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UNIQUE CONTINUATION FOR FAST DIFFUSION

By

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Abstract. We consider the non-characteristic Cauchy problem for the degenerate nonlinear parabolic equation $|u|^{\alpha}u_t = \triangle u$, where we assume that $1/2 < \alpha < 1$. This equation is based on the fast diffusion model. And we prove the unique continuation property for the above problem.

1 Introduction

The non-characteristic Cauchy problem for the parabolic equation is not well-posed in the C^{∞} class of functions, but the unique continuation property holds a fact, which was proved first by Mizohata [12]. More precisely, his result is as follows: Let u be any solution of the second order parabolic equation with linear principal parts defined in a neighborhood of the non-characteristic Cauchy surface Γ. Then if its Cauchy data equals zero on Γ, u vanishes identically along the horizontal zone of Γ. A model is the semilinear equation

$$u_t = \triangle u + f(x,u), \tag{1.1}$$

where t is the time variable and $\triangle = \partial_{x_1}^2 + \cdots + \partial_{x_N}^2$ with the space variable $x = (x_1, \ldots, x_N)$. If the nonlinear part f satisfies

$$|f(x,u)| \le M|u|, \quad M < \infty,$$

the result in [12] remains true. If $f(x,u) = V(x)u$ and V is not locally bounded, the situation is difficult. When $V \in L_{loc}^{(N+2)/2}$, Lin [10] proved the following: If the solution u of (1.1) vanishes at (x^0, t^0) of infinite order with respect to the x-variable, then $u(x, t^0) = 0$ identically in the horizontal plane. Mizohata's proof in [12] is to make use of the theory of singular integral operators, from which the

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theory of pseudo differential operators origins. Another elementary proof was given by Saut and Scheurer [16].

In place of (1.1) we consider the non-characteristic Cauchy problem for the equation

$$|u|^{\alpha}u_t = \Delta u. \tag{1.2}$$

Throughout this paper we assume that $\alpha > 0$. Thus (1.2) is a degenerate nonlinear parabolic equation. In this paper we treat the equation (1.3) more general than (1.2), but our leading equation is (1.2) itself.

From the viewpoint of physics (1.2) is known as a model of the fast diffusion. The function u means some positive power of the density of some substance. For such a case it is natural to assume that u is nonnegative. We shall consider the initial value problem for (1.2). If the initial function is nonnegative, the solution u of (1.2) is so. The precise result is referred to Kalashnikov's work [7], which is a survey on the theory of nonlinear parabolic equations. We note that (1.2) means the porous media equation when $-1 < \alpha < 0$. In this case the function $|u|^{\alpha}u$ is a solution of the original porous media equation. Even if the initial value is nonnegative and smooth, the solution is not always regular. More precisely, an interface occurs. But if $N = 1$ and $\frac{-\alpha}{\alpha+1}$ is an even natural number, the local existence and the C^{∞} regularity property of solutions hold. This result is due to [14].

We consider the non-characteristic Cauchy problem for (1.2) in the category of nonnegative solutions. By the result of Sabinina [15] it is known that $u(x, t^0) = 0$ on the horizontal plane, if $u(x^0, t^0) = 0$ for some point $(x^0, t^0) \in R^{N+1}$. The method in [15] is to use a technique similar to the maximum principle. This implies immediately that the unique continuation property holds for nonnegative solutions of (1.2), concerning the non-characteristic Cauchy problem.

In general in order to show the unique continuation property we require an estimate, which is called Carleman's inequality. In several cases, from this inequality we can deduce an estimate on the continuous dependence of solutions under their prescribed bound and the bounds of their Cauchy data. But this is not almost correct. For the present we call such an estimate by the "well-behaved" estimate in accordance with John [6], where the elliptic case was treated. On the non-characteristic Cauchy problem for (1.1) Cannon [2] proved a well-behaved estimate for the heat equation. There are several results concerning it. These are referred to [17]. For (1.2) the author and Yamashiro [3] proved a well-behaved estimate for non-negative solutions, under the assumption with $0 < \alpha < 1$. On the main theorem in [3] we assumed that $N = 1$ and the non-characteristic surface is strictly convex. Then an estimation of the L^2-norms of u, u_x and $u^\alpha u_t$ were obtained.

We now drop the assumption of the nonnegative definiteness of solutions. Then (1.2) is considerd from the viewpoint of pure mathematics. Brezis and Friedman [1] proved the existence of weak solutions of some related equations of (1.2) concerning the initial value problem. The regularity of solutions cannot be assured any more. But under some assumptions on N and α, there are infinitely many classical solutions of (1.2) taking both positive and negative values. This is stated at the end of this section. Thus it seems to us that it is meaningful to consider only classical solutions of (1.2) without nonnegative definiteness.

In place of (1.2) we consider the non-characteristic Cauchy problem for the equation

$$(1.3) \qquad |u|^\alpha u_t = \triangle u + \gamma |u|^{-\beta} u,$$

whose lower order term contains the sublinear case.

Using the method in [16], the author [4] has proved a well-behaved estimate for solutions of (1.3) (see [5] too). In [4] some assumptions on α, β, and γ are imposed. In particular it is assumed that $\alpha \geq 1$. But in our approach the L^2-norms of u, ∇u and $|u|^\alpha u_t$ can be estimated. Clearly it is impossible to weaken the assumption $\alpha \geq 1$ by our method in [4].

In this paper our aim is to weaken the above assumption. We can replace it with the assumption $\frac{1}{2} < \alpha < 1$ (see Theorems 1 and 2). The method is quite different from that in [4] ([5]). Furthermore the computation is more simple. However we need to suppose that the Cauchy datas are all zero. That is, the "well-behaved" estimate cannot be proved any longer. This is the reason why Lemma 2 in Section 3 is necessary for the proof of Theorem 1.

In our result we treat only any solution u of (1.3), but not any difference of two solutions of (1.3), namely $u - v$, where u and v are both solutions of (1.3). From the viewpoint of the uniqueness property it is desirable to consider such a difference. But up to now there is no such a consideration. We raise the following two examples.

Masuda [11] proved the backward unique continuation for Navier-Stokes equation. His result is as follows: Let u be a vector valued solution of Navier-Stokes equation with homogeneous boundary condition in a cylindrical domain. Let $u(\cdot, t^0) = 0$ for some t^0. Then $u(\cdot, t)$ vanishes identically for $t \leq t^0$. Moreover Kazdan [8] suggested the following conjecture at the end of his paper: Let u be a p-harmonic function. Let u vanish at a point with infinitely fast order. Then does u vanish identically?

Our method is to yield Carleman's inequality with a weight function. The weight function to be used here is the primitive form in [13].

Finally we show that there are infinitely many classical solutions of (1.2) taking both positive and negative values, under some conditions on N and α. Let $A > 0$ and $\lambda < 0$ be two given numbers. We define

$$u(x,t) = (A + \alpha\lambda t)^{1/\alpha} w(x).$$

If w satisfies

$$\triangle w = \lambda |w|^{\alpha} w \quad \text{in } R^N, \tag{1.4}$$

then we see that u is a solution of (1.2) in $R^N \times (-\infty, A/(\alpha|\lambda|))$. By virtue of Kusano and Naito [9] it is known that (1.4) has infinitely many classical entire solutions taking both positive and negative values, if $N \geq 3$ and $0 < \alpha < 4/(N-2)$.

2 Theorems

Let $x = (x_1, \ldots, x_N)$ be the space variable in R^N, and t be the time variable in R^1. We write $R^{N+1} = R_x^N \times R_t^1$. The origin in $R_x^N (R_x^N \times R_t^1)$ is denoted by $O((O,0))$, respectively.

Throughout this paper let Ω be a domain of class C^1 contained in the half space $\{x_N > 0\}$ of R^{N+1} such that $\partial\Omega \ni (O,0)$. We say that Ω is strongly convex at $(O,0)$, if there exists a positive number δ_0 such that $\Omega \cap \{x_N = \delta\}$ is a bounded domain in R^N for δ with $0 < \delta < \delta_0$ and its diameter tends to 0 as $\delta \to 0$.

Our first aim is to prove

THEOREM 1. *Let Ω be strongly convex at $(O,0)$. Let $u \in C^3(\bar{\Omega})$ satisfy* (1.3) *in Ω. Suppose that $\frac{1}{2} < \alpha < 1$ and $\alpha + \beta < 1$. Suppose that either $\beta\gamma > 0$ or $\gamma = 0$. Then u vanishes identically in a neighborhood of $(O,0)$, if $u = u_t = u_{x_i} = 0$, $i = 1, \ldots, N$, on $\partial\Omega \cap \{x_N < \delta_0\}$.*

Next we don't assume the strong convexity of Ω at $(O,0)$. Then we need to restrict the conclusion to the case of $N = 1$. But on the next theorem the assumptions on α, β and γ are weaker than that of Theorem 1. Writing x_1 as x simply, we have

THEOREM 2. *Let $\partial\Omega \cap \{x = 0\}$ be an open interval in R^1 containing the origin. Let $u \in C^3(\bar{\Omega})$ satisfy* (1.3) *in Ω. Suppose that $\frac{1}{2} < \alpha < 1$, $\alpha + \beta < 1$ and $\beta\gamma \geq 0$. Then u vanishes identically in a neighborhood of the origin, if $u = u_x = 0$ on $\partial\Omega \cap \{x = 0\}$.*

REMARK. Naturally the question arises: Does the conclusion of Theorem 2 hold for general $N \geq 2$? We cannot yet prove this at the present stage. The reason is as follows:

After the Holmgren's transformation, the new variables are denoted by the previous ones. We set $v = \exp\{\lambda x_N\}u$ for the function u in Theorem 2. Our aim is to estimate the L^2-norm of v_{x_N} from above. But if $N \geq 2$, the term of the following L^1-norm of $(1 + \sum_{i=1}^{N-1} x_i^2)^{-1}(\sum_{i=1}^{N-1} x_i v_{x_i x_N})v_{x_N}$ appears and we cannot find any method to estimate it.

3 Preliminaries

We prepare some lemmas. First we have

LEMMA 1. *Let u belong to $C^1[a,b] \cap C^2(a,b)$. Suppose that $u, u' \neq 0$ in (a,b) and $u'(a) = u'(b) = 0$. Then for $\kappa \geq 2$, it holds that*

$$|u'(t)|^\kappa \leq \kappa |u(t)| \sup_{(a,b)}(|u'|^{\kappa-2}|u''|), \quad t \in (a,b). \tag{3.1}$$

PROOF. First assume that $u, u' > 0$ in (a,b). By Cauchy's theorem on the mean value theorem, we see that for any $t \in (a,b)$ there exists a number c such that $a < c < t$ and

$$\begin{aligned}(|u'(t)|^\kappa - |u'(a)|^\kappa)/(u(t) - u(a)) &= (|u'|^\kappa)'(c)/u'(c) \\ &= \kappa |u'(c)|^{\kappa-2} u''(c).\end{aligned} \tag{3.2}$$

From our assumption, $0 < u(t) - u(a) \leq u(t)$. Thus we obtain the required inequality.

Next let $u > 0$ and $u' < 0$ in (a,b). Then the left-hand side of (3.2) can be replaced with

$$(|u'(t)|^\kappa - |u'(b)|^\kappa)/(u(t) - u(b)).$$

Finally let $u < 0$ in (a,b). Then it is enough to replace u with $-u$. Q.E.D.

Generalizing Lemma 1, we have

LEMMA 2. *Let u belong to $C^1[a,b] \cap C^2(a,b)$. Let $u(a) = u'(a) = u(b) = u'(b) = 0$. Suppose that $u'(\xi) = 0$ if $u(\xi) = 0$, $a < \xi < b$. Then (3.1) holds for $\kappa \geq 2$.*

PROOF. It is written that

$$(a,b) = \left(\bigcup_{i=1}^{\infty}(a_i,b_i)\right) \cup E, \quad (a_i,b_i) \cap (a_j,b_j) = \varnothing \quad (i \neq j),$$

where $u = 0$ on E, $u(a_i) = u(b_i) = 0$ and $u \neq 0$ in (a_i, b_i). From our assumption, $u'(a_i) = u'(b_i) = 0$ and $u' = 0$ on E

We rewrite (a_i, b_i) with (a', b') newly. We can rewrite as

$$(a',b') = \left(\bigcup_{i=1}^{\infty}(a_i',b_i')\right) \cap E',$$

where $u' = 0$ on E', $u'(a_i') = u'(b_i') = 0$ and $u' \neq 0$ in (a_i, b_i). By Lemma 1 we see that (3.1) holds on each (a_i', b_i'). This completes the proof. Q.E.D.

Next we give some property for classical solutions of (1.3). Let D be a domain in R^{N+1}.

LEMMA 3. *Let $u \in C^3(D)$ satisfy* (1.3) *in D and suppose that* $0 < \alpha < 1$ *and* $\alpha + \beta < 1$. *Then $u_t(P) = 0$, if $u(P) = 0$ for $P \in D$.*

PROOF. Without loss of generality we may assume that $P = (O, 0)$. We set $v(t) = u(O, t)$. Since $v(0) = 0$, it is written that

$$v(t) = At + o(t) \quad (t \to 0).$$

We often write as $o(t^k)$ simply, Landau's notation $o(t^k)$ $(t \to 0)$. From (1.3), it follows that $(\triangle u)(O, 0) = 0$. Hence it follows that

$$(\triangle u)(O,t) = Bt + o(t).$$

It is enough to show that $A = 0$. Since $v'(t) = A + o(1)$, we have from (1.3) that

$$|At + o(t)|^{\alpha}(A + o(1)) = Bt + o(t) + \gamma|At + o(t)|^{-\beta}(At + o(t)).$$

Hence

$$|A + o(1)|^{\alpha}(A + o(1)) = (B + o(1))|t|^{-\alpha}t + \gamma|A + o(1)|^{-\beta}(A + o(1))|t|^{-\alpha-\beta}t,$$

if $A \neq 0$. But it is a contradiction, because the limits of the both sides are different if $t \to 0$. This means that $u_t(O, 0) = 0$. Q.E.D.

Lastly we have

LEMMA 4. *Suppose that* $\frac{1}{2} \leq \alpha_0 < \alpha$ *and* $\alpha + \beta < 1$. *Let* Ω *be the domain in* Theorem 1. *Let* $u \in C^3(\bar{\Omega})$ *satisfy* (1.3) *in* Ω *and* $u = u_t = u_{x_i} = 0$, $i = 1, \dots, N$, *on* $\partial\Omega \cap \{x_N < \delta_0\}$. *Then it holds that*

$$|u|^{\alpha-1}|u_t| \leq C_0|u|^{\alpha-\alpha_0} \quad \text{in } \Omega \cap \{x_N < \delta_0\},$$

where C_0 *is a positive constant depending only on* α *and* u.

PROOF. If we set $\kappa = \frac{1}{1-\alpha_0} \geq 2$ and $v(t) = u(x,t)$ for any fixed x, then by virtue of Lemma 3, v satisfies the assumption in Lemma 2. Hence (3.1) holds, namely

$$|u_t(x,t)|^\kappa \leq \kappa |u(x,t)| \sup_\Omega (|u_t|^{\kappa-2}|u_{tt}|) \quad (x,t) \in \Omega \cap \{x_N < \delta_0\}.$$

From this we obtain immediately the required inequality. Q.E.D.

4 Proof of Theorems

First we prove Theorem 1.
(Proof of Theorem 1)
We define for $\delta < \delta_0$:

$$\Omega_\delta = \Omega \cap \{x_N < \delta\}, \quad \Gamma_\delta = \partial\Omega \cap \{x_N < \delta\} \quad \text{and} \quad H_\delta = \Omega \cap \{x_N = \delta\}.$$

We put $y = x_N$, $x' = (x_1, \dots, x_{N-1})$ and

$$\triangle' = \sum_{i \neq N} \partial_{x_i}^2, \quad \nabla' = (\partial_{x_1}, \dots, \partial_{x_{N-1}}).$$

For $\lambda < -1$ we set $v = e^{\lambda y}u$. Then from (1.3)

$$v_{yy} + \triangle' v - 2\lambda v_y + \lambda^2 v + \gamma e^{\beta\lambda y}|v|^{-\beta}v - e^{-\alpha\lambda y}|v|^\alpha v_t = 0. \tag{4.1}$$

We retake λ in such a way that $|\lambda|$ is sufficiently large, if necessary.

From now on we denote by (,) the $L^2(\Omega_\delta)$-inner product. We often use the integration by parts without saying. From (4.1) it follows that

$$-(v_{yy} + \triangle' v + \lambda^2 v + \gamma e^{\beta\lambda y}|v|^{-\beta}v, 2\lambda v_y + e^{-\alpha\lambda y}|v|^\alpha v_t) \leq 0. \tag{4.2}$$

We write the left-hand side of (4.2) as I. Then

$$I = -(v_{yy} + \triangle' v + \lambda^2 v, 2\lambda v_y + e^{-\alpha\lambda y}|v|^\alpha v_t) - 2\gamma\lambda(e^{\beta\lambda y}|v|^{-\beta}v, v_y) - \gamma(e^{(\beta-\alpha)\lambda y}|v|^{\alpha-\beta}v, v_t).$$

Clearly

$$-(e^{\beta\lambda y}|v|^{-\beta}v, v_y) = -\frac{1}{2-\beta}(e^{\beta\lambda y}, (|v|^{2-\beta})_y)$$

$$= \frac{\beta}{2-\beta}\lambda(e^{\beta\lambda y}, |v|^{2-\beta}) - \frac{1}{2-\beta}e^{\beta\delta\lambda}\int_{H_\delta}|v|^{2-\beta}\,dx'dt$$

and

$$(e^{(\beta-\alpha)\lambda y}|v|^{\alpha-\beta}v, v_t) = \frac{1}{\alpha-\beta+2}(e^{(\beta-\alpha)\lambda y}, (|v|^{\alpha-\beta+2})_t) = 0.$$

We set

$$I' = -(v_{yy} + \triangle' v + \lambda^2 v, 2\lambda v_y + e^{-\alpha\lambda y}|v|^\alpha v_t).$$

Then from the above

$$I = I' + \frac{2\beta\gamma}{2-\beta}\lambda^2(e^{\beta\lambda y}, |v|^{2-\beta}) - \frac{2\gamma}{2-\beta}\lambda e^{\beta\delta\lambda}\int_{H_\delta}|v|^{2-\beta}\,dx'dt. \tag{4.3}$$

Now we calculate I'. We write

$$\begin{aligned} I' = &-2\lambda(v_{yy}, v_y) - 2\lambda(\triangle' v, v_y) - 2\lambda^3(v, v_y) - (v_{yy}, e^{-\alpha\lambda y}|v|^\alpha v_t) \\ &- (\triangle' v, e^{-\alpha\lambda y}|v|^\alpha v_t) - \lambda^2(e^{-\alpha\lambda y}, |v|^\alpha v v_t). \end{aligned} \tag{4.4}$$

The last term vanishes as previously. We see that

$$\begin{aligned} -(v_{yy}, e^{-\alpha\lambda y}|v|^\alpha v_t) = &-\alpha\lambda(v_y e^{-\alpha\lambda y}, |v|^\alpha v_t) + \alpha(v_y^2, e^{-\alpha\lambda y}|v|^{\alpha-2}v v_t) \\ &+ (v_y e^{-\alpha\lambda y}, |v|^\alpha v_{ty}) - e^{-\alpha\delta\lambda}\int_{H_\delta}|v|^\alpha v_y v_t\,dx'dt. \end{aligned} \tag{4.5}$$

We consider the second term on the right-hand side of (4.5). Since $e^{-\alpha\lambda y}|v|^{\alpha-2}vv_t = |u|^{\alpha-2}uu_t$, we can use Lemma 4. Thus the term $(v_y^2, e^{-\alpha\lambda y}|v|^{\alpha-2}vv_t)$ is finite. More carefully we examine (4.5). By replacing $|v|^\alpha$ with $(v^2+\varepsilon)^{\alpha/2}$ and integrating by parts the left-hand side of (4.5), we take $\varepsilon \to +0$. Then we see that (4.5) is correct.

Furthermore

$$(v_y e^{-\alpha\lambda y}, |v|^\alpha v_{ty}) = \frac{1}{2}(e^{-\alpha\lambda y}|v|^\alpha, (v_y^2)_t) = -\frac{\alpha}{2}(e^{-\alpha\lambda y}|v|^{\alpha-2}vv_t, v_y^2).$$

From (4.1), it follows that

$$(v_y e^{-\alpha\lambda y}, |v|^\alpha v_t) = (v_y, v_{yy}) + (v_y, \triangle' v) - 2\lambda(1, v_y^2) + \lambda^2(v_y, v) + \gamma(v_y, e^{\beta\lambda y}|v|^{-\beta}v).$$

Hence (4.5) becomes

$$(4.6)\qquad -(v_{yy}, e^{-\alpha\lambda y}|v|^{\alpha}v_t) = -\alpha\lambda(v_y, v_{yy}) - \alpha\lambda(v_y, \triangle' v) + 2\alpha\lambda^2(1, v_y^2) - \alpha\lambda^3(v_y, v) - \alpha\gamma\lambda(v_y, e^{\beta\lambda y}|v|^{-\beta}v) + \frac{\alpha}{2}(e^{-\alpha\lambda y}|v|^{\alpha-2}vv_t, v_y^2) - e^{-\alpha\delta\lambda}\int_{H_\delta} |v|^{\alpha}v_y v_t \, dx'dt.$$

Combining (4.4) with (4.6), we obtain

$$(4.7)\qquad I' = -(2+\alpha)\lambda(v_y, v_{yy}) - (2+\alpha)\lambda(v_y, \triangle' v) + 2\alpha\lambda^2(1, v_y^2) - (2+\alpha)\lambda^3(v_y, v) - \alpha\gamma\lambda(v_y, e^{\beta\lambda y}|v|^{-\beta}v) + \frac{\alpha}{2}(e^{-\alpha\lambda y}|v|^{\alpha-2}vv_t, v_y^2) - (\triangle' v, e^{-\alpha\lambda y}|v|^{\alpha}v_t) - e^{-\alpha\delta\lambda}\int_{H_\delta} |v|^{\alpha}v_y v_t \, dx'dt.$$

Obviously

$$(v_y, v) = \frac{1}{2}(1, (v^2)_y) = \frac{1}{2}\int_{H_\delta} v^2 \, dx'dt, \quad (v_y, v_{yy}) = \frac{1}{2}(1, (v_y^2)_y) = \frac{1}{2}\int_{H_\delta} v_y^2 \, dx'dt$$

and

$$(v_y, \triangle' v) = -(\nabla' v_y, \nabla' v) = -\frac{1}{2}(1, (|\nabla' v|^2)_y) = -\frac{1}{2}\int_{H_\delta} |\nabla' v|^2 \, dx'dt.$$

We easily see that

$$(v_y e^{\beta\lambda y}, |v|^{-\beta}v) = \frac{1}{2-\beta}(e^{\beta\lambda y}, (|v|^{2-\beta})_y) = -\frac{\beta\lambda}{2-\beta}(e^{\beta\lambda y}, |v|^{2-\beta}) + \frac{1}{2-\beta}e^{\beta\delta\lambda}\int_{H_\delta} |v|^{2-\beta} \, dx'dt.$$

Further

$$(4.8)\qquad -(\triangle' v, e^{-\alpha\lambda y}|v|^{\alpha}v_t) = \alpha(|\nabla' v|^2 e^{-\alpha\lambda y}, |v|^{\alpha-2}vv_t) + (e^{-\alpha\lambda y}|v|^{\alpha}, \nabla' v\cdot\nabla' v_t)$$

and

$$(4.9)\qquad (e^{-\alpha\lambda y}|v|^{\alpha}, \nabla' v\cdot\nabla' v_t) = \frac{1}{2}(e^{-\alpha\lambda y}|v|^{\alpha}, (|\nabla' v|^2)_t) = -\frac{\alpha}{2}(e^{-\alpha\lambda y}|v|^{\alpha-2}v, v_t|\nabla' v|^2).$$

Hence

$$-(\triangle' v, e^{-\alpha\lambda y}|v|^{\alpha} v_t) = \frac{\alpha}{2}(e^{-\alpha\lambda y}|v|^{\alpha-2} v, v_t|\nabla' v|^2).$$

Combining the above with (4.7), we obtain

$$\begin{aligned} I' &= 2\alpha\lambda^2(1, v_y^2) + \frac{\alpha\beta\gamma}{2-\beta}\lambda^2(e^{\beta\lambda y}, |v|^{2-\beta}) + \frac{\alpha}{2}(e^{-\alpha\lambda y}|v|^{\alpha-2} v, v_t|\nabla v|^2) \\ &\quad - \frac{1}{2}(2+\alpha)\lambda \int_{H_\delta} (v_y^2 - |\nabla' v|^2 + \lambda^2 v^2)\, dx' dt - \frac{\alpha\gamma}{2-\beta}\lambda e^{\beta\delta\lambda} \int_{H_\delta} |v|^{2-\beta}\, dx' dt \\ &\quad - e^{-\alpha\delta\lambda} \int_{H_\delta} |v|^{\alpha} v_y v_t\, dx' dt. \end{aligned}$$

Therefore from (4.2) and (4.3) it holds that

$$\begin{aligned} (4.10)\quad & 2\alpha\lambda^2(1, v_y^2) + \frac{2+\alpha}{2-\beta}\beta\gamma\lambda^2(e^{\beta\lambda y}, |v|^{2-\beta}) + \frac{\alpha}{2}(e^{-\alpha\lambda y}|v|^{\alpha-2} v, v_t|\nabla v|^2) \\ &\le \frac{1}{2}(2+\alpha)\lambda \int_{H_\delta} (|\nabla v|^2 + \lambda^2 v^2)\, dx' dt + \frac{2+\alpha}{2-\beta}\gamma\lambda e^{\beta\delta\lambda} \int_{H_\delta} |v|^{2-\beta}\, dx' dt \\ &\quad + e^{-\alpha\delta\lambda} \int_{H_\delta} |v|^{\alpha} v_y v_t\, dx' dt. \end{aligned}$$

As previously the third term on the left-hand sided of (4.10) equals $\frac{\alpha}{2}(|u|^{\alpha-2} u u_t, |\nabla v|^2)$, which is finite. This means that

$$(4.11)\qquad |(e^{-\alpha\lambda y}|v|^{\alpha-2} v, v_t|\nabla v|^2)| \le C_0(1, |\nabla v|^2).$$

From now on we denote by C_i, $i = 1, 2, \dots$, all positive constants independent of λ. By virtue of (4.11), (4.10) becomes

$$(4.12)\qquad \lambda^2(1, v_y^2) + \beta\gamma\lambda^2(e^{\beta\lambda y}, |v|^{2-\beta}) \le C_1(|\lambda|^3 e^{2\delta\lambda} + (1, |\nabla v|^2)).$$

Lastly we estimate $(1, |\nabla' v|^2)$. Multiplying the both sides of (4.1) by v, we have

$$(v_{yy}, v) + (\triangle' v, v) + \lambda^2(1, v^2) + \gamma(e^{\beta\lambda y}|v|^{-\beta}, v^2) - 2\lambda(v_y, v) - (e^{-\alpha\lambda y}|v|^{\alpha} v_t, v) = 0.$$

Using the previous equalities, we see that

$$(1, |\nabla' v|^2) \le \lambda^2(1, v^2) + \gamma(e^{\beta\lambda y}, |v|^{2-\beta}) + \int_{H_\delta} v v_y\, dx' dt - \lambda \int_{H_\delta} v^2\, dx' dt.$$

By Poincaré's inequality, $(1, v^2) \le C_2(\delta)(1, v_y^2)$, where $C_2(\delta) \to 0$ as $\delta \to 0$. Hence

$$C_1(1, |\nabla' v|^2) \le C_1 C_2(\delta)\lambda^2(1, v_y^2) + C_1\gamma(e^{\beta\lambda y}, |v|^{2-\beta}) + C_3|\lambda| e^{2\delta\lambda}.$$

Combining this with (4.12), we obtain

$$(\lambda^2 - C_1 C_2(\delta)\lambda^2 - C_1)(1, v_y^2) + (\beta\lambda^2 - C_1)\gamma(e^{\beta\lambda y}, |v|^{2-\beta}) \le C_4|\lambda|^3 e^{2\delta\lambda}.$$

Taking δ with $C_1 C_2(\delta) < 1$, we fix it. Moreover taking $-\lambda$ as sufficiently large, we conclude that

$$(1, v^2) \le C_5|\lambda| e^{2\delta\lambda} \le C_6 e^{3\delta\lambda/2}.$$

We proceed as in the usual way. That is, on the left-hand side we replace the integral domain Ω_δ with $\Omega_{\delta/2}$. Then

$$\int_{\Omega_{\delta/2}} u^2 \, dxdt \le C_6 e^{\delta\lambda/2}.$$

Letting $\lambda \to -\infty$, we conclude that $u = 0$ in $\Omega_{\delta/2}$. Thus we have finished the proof of Theorem 1.

Next we prove Theorem 2.

(Proof of Theorem 2)

We use the Holmgren's transformation: $t' = t$, $y' = y + t^2$. Then $u_{y'} = u_y$, $u_{t'} = u_t - 2tu_y$. So (1.3) becomes

$$|u|^\alpha u_{t'} = u_{y'y'} - 2t'|u|^\alpha u_{y'} + \gamma|u|^{-\beta}u. \tag{4.13}$$

We can retake the domain Ω_δ as follows: $\Omega_\delta = \{(y', t') \mid t'^2 < y' < \delta\}$. Then $u = u_{t'} = u_{y'} = 0$ on $\partial\Omega_\delta \cap \{y' = t'^2\}$. In the proof of Lemma 3 we replace $\triangle u$ with $u_{y'y'} - 2t'|u|^\alpha u_{y'}$ and we proceed similarly. Then from (4.13) we see that $u_{t'}(P) = 0$, if $u(P) = 0$ for $P \in \Omega$. Hence the assumption in Lemma 2 is satisfied. This means that Lemma 4 is applicable for such a u.

From now on we denote (y', t') be (y, t) simply. We put $F = 2t|u|^\alpha(v_y - \lambda v)$, where v is the function in the proof of Theorem 1. Then (4.13) is rewritten as

$$(v_{yy} + \lambda^2 v + \gamma e^{\beta\lambda y}|v|^{-\beta}v) - (2\lambda v_y + e^{-\alpha\lambda y}|v|^\alpha v_t) = F. \tag{4.14}$$

We proceed along the proof of Theorem 1. In place of (4.2) we have

$$-(v_{yy} + \lambda^2 v + \gamma e^{\beta\lambda y}|v|^{-\beta}v, 2\lambda v_y + e^{-\alpha\lambda y}|v|^\alpha v_t) \le \frac{1}{2}(1, F^2). \tag{4.15}$$

From (4.14) we obtain

$$(v_y e^{-\alpha\lambda y}, |v|^\alpha v_t) = (v_y, v_{yy}) - 2\lambda(1, v_y^2) + \lambda^2(v_y, v) + \gamma(v_y, e^{\beta\lambda y}|v|^{-\beta}v) - (v_y, F).$$

Thus in place of (4.6)

$$-(v_{yy}, e^{-\alpha\lambda y}|v|^{\alpha}v_t) = -\alpha\lambda(v_y, v_{yy}) + 2\alpha\lambda^2(1, v_y^2) - \alpha\lambda^3(v_y, v)$$
$$- \alpha\gamma\lambda(v_y, e^{\beta\lambda y}|v|^{-\beta}v) + \frac{\alpha}{2}(e^{-\alpha\lambda y}|v|^{\alpha-2}vv_t, v_y^2)$$
$$- e^{-\alpha\delta\lambda}\int_{H_\delta}|v|^{\alpha}v_yv_t\,dt + \alpha\lambda(v_y, F).$$

As previously we denote by I the left-hand side of (4.15). In addition if we set similarly

$$I' = -(v_{yy} + \lambda^2 v, 2\lambda v_y + e^{-\alpha\lambda y}|v|^{\alpha}v_t),$$

the same inequality as in (4.3) holds.

In place of (4.10) we have

$$2\alpha\lambda^2(1, v_y^2) + \frac{2+\alpha}{2-\beta}\beta\gamma\lambda^2(e^{\beta\lambda y}, |v|^{2-\beta}) + \frac{\alpha}{2}(e^{-\alpha\lambda y}|v|^{\alpha-2}v, v_tv_y^2)$$
$$\leq \frac{1}{2}(2+\alpha)\lambda\int_{H_\delta}(v_y^2 + \lambda^2v^2)\,dt + \frac{2+\alpha}{2-\beta}\gamma\lambda e^{\beta\delta\lambda}\int_{H_\delta}|v|^{2-\beta}\,dt + e^{-\alpha\delta\lambda}\int_{H_\delta}|v|^{\alpha}v_yv_t\,dt$$
$$- \alpha\lambda(v_y, F) + \frac{1}{2}(1, F^2).$$

Here we note that $|(v_y, F)| \leq C_7\sqrt{\delta}|\lambda|((1, v_y^2) + (1, v^2))$. Furthermore $(1, F^2) \leq C_7(\lambda^2(1, v^2) + (1, v_y^2))$. Therefore as previously we conclude that

$$\lambda^2(1 - C_7\sqrt{\delta} - C_7C_2(\delta))(1, v_y^2) \leq C_7|\lambda|^3e^{2\delta\lambda}.$$

Thus we have completed the proof of Theorem 2.

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