2),(v_1,v_2)] := (\nabla_{g_1}u_1, \nabla_{g_1}v_1)_{TM_1} - (\nabla_{g_2}u_2, \nabla_{g_2}v_2)_{TM_2} + \kappa^2((\varepsilon u_1, v_1)_{M_1} - (\delta u_2, v_2)_{M_2}) - (\zeta u_1, v_1)_{\Gamma}$$

for all $(u_1,u_2),(v_1,v_2) \in \mathbf{H}_0$. In addition, we define a bounded operator $\mathscr{I}_{\kappa,\delta,\varepsilon}$ on $\mathbf{H}_0$ by

$$(\mathscr{I}_{\kappa,\varepsilon,\delta}(u_1,u_2),(v_1,v_2))_{\mathbf{H}} := A_{i\kappa,\varepsilon,\delta}[(u_1,u_2), T(v_1,v_2)]$$

for all $(u_1,u_2),(v_1,v_2) \in \mathbf{H}_0$.

Now in order to reduce the ITE problem (4.1)–(4.4) to the eigenvalue problem for a certain compact operator, we state the following key lemma.

Lemma 5.4. *Let $n_l \in L^\infty(M_l)$ $(l = 1,2)$ and $\zeta \in L^\infty(\Gamma)$ be complex valued functions. We assume*

$$\frac{g_2}{\sqrt{G_2}} \le c\frac{g_1}{\sqrt{G_1}} \quad \text{on } \Sigma \tag{5.2}$$

for some constant $0 < c < 1$. Then there exist a point $\zeta_0 > 0$ and a constant $C > 0$ such that for ζ with $\operatorname{Re}\zeta \ge -\zeta_0$, the inequality

$$\operatorname{Re}(\mathscr{I}_{\kappa,\varepsilon,\delta}(u_1,u_2),(u_1,u_2))_{\mathbf{H}} \ge C\|(u_1,u_2)\|^2_{\mathbf{H}}, \quad (u_1,u_2) \in \mathbf{H}_0 \tag{5.3}$$

holds.

Proof. We have the equality

$$\begin{aligned} &\operatorname{Re}(\mathscr{I}_{\kappa,\varepsilon,\delta}(u_1,u_2),(u_1,u_2))_{\mathbf{H}} \\ &= \int_{M_1\setminus\Sigma} |\nabla_{g_1}u_1|^2_{g_1}\, dV_{g_1} + \int_{M_2\setminus\Sigma} |\nabla_{g_2}u_2|^2_{g_2}\, dV_{g_2} \end{aligned} \tag{5.4}$$

$$+\kappa^2(\varepsilon\|u_1\|^2_{M_1\setminus\Sigma}+\delta\|u_2\|^2_{M_2\setminus\Sigma})+\int_\Sigma|\nabla_{g_1}u_1|^2_{g_1}\,dV_{g_1}$$

$$+\int_\Sigma|\nabla_{g_2}u_2|^2_{g_2}\,dV_{g_2}+\kappa^2\left(\varepsilon\int_\Sigma|u_1|^2\,dV_{g_1}+\int_\Sigma\delta|u_2|^2\,dV_{g_2}\right)$$

$$-2\,\mathrm{Re}(\nabla_{g_1}u_1,\nabla_{g_1}(\chi u_2))_{TM_1}-2\kappa^2\varepsilon\,\mathrm{Re}(u_1,\chi u_2)_{M_1}+\mathrm{Re}(\zeta u_1,u_1)_\Gamma$$

for all $(u_1,u_2)\in\mathbf{H}_0$. Using Young's inequality and (5.2), we have

$$\begin{aligned}
&2\,\mathrm{Re}(\nabla_{g_1}u_1,\nabla_{g_1}(\chi u_2))_{TM_1}\\
&\quad\le(\alpha+\beta)\int_\Sigma|\nabla_{g_1}u_1|^2_{g_1}\,dV_{g_1}+\alpha^{-1}\int_\Sigma|\nabla_{g_1}u_2|^2_{g_1}\,dV_{g_1}\\
&\qquad+\beta^{-1}\int_\Sigma|\nabla_{g_1}\chi|^2_{g_1}|u_2|^2\,dV_{g_1}\\
&\quad\le(\alpha+\beta)\int_\Sigma|\nabla_{g_1}u_1|^2_{g_1}\,dV_{g_1}+c\alpha^{-1}\int_\Sigma|\nabla_{g_2}u_2|^2_{g_2}\,dV_{g_2}\\
&\qquad+\beta^{-1}\sup_\Sigma\left(|\nabla_{g_1}\chi|^2_{g_1}\sqrt{\frac{G_1}{G_2}}\right)\int_\Sigma|u_2|^2\,dV_{g_2}
\end{aligned}\tag{5.5}$$

and

$$2\kappa^2\varepsilon\,\mathrm{Re}(u_1,\chi u_2)_{M_1}\le\kappa^2\varepsilon\gamma\int_\Sigma|u_1|^2\,dV_{g_1}+\kappa^2\int_\Sigma\frac{1}{\sqrt{G_2}}\gamma^{-1}\sqrt{G_1}\varepsilon|u_2|^2\,dV_{g_2}\tag{5.6}$$

for all $\alpha,\beta,\gamma>0$. Plugging (5.5) and (5.6) into (5.4), we obtain

$$\begin{aligned}
&\mathrm{Re}(\mathscr{I}_{\kappa,\varepsilon,\delta}(u_1,u_2),(u_1,u_2))_{\mathbf{H}}\\
&\quad\ge\int_{M_1\setminus\Sigma}|\nabla_{g_1}u_1|^2_{g_1}\,dV_{g_1}+\int_{M_2\setminus\Sigma}|\nabla_{g_2}u_2|^2_{g_2}\,dV_{g_2}+\kappa^2(\varepsilon\|u_1\|^2_{M_1\setminus\Sigma}+\delta\|u_2\|^2_{M_2\setminus\Sigma})\\
&\qquad+(1-\alpha-\beta)\int_\Sigma|\nabla_{g_1}u_1|^2_{g_1}\,dV_{g_1}+(1-c\alpha^{-1})\int_\Sigma|\nabla_{g_2}u_2|^2_{g_2}\,dV_{g_2}\\
&\qquad+\kappa^2\varepsilon(1-\gamma)\int_\Sigma|u_1|^2\,dV_{g_1}+\kappa^2\int_\Sigma\frac{1}{\sqrt{G_2}}(\delta_*-\gamma^{-1}\varepsilon^*)|u_2|^2\,dV_{g_2}\\
&\qquad-\beta^{-1}\sup_\Sigma\left(|\nabla_{g_1}\chi|^2_{g_1}\sqrt{\frac{G_1}{G_2}}\right)\int_\Sigma|u_2|^2\,dV_{g_2}-\zeta_0\|u_1\|^2_\Gamma.
\end{aligned}$$

Taking γ such that $\varepsilon^*/\delta_*<\gamma<1$, we have

$$
\begin{aligned}
&\mathrm{Re}(\mathscr{I}_{\kappa,\varepsilon,\delta}(u_1,u_2),(u_1,u_2))_{\mathbf{H}} \\
&\quad \geq \int_{M_1\setminus\Sigma} |\nabla_{g_1} u_1|_{g_1}^2 \, dV_{g_1} + \int_{M_2\setminus\Sigma} |\nabla_{g_2} u_2|_{g_2}^2 \, dV_{g_2} \\
&\qquad + \kappa^2(\varepsilon\|u_1\|_{M_1\setminus\Sigma}^2 + \delta\|u_2\|_{M_2\setminus\Sigma}^2) + (1-\alpha-\beta)\int_\Sigma |\nabla_{g_1} u_1|_{g_1}^2 \, dV_{g_1} \\
&\qquad + (1-c\alpha^{-1})\int_\Sigma |\nabla_{g_2} u_2|_{g_2}^2 \, dV_{g_2} + \kappa^2\varepsilon(1-\gamma)\int_\Sigma |u_1|^2 \, dV_{g_1} \\
&\qquad + (\kappa^2 c_1(\delta_* - \gamma^{-1}\varepsilon^*) - c_2\beta^{-1})\int_\Sigma |u_2|^2 \, dV_{g_2} - \zeta_0\|u_1\|_\Gamma^2
\end{aligned}
$$

for some $c_1, c_2 > 0$. Using the trace theorem, we obtain

$$
\|u_1\|_\Gamma \leq c_3\|u_1\|_{H^1(M_1)} \tag{5.7}
$$

for some $c_3 > 0$. By taking α, β such that

$$
c < \alpha < 1, \quad 0 < \beta < 1 - \alpha
$$

and using (5.7), letting $|\kappa| > 0$ large enough and $\zeta_0 > 0$ small enough, more precisely taking

$$
\kappa^2 > \frac{c_2\beta^{-1}}{c_1(\delta_* - \gamma^{-1}\varepsilon^*)}, \quad 0 < \zeta_0 < c_3^{-1}\min\{1-\alpha-\beta, \kappa^2\varepsilon(1-\gamma)\},
$$

we can easily show that there exists a constant $C > 0$ such that the inequality (5.3) holds. □

Remark 5.5. For example, we take

$$
\alpha = \frac{c+1}{2}, \quad \beta = \frac{1-c}{4}, \quad \gamma = \frac{\delta_* + \varepsilon^*}{2\delta_*}, \quad \zeta_0 = \frac{1-c}{8c_3^2}
$$

and

$$
\kappa^2 = \max\left\{\frac{2\delta_*}{\varepsilon(\delta_* - \varepsilon^*)}, \frac{1}{\delta}, \frac{\delta_* + \varepsilon^*}{c_1\delta_*(\delta_* - \varepsilon^*)}\left(1 + \frac{4c_2}{1-c}\right)\right\}.
$$

Then the constant $C > 0$ appeared in (5.3) is equal to $(1-c)/8$.

Remark 5.6. As stated above, using the isomorphism T given by (5.1), we can avoid the difficulty arising from the non-ellipticity of the sesquilinear form $A_k[\cdot\,,\cdot]$. Such a method is called the T-coercivity method.

Using the above lemma, we can write $\mathscr{A}^T(k)$ as the sum of an isomorphism and a compact operator as follows.

Proposition 5.7. *Let $n_l \in L^\infty(M_l)$ $(l = 1, 2)$ and $\zeta \in L^\infty(\Gamma)$ be complex valued functions. We assume* (5.2) *for some constant $0 < c < 1$. Then there exists a point $\zeta_0 > 0$ such that for ζ with $\operatorname{Re} \zeta \geq -\zeta_0$ and for all $k \in \mathbf{C}$, the operator $\mathscr{A}^T(k)$ is written in the form $\mathscr{A}^T(k) = \mathscr{I} + \mathscr{K}$ where $\mathscr{I}$ is an isomorphism on $\mathbf{H}_0$ and $\mathscr{K}$ is a compact operator on $\mathbf{H}_0$. As a result, $\mathscr{A}^T(k)$ is a Fredholm operator on $\mathbf{H}_0$ for all $k \in \mathbf{C}$.*

Proof. By Lemma 5.4, the inequality (5.3) holds. Applying Theorem 5.3 to the bounded linear operator $\mathscr{I}_{\kappa,\varepsilon,\delta}$, we find that $\mathscr{I}_{\kappa,\varepsilon,\delta}$ is an isomorphism on $\mathbf{H}_0$. Recall that $\mathscr{A}^T(k)$ and $\mathscr{I}_{\kappa,\varepsilon,\delta}$ are written as

$$
\begin{aligned}
&(\mathscr{A}^T(k)(u_1, u_2), (v_1, v_2))_{\mathbf{H}} \\
&\quad = (\nabla_{g_1} u_1, \nabla_{g_1} v_1)_{TM_1} + (\nabla_{g_2} u_2, \nabla_{g_2} v_2)_{TM_2} - 2(\nabla_{g_1} u_1, \nabla_{g_1}(\chi v_2))_{TM_1} \\
&\qquad - k^2((n_1 u_1, v_1)_{M_1} + (n_2 u_2, v_2)_{M_2} - 2(n_1 u_1, \chi v_2)_{M_1}) - (\zeta u_1, v_1)_\Gamma
\end{aligned}
$$

and

$$
\begin{aligned}
&(\mathscr{I}_{\kappa,\varepsilon,\delta}(u_1, u_2), (v_1, v_2))_{\mathbf{H}} \\
&\quad = (\nabla_{g_1} u_1, \nabla_{g_1} v_1)_{TM_1} + (\nabla_{g_2} u_2, \nabla_{g_2} v_2)_{TM_2} - 2(\nabla_{g_1} u_1, \nabla_{g_1}(\chi v_2))_{TM_1} \\
&\qquad + \kappa^2((\varepsilon u_1, v_1)_{M_1} + (\delta u_2, v_2)_{M_2} - 2(\varepsilon u_1, \chi v_2)_{M_1}) - (\zeta u_1, v_1)_\Gamma
\end{aligned}
$$

for $(u_1, u_2), (v_1, v_2) \in \mathbf{H}_0$, respectively. We put $\mathscr{K} := \mathscr{A}^T(k) - \mathscr{I}_{\kappa,\varepsilon,\delta}$. Then the operator $\mathscr{K}$ satisfies

$$
\begin{aligned}
&(\mathscr{K}(u_1, u_2), (v_1, v_2))_{\mathbf{H}} \\
&\quad = -k^2((n_1 u_1, v_1)_{M_1} + (n_2 u_2, v_2)_{M_2} - 2(n_1 u_1, \chi v_2)_{M_1}) \\
&\qquad - \kappa^2((\varepsilon u_1, v_1)_{M_1} + (\delta u_2, v_2)_{M_2} - 2(\varepsilon u_1, \chi v_2)_{M_1})
\end{aligned}
$$

for all $(u_1, u_2), (v_1, v_2) \in \mathbf{H}_0$. Therefore, the inequality

$$
|(\mathscr{K}(u_1, u_2), (v_1, v_2))_{\mathbf{H}}| \leq C \|(u_1, u_2)\|_{L^2(M_1) \times L^2(M_2)} \|(v_1, v_2)\|_{\mathbf{H}}
$$

holds for some constant $C > 0$ depending on k. Here, $\|\cdot\|_{L^2(M_1) \times L^2(M_2)}$ is a norm of the Hilbert space $L^2(M_1) \times L^2(M_2)$ and denotes

$$
\|(u_1, u_2)\|_{L^2(M_1) \times L^2(M_2)} = (\|u_1\|_{M_1}^2 + \|u_2\|_{M_2}^2)^{1/2}
$$

for $(u_1, u_2) \in L^2(M_1) \times L^2(M_2)$. The above inequality is equivalent to

$$\|\mathscr{K}(u_1, u_2)\|_{\mathbf{H}} \leq C\|(u_1, u_2)\|_{L^2(M_1)\times L^2(M_2)} \tag{5.8}$$

for all $(u_1, u_2) \in \mathbf{H}_0$. By the Rellich–Kondrashov theorem (see e.g., [1, Page 168, Theorem 6.3]), a bounded sequence in $\mathbf{H}_0$ has a Cauchy subsequence in $L^2(M_1) \times L^2(M_2)$. Let $\{(u_{1n}, u_{2n})\}_{n=1}^{\infty}$ be such a subsequence. Using the inequality (5.8), we have

$$\|\mathscr{K}(u_{1n}, u_{2n}) - \mathscr{K}(u_{1m}, u_{2m})\|_{\mathbf{H}} \leq C\|(u_{1n}, u_{2n}) - (u_{1m}, u_{2m})\|_{L^2(M_1)\times L^2(M_1)}.$$

This means that $\{\mathscr{K}(u_{1n}, u_{2n})\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbf{H}_0$. Thus, $\mathscr{K}$ is a compact operator on $\mathbf{H}_0$. If we take $\mathscr{I} = \mathscr{I}_{\kappa,\varepsilon,\delta}$, then we have $\mathscr{A}^T(k) = \mathscr{I} + \mathscr{K}$, which proves the assertion. □

6. Proof of the Main Theorems

First, we prove Theorem 4.4.

Proof of Theorem 4.4. Let us define two operators $\mathscr{F}$ and $\mathscr{G}_{\kappa,\varepsilon,\delta}$ on $\mathbf{H}_0$ by

$$(\mathscr{F}(u_1, u_2), (v_1, v_2))_{\mathbf{H}} = (n_1u_1, v_1)_{M_1} + (n_2u_2, v_2)_{M_2} - 2(n_1u_1, \chi v_2)_{M_1}$$

and

$$(\mathscr{G}_{\kappa,\varepsilon,\delta}(u_1, u_2), (v_1, v_2))_{\mathbf{H}} = \kappa^2((\varepsilon u_1, v_1)_{M_1} + (\delta u_2, v_2)_{M_2} - 2(\varepsilon u_1, \chi v_2)_{M_1})$$

for all $(u_1, u_2), (v_1, v_2) \in \mathbf{H}_0$, respectively. By the same argument as in the proof of Proposition 5.7, we can show that $\mathscr{F}$ and $\mathscr{G}_{\kappa,\varepsilon,\delta}$ are also compact operators on $\mathbf{H}_0$. Using these operators, we rewrite $\mathscr{A}^T(k)$ as

$$\mathscr{I}_{\kappa,\varepsilon,\delta} - k^2\mathscr{F} - \mathscr{G}_{\kappa,\varepsilon,\delta}.$$

Let us take $\varepsilon, \delta > 0$ such that $\sup_{\Sigma}(\sqrt{G_1})\varepsilon < \inf_{\Sigma}(\sqrt{G_2})\delta$. Next, we choose ε and δ small enough such that $\|\mathscr{I}_{\kappa,\varepsilon,\delta}^{-1}\mathscr{G}_{\kappa,\varepsilon,\delta}\|_{\mathbf{H}_0\to\mathbf{H}_0} < 1$. Here, $\|\cdot\|_{\mathbf{H}_0\to\mathbf{H}_0}$ denotes the operator norm for bounded linear operators on $\mathbf{H}_0$. Then we can easily show that $I_{\mathbf{H}} - \mathscr{I}_{\kappa,\varepsilon,\delta}^{-1}\mathscr{G}_{\kappa,\varepsilon,\delta}$ is a bijection on $\mathbf{H}_0$ and has a bounded inverse. Therefore, an interior transmission eigenfunction $(u_1, u_2) \in \mathbf{H}_0$ associated with $k \in \sigma_I$ satisfies

$$\begin{aligned} 0 &= \mathscr{I}_{\kappa,\varepsilon,\delta}^{-1}\mathscr{A}^T(k)(u_1, u_2) \\ &= (I_{\mathbf{H}} - \mathscr{I}_{\kappa,\varepsilon,\delta}^{-1}\mathscr{G}_{\kappa,\varepsilon,\delta})(u_1, u_2) - k^2\mathscr{I}_{\kappa,\varepsilon,\delta}^{-1}\mathscr{F}(u_1, u_2). \end{aligned} \tag{6.1}$$

Put $\mathscr{B} = (I_{\mathbf{H}} - \mathscr{I}_{\kappa,\varepsilon,\delta}^{-1}\mathscr{G}_{\kappa,\varepsilon,\delta})^{-1}\mathscr{I}_{\kappa,\varepsilon,\delta}^{-1}$. Obviously, $\mathscr{B}$ is a bounded operator on $\mathbf{H}_0$ and is independent of k. Thus, $\mathscr{B}\mathscr{F}$ is also a compact operator on $\mathbf{H}_0$. Moreover, it follows easily from (6.1) that

$$\mathscr{B}\mathscr{F}(u_1, u_2) = k^{-2}(u_1, u_2)$$

for all $(u_1, u_2) \in \mathbf{H}_0 \backslash \{(0,0)\}$. As a conclusion, $(u_1, u_2) \in \mathbf{H}_0$ is an interior transmission eigenfunction associated with $k \in \sigma_I \backslash \{0\}$ if and only if $k^{-2} \in \mathbf{C}$ is an eigenvalue of the compact operator $\mathscr{B}\mathscr{F}$ on $\mathbf{H}_0$ and $(u_1, u_2) \in \mathbf{H}_0$ is the corresponding eigenfunction associated with k^{-2}. As is well-known in the theory of compact operators, 0 is the only possible accumulation point of eigenvalues of a compact operator. Therefore, we obtain the assertion of Theorem 4.4. □

Next, we prove Theorem 4.6.

Proof of Theorem 4.6. It is sufficient to prove that there exist constants $r > 0$ and $\theta \in (0, \pi/2]$ such that for all $k \in N(r, \theta)$ and for some constant $C > 0$, the inequality

$$\operatorname{Re}(\mathscr{A}^T(k)(u_1, u_2), (u_1, u_2)) \geq C\|(u_1, u_2)\|_{\mathbf{H}}^2, \quad (u_1, u_2) \in \mathbf{H}_0 \tag{6.2}$$

holds. Indeed, applying Theorem 5.3 to the bounded linear operator $\mathscr{A}^T(k)$, we find that for $k \in N(r, \theta)$, $\mathscr{A}^T(k)$ is an isomorphism on $\mathbf{H}_0$ and has a trivial kernel. Hence, such a complex number k is not in σ_I.

We put

$$n_1^* := \sup_{\Sigma}(\sqrt{G_1}(\operatorname{Re} n_1)), \quad n_{2*} := \inf_{\Sigma}(\sqrt{G_2}(\operatorname{Re} n_2)).$$

We assume that n_1 satisfies

$$|\operatorname{Im} n_1| < \varepsilon_0 \quad \text{in } \Sigma$$

for some constant $\varepsilon_0 > 0$. Then we derive the estimate

$$\begin{aligned} &2 \operatorname{Re}(n_1 u_1, \chi u_2)_{M_1} \\ &\quad \leq \gamma \int_{\Sigma} (\operatorname{Re} n_1)|u_1|^2 \, dV_{g_1} + \int_{\Sigma} \frac{1}{\sqrt{G_2}} \gamma^{-1} (\sqrt{G_1} \operatorname{Re} n_1)|u_2|^2 \, dV_{g_2} \\ &\qquad + \varepsilon_0 \int_{\Sigma} |u_1|^2 \, dV_{g_1} + \varepsilon_0 \sup_{\Sigma}\left(\sqrt{\frac{G_1}{G_2}}\right) \int_{\Sigma} |u_2|^2 \, dV_{g_2} \end{aligned} \tag{6.3}$$

for all $\gamma > 0$. Let $\rho \in \mathbf{R} \backslash \{0\}$. Using (4.5), (5.5) and (6.3), we obtain

$$
\begin{aligned}
&\operatorname{Re}(\mathscr{A}^T(i\rho)(u_1,u_2),(u_1,u_2))_{\mathbf{H}}\\
&\quad \ge \int_{M_1\setminus\Sigma} |\nabla_{g_1}u_1|_{g_1}^2\,dV_{g_1} + \int_{M_2\setminus\Sigma} |\nabla_{g_2}u_2|_{g_2}^2\,dV_{g_2}\\
&\qquad + \rho^2\left(\inf_{M_1\setminus\Sigma}(\operatorname{Re} n_1)\|u_1\|_{M_1\setminus\Sigma}^2 + \inf_{M_2\setminus\Sigma}(\operatorname{Re} n_2)\|u_2\|_{M_2\setminus\Sigma}^2\right)\\
&\qquad + (1-\alpha-\beta)\int_\Sigma |\nabla_{g_1}u_1|_{g_1}^2\,dV_{g_1} + (1-c\alpha^{-1})\int_\Sigma |\nabla_{g_2}u_2|_{g_2}^2\,dV_{g_2}\\
&\qquad + \rho^2\int_\Sigma (1-\gamma)(\operatorname{Re} n_1)|u_1|^2\,dV_{g_1} - \rho^2\varepsilon_0\int_\Sigma |u_1|^2\,dV_{g_1}\\
&\qquad + \rho^2\int_\Sigma \frac{1}{\sqrt{G_2}}(n_{2*}-\gamma^{-1}n_1^*)|u_2|^2\,dV_{g_2} - \rho^2\varepsilon_0\sup_\Sigma\left(\sqrt{\frac{G_1}{G_2}}\right)\int_\Sigma |u_2|^2\,dV_{g_2}\\
&\qquad - \beta^{-1}\sup_\Sigma\left(|\nabla_{g_1}\chi|_{g_1}^2\sqrt{\frac{G_1}{G_2}}\right)\int_\Sigma |u_2|^2\,dV_{g_2} - \zeta_0\|u_1\|_\Gamma^2.
\end{aligned}
$$

for all $\alpha,\beta,\gamma>0$. Taking γ such that $n_1^*/n_{2*}<\gamma<1$, we have

$$
\begin{aligned}
&\operatorname{Re}(\mathscr{A}^T(i\rho)(u_1,u_2),(u_1,u_2))_{\mathbf{H}}\\
&\quad \ge \int_{M_1\setminus\Sigma} |\nabla_{g_1}u_1|_{g_1}^2\,dV_{g_1} + \int_{M_2\setminus\Sigma} |\nabla_{g_2}u_2|_{g_2}^2\,dV_{g_2}\\
&\qquad + \rho^2\left(\inf_{M_1\setminus\Sigma}(\operatorname{Re} n_1)\|u_1\|_{M_1\setminus\Sigma}^2 + \inf_{M_2\setminus\Sigma}(\operatorname{Re} n_2)\|u_2\|_{M_2\setminus\Sigma}^2\right)\\
&\qquad + (1-\alpha-\beta)\int_\Sigma |\nabla_{g_1}u_1|_{g_1}^2\,dV_{g_1} + (1-c\alpha^{-1})\int_\Sigma |\nabla_{g_2}u_2|_{g_2}^2\,dV_{g_2}\\
&\qquad + \rho^2\left((1-\gamma)\inf_\Sigma(\operatorname{Re} n_1)-\varepsilon_0\right)\int_\Sigma |u_1|^2\,dV_{g_1}\\
&\qquad + (\rho^2(c_1(n_{2*}-\gamma^{-1}n_1^*)-c_4\varepsilon_0)-c_2\beta^{-1})\int_\Sigma |u_2|^2\,dV_{g_2} - \zeta_0\|u_1\|_\Gamma^2
\end{aligned}
$$

for some $c_1,c_2,c_4>0$. Using the same argument as in the proof of Lemma 5.4, for a suitable choice of constants α, β, γ and a small constant $\varepsilon_0>0$ and a large constant $r>0$ and letting $|\rho|>r$, we have

$$
\begin{aligned}
(6.4)\qquad &\operatorname{Re}(\mathscr{A}^T(i\rho)(u_1,u_2),(u_1,u_2))_{\mathbf{H}}\\
&\quad \ge C_1(\|\nabla_{g_1}u_1\|_{TM_1}^2+\|\nabla_{g_2}u_2\|_{TM_2}^2)+C_2\rho^2(\|u_1\|_{M_1}^2+\|u_2\|_{M_2}^2)-\zeta_0\|u_1\|_\Gamma^2
\end{aligned}
$$

for some constants $C_1, C_2 > 0$. On the other hand, taking $k = i\rho e^{i\varphi}$ with $0 \le \varphi < \pi/2$, we find that there exists a constant $C_3 > 0$ such that

$$\begin{aligned} \operatorname{Re}((\mathscr{A}^T(i\rho) - \mathscr{A}^T(k))(u_1, u_2), (u_1, u_2))_{\mathbf{H}} \\ \le C_3\rho^2|1 - e^{2i\varphi}|(\|u_1\|_{M_1}^2 + \|u_2\|_{M_2}^2) \end{aligned} \tag{6.5}$$

for all $(u_1, u_2) \in \mathbf{H}_0$. Combining (6.4) with (6.5), we obtain

$$\begin{aligned} &\operatorname{Re}(\mathscr{A}^T(k)(u_1, u_2), (u_1, u_2))_{\mathbf{H}} \\ &\quad \ge C_1(\|\nabla_{g_1} u_1\|_{TM_1}^2 + \|\nabla_{g_2} u_2\|_{TM_2}^2) \\ &\qquad + (C_2 - C_3|1 - e^{2i\varphi}|)\rho^2(\|u_1\|_{M_1}^2 + \|u_2\|_{M_2}^2) - \zeta_0\|u_1\|_\Gamma^2 \end{aligned}$$

for all $(u_1, u_2) \in \mathbf{H}_0$. By choosing $\varphi, \zeta_0 > 0$ small enough and using (5.7), we have

$$\operatorname{Re}(\mathscr{A}^T(k)(u_1, u_2), (u_1, u_2))_{\mathbf{H}} \ge C\|(u_1, u_2)\|_{\mathbf{H}}^2$$

for some constant $C > 0$. We put $\theta := \pi/2 - \varphi$. Then for all $k \in N(r, \theta)$, the inequality (6.2) holds. Therefore, we obtain the assertion of the Theorem 4.6. □

7. Final Remarks

1. In this paper, we have presented spectral properties of interior transmission eigenvalues corresponding to scattering by an inhomogeneous medium on Riemannian manifolds. In particular, we have studied the discreteness and localization of interior transmission eigenvalues corresponding to the case of locally anisotropic type. In this case, we used the T-coercivity method. This method was first introduced by Bonnet-Ben Dhia, Ciarlet and Zwölf [5]. Using the idea of T-coercivity, they proved that the electromagnetic wave transmission problem is well-posed when dielectric constant changes its sign. In [4], Bonnet-Ben Dhia, Chesnel and Haddar first applied T-coercivity method to the study of the discreteness of interior transmission eigenvalues. They considered the anisotropic ITE problem in a bounded domain of $\mathbf{R}^d$. Making use of T-coercivity method, they proved that the set of interior transmission eigenvalues forms a discrete set under a certain condition on refractive index, which is crucial to use analytic Fredholm theorem.

2. The ITE problem on Riemannian manifolds is also studied in [11]. In a similar setting to ours, however, they deal with the case when $g_1(x) = g_2(x)$ for

all $x \in \Gamma$. Here, the ITE problem (4.1)–(4.4) corresponding to this case is said to be *locally isotropic type* on Γ. In this case, they employ the method of Dirichlet-to-Neumann operators to prove the discreteness, existence and Weyl asymptotics of interior transmission eigenvalues. Their Weyl asymptotics is estimated from below by the counting functions of the corresponding Dirichlet eigenvalues on compact Riemannian manifolds.

Acknowledgements

The author would like to express his gratitude to Professor Tomoyuki Kakehi for his constant advice and suggestions. The author would also like to thank Professor Hiroshi Isozaki for his valuable advice. The author would like to thank Professor Hisashi Morioka for fruitful discussions and encouragement.

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