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SIMPLE CONTINUED FRACTION EXPANSIONS OF SOME VALUES OF CERTAIN HYPERGEOMETRIC FUNCTIONS

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Abstract. The hypergeometric series reduce to many elementary functions. Many of them are known to have the continued fraction expansions. Few of them for certain values, however, are known to have the simple continued fraction expansions which keep regularities. In this paper we show more simple continued fraction expansions holding regularities.

1. Introduction

The *hypergeometric function* $F(a,b,c;z)$ is defined by the power series

$$F(a,b,c;z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \cdots,$$

where a, b and c are any complex constants and c is not any negative integer or 0. It reduces to many elementary functions, for example, $zF(1,1,2;-z) = \log(1+z)$, $F(-k,1,1;-z) = (1+z)^k$ and $zF(1/2,1,3/2,-z^2) = \arctan z$. The *confluent hypergeometric functions* are defined by the entire functions

$$\Phi(b,c;z) = 1 + \frac{b}{c}z + \frac{b(b+1)}{c(c+1)}\frac{z^2}{2!} + \frac{b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!} + \cdots$$

or

$$\Psi(c;z) = 1 + \frac{1}{c}z + \frac{1}{c(c+1)}\frac{z^2}{2!} + \frac{1}{c(c+1)(c+2)}\frac{z^3}{3!} + \cdots,$$

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where a, b and c are any complex constants and c is not any negative integer or 0. They also reduce to many elementary functions, for example,

$$\Phi(1,1;z)=e^z,\quad \frac{z\Psi(3/2;z/4)^2}{\Psi(1/2;z/4)^2}=\tanh z,\quad \frac{z\Psi(3/2;-z^2/4)}{\Psi(1/2;-z^2/4)}=\frac{\sin z}{\cos z}=\tan z$$

and

$$\frac{2(\lambda+1)}{z}\frac{\Psi(\lambda+1;-z^2/4)}{\Psi(\lambda+2;-z^2/4)}=\frac{J_{\lambda+1}(z)}{J_\lambda(z)},$$

where $J_\lambda(z)$ is the Bessel function of the first kind of order λ.

A lot of these functions can be expressed as the equivalence of the continued fraction expansions. The typical one is the *continued fraction of Gauss*:

$$\frac{F(a,b,c;z)}{F(a,b+1,c+1;z)}=1-\cfrac{b_1^*z}{1-\cfrac{b_2^*z}{1-\cfrac{b_3^*z}{1-\cdots}}},$$

where

$$b_{2n+1}^*=\frac{(a+n)(c-b+n)}{(c+2n)(c+2n+1)}\quad (n=0,1,2,\dots)$$

and

$$b_{2n}^*=\frac{(b+n)(c-a+n)}{(c+2n-1)(c+2n)}\quad (n=1,2,3,\dots).$$

See [2], Chapter 6 or [9], Chapter 18, 19 for details in the continued fractions of hypergeometric functions. In this paper we shall refer mainly to [2] though the similar content is described in [9] too.

Arithmetical properties of certain values of the hypergeometric functions by using the continued fraction expansions can be seen in e.g., [7] or [8]. Most of the reduced elementary functions in this paper were considered in [7] too. Much more works in arithmetical properties are, however, handled by using the *simple* continued fraction expansion because it is easier to be handled. But, up to present, only a few of them for some values have been known to have the simple continued fraction expansions holding regularities. For example, for positive integers u and v the simple continued fraction expansions of

$$e^{2/u},\quad \sqrt{\frac{v}{u}}\tanh\frac{1}{\sqrt{uv}},\quad \sqrt{\frac{v}{u}}\tan\frac{1}{\sqrt{uv}},\quad \frac{J_{(v/u)+1}\left(\frac{2}{u}\right)}{J_{v/u}\left(\frac{2}{u}\right)}$$

and

$$\frac{\Psi(u;1)}{\Psi'(u;1)} = [u; u+1, u+2, u+3, \dots]$$

are well-known. All of these values have the so-called *quasi-periodic* or *Hurwitzian* continued fraction expansions. Namely, they are defined as the form

$$[c_0; c_1, \dots, c_n, \overline{Q_1(k), \dots, Q_p(k)}]_{k=1}^{\infty},$$

where c_0 is an integer, $c_1, \dots, c_n$ are positive integers, $Q_1(k), \dots, Q_p(k)$ are polynomials with rational coefficients which take positive integral values for $k = 1, 2, \dots$ and at least one of the polynomials is not constant ([1], [4], [5]). But, it seems that no case has been known for $\deg_k Q_j(k) > 1$. The cases where $Q_j(k)$ is exponential are initiated in [8] and are appeared properly in [3], e.g.,

$$\frac{\sum_{s=0}^{\infty} a^{-(s+1)^2} \prod_{i=1}^{s} (a^{2i} - 1)^{-1}}{\sum_{s=0}^{\infty} a^{-s^2} \prod_{i=1}^{s} (a^{2i} - 1)^{-1}} = [0; a, a^2, a^3, a^4, \dots].$$

In this paper we shall show more simple continued fraction expansions of some values of certain hypergeometric functions.

2. Main Results

We write

$$\{Q_1(k), \dots, Q_p(k)\}_{k=1}^{N} = Q_1(1), \dots, Q_p(1), Q_1(2), \dots, Q_p(2), \dots, Q_1(N), \dots, Q_p(N)$$

for simplicity. Denote $(2k-1)!! = (2k-1)(2k-3)\cdots 3 \cdot 1$ and $(2k)!! = (2k)(2k-2)\cdots 4 \cdot 2$ $(k = 1, 2, \dots)$ with $(-1)!! = 0!! = 1$.

THEOREM 1.

$$\frac{1}{2}\log\frac{A_N + 1}{A_N - 1} = [0; Q(1) - 1, 1, \{Q(k) - 2, 1\}_{k=2}^{N}, \dots,],$$

where $Q(k) = (2k-1)((k-2)!!/(k-1)!!)^2 A_N$ $(k = 1, 2, \dots, N)$ *with* $A_N = (N!)^2$.

EXAMPLE 1. Theorem 1 shows that the simple continued fraction expansion of $(1/2)\log((A_N + 1)/(A_N - 1))$ has regularity at least up to $(2N+2)$-th partial quotient. For example, for $N = 5$ we have

$$\frac{1}{2}\log\frac{A_5+1}{A_5-1}=\left[0;A_5-1,1,3A_5-2,1,5\left(\frac{1}{2}\right)^2A_5-2,1,\right.$$
$$7\left(\frac{2}{3}\right)^2A_5-2,1,9\left(\frac{3!!}{4!!}\right)^2A_5-2,1,$$
$$\left.11\left(\frac{4!!}{5!!}\right)^2A_5-2,1,13\left(\frac{5!!}{6!!}\right)^2A_5-2,\dots\right]$$
$$=[0;14399,1,43198,1,17998,1,44798,1,18223,1,45054,$$
$$1,18280,4,2820,1,38,5,7,2,2,1,3,1,15,1,2,1,3,1,5,$$
$$3,7,2,3,3,1,5,1,1,55,2,2,1,1,5,1,\dots].$$

There is a regularity up to the 12-th partial quotient. However, the 13-th one, $13(5!!/6!!)^2A_5-2=18279+1/4$, is not an integer. Thus, regularity is collapsed after that, showing $18280,4,2820,1,\dots$.

REMARK. Taking N as infinity in Theorem 1, we might have

$$\lim_{N\to\infty}\frac{1}{2}\log\frac{A_N+1}{A_N-1}=[0;Q(1)-1,\overline{1,Q(k)-2}]_{k=2}^{\infty}.$$

But it is nonsense because it is trivially seen that both sides tend to 0.

THEOREM 2.

$$\arctan A_N^{-1}=[0;\{Q(k)\}_{k=1}^{N},\dots],$$

where $Q(k)=(2k-1)((k-2)!!/(k-1)!!)^2A_N$ $(k=1,2,\dots,N)$ *with* $A_N=(N!)^2$.

THEOREM 3.

$$\frac{\arcsin B_N^{-1}}{\sqrt{1-B_N^{-2}}}=\left[0;B_N-1,\left\{1,\frac{4k-1}{2k(2k-1)}B_N-2,1,(4k+1)B_N-2\right\}_{k=1}^{\lfloor N/2\rfloor},\dots\right],$$

where $B_N=N!$.

THEOREM 4. *If* $n\ge 2$ *then*

$$\int_0^{C_N^{-1}} \frac{dt}{1+t^n} = \frac{1}{C_N} F\left(\frac{1}{n}, 1, 1+\frac{1}{n}; -\frac{1}{C_N^n}\right)$$

$$= \left[0; \left\{ ((2k-2)n+1)C_N \prod_{i=1}^{k-1} \left(\frac{(i-1)n+1}{in}\right)^2, \right.\right.$$

$$\left.\left. ((2k-1)n+1)C_N^{n-1} \prod_{i=1}^{k-1} \left(\frac{in}{in+1}\right)^2 \right\}_{k=1}^{N}, \dots \right],$$

where $C_N = \prod_{i=1}^N (in)^2(in+1)^2$.

3. Lemmas

Let α be real. We denote the *continued fraction expansion* of α by

$$a_0^* + \frac{b_1^*}{a_1^* +} \frac{b_2^*}{a_2^* +} \frac{b_3^*}{a_3^* +} \cdots = a_0^* + \cfrac{b_1^*}{a_1^* + \cfrac{b_2^*}{a_2^* + \cfrac{b_3^*}{a_3^* + \cdots}}}.$$

The *simple* (or *regular*) continued fraction expansion of α is given by $\alpha = [a_0; a_1, a_2, \dots]$, satisfying

$$\alpha = a_0 + (1/\alpha_1), \qquad a_0 = \lfloor \alpha \rfloor,$$

$$\alpha_n = a_n + (1/\alpha_{n+1}), \quad \alpha_n = \lfloor \alpha_n \rfloor \quad (n \ge 1).$$

Hence, when $b_1^* = b_2^* = b_3^* = \cdots = 1$ and $a_0^* \in \boldsymbol{Z}$ and $a_i^* \in \boldsymbol{N}$ $(i \ge 1)$, we can write

$$a_0^* + \frac{b_1^*}{a_1^* +} \frac{b_2^*}{a_2^* +} \frac{b_3^*}{a_3^* +} \cdots = [a_0^*; a_1^*, a_2^*, \dots].$$

An equivalence formation between the continued fraction expansion and the simple continued fraction expansion is well-known.

Lemma 1.

$$[a_0; a_1, a_2, \dots] = a_0^* + \frac{b_1^*}{a_1^* +} \frac{b_2^*}{a_2^* +} \frac{b_3^*}{a_3^* +} \cdots \tag{1}$$

if $a_0 = a_0^*$, $a_1 = a_1^*/b_1^*$ *and for* $k = 1, 2, \dots$

$$a_{2k} = \frac{b_{2k-1}^* b_{2k-3}^* \cdots b_1^*}{b_{2k}^* b_{2k-2}^* \cdots b_2^*} a_{2k}^* \quad \textit{and} \quad a_{2k+1} = \frac{b_{2k}^* b_{2k-2}^* \cdots b_2^*}{b_{2k+1}^* b_{2k-1}^* \cdots b_1^*} a_{2k+1}^*.$$

PROOF. See [2], (2.3.23) (p. 35). [2], Theorem 2.6 shows that (1) holds if and only if there is a sequence of non-zero constants $\{r_n\}$ with $r_0 = 1$ such that $b_n^* = r_n r_{n-1}$ $(n = 1, 2, \dots)$ and $a_n^* = r_n a_n$ $(n = 0, 1, \dots)$. Therefore, $a_0 = a_0^*$, $r_1 = b_1^*$, for $k = 1, 2, \dots$

$$r_{2k} = \frac{b_{2k}^* b_{2k-2}^* \cdots b_2^*}{b_{2k-1}^* b_{2k-3}^* \cdots b_1^*} \quad \text{and} \quad r_{2k+1} = \frac{b_{2k+1}^* b_{2k-1}^* \cdots b_1^*}{b_{2k}^* b_{2k-2}^* \cdots b_2^*},$$

yielding $a_1 = a_1^*/r_1 = a_1^*/b_1^*$, and for $k = 1, 2, \dots$

$$a_{2k} = \frac{a_{2k}^*}{r_{2k}} = \frac{b_{2k-1}^* b_{2k-3}^* \cdots b_1^*}{b_{2k}^* b_{2k-2}^* \cdots b_2^*} a_{2k}^*$$

and

$$a_{2k+1} = \frac{a_{2k+1}^*}{r_{2k+1}} = \frac{b_{2k}^* b_{2k-2}^* \cdots b_2^*}{b_{2k+1}^* b_{2k-1}^* \cdots b_1^*} a_{2k+1}^*.$$

If some partial quotients are inadmissible, the following lemma is also useful. Denote

$$^{-}[0; a, b, c, d, \dots] = \frac{1}{a} \, \frac{}{-} \, \frac{1}{b} \, \frac{}{-} \, \frac{1}{c} \, \frac{}{-} \, \frac{1}{d} \, \frac{}{-} \cdots.$$

LEMMA 2.

$$\begin{aligned} ^{-}[0; a, b, c, d, \dots] &= [0; a, -b, c, -d, \dots] \\ &= [0; a-1, 1, b-2, 1, c-2, 1, d-2, 1, \dots]. \end{aligned}$$

PROOF. Since $b_1^* = 1$ and $b_2^* = b_3^* = \cdots = -1$, by Lemma 1 we have for $k \ge 1$

$$a_{2k-1} = \frac{\overbrace{(-1)\cdots(-1)}^{k-1}}{\underbrace{(-1)\cdots(-1)}_{k-1} \cdot 1} a_{2k-1}^* = a_{2k-1}^* \quad \text{and} \quad a_{2k} = \frac{\overbrace{(-1)\cdots(-1)}^{k-1} \cdot 1}{\underbrace{(-1)\cdots(-1)}_{k}} a_{2k}^* = -a_{2k}^*.$$

Hence, $^{-}[0; a, b, c, d, \dots] = [0; a, -b, c, -d, \dots]$. By repeating the relation

$$[\dots, a, -b, \gamma] = [\dots, a-1, 1, b-1, -\gamma]$$

in [6], Section 6, we obtain

$$\begin{aligned} [0; a, -b, c, -d, \dots] &= [0; a-1, 1, b-1, -c, d, \dots] \\ &= [0; a-1, 1, b-2, 1, c-1, -d, \dots] \\ &= [0; a-1, 1, b-2, 1, c-2, 1, d-1, \dots] = \cdots. \end{aligned}$$

4. Proofs of the Theorems

PROOF OF THEOREM 1. From [2], (6.1.18) (p. 203)

$$\log\frac{1+z}{1-z} = 2zF\left(\frac{1}{2},1,\frac{3}{2};z^2\right)$$

$$= \frac{2z}{1\ -}\ \frac{1^2z^2}{3\ -}\ \frac{2^2z^2}{5\ -}\ \frac{3^2z^2}{7\ -}\ \frac{4^2z^2}{9\ -}\ \cdots.$$

Since $a_0^* = 0$, $a_{2k-1}^* = 4k-3$, $a_{2k}^* = 4k-1$ $(k \geq 1)$, $b_1^* = 2z$, $b_{2k}^* = -(2k-1)^2z^2$ and $b_{2k+1}^* = -(2k)^2z^2$ $(k \geq 1)$, we have $a_0 = 0$, for $k \geq 1$

$$a_{2k-1} = \frac{(-(2k-3)^2z^2)(-(2k-5)^2z^2)\cdots(-1^2z^2)}{(-(2k-2)^2z^2)(-(2k-4)^2z^2)\cdots(-2^2z^2)\cdot 2z}(4k-3)$$

$$= \frac{4k-3}{2z}\left(\frac{(2k-3)!!}{(2k-2)!!}\right)^2$$

and

$$a_{2k} = \frac{(-(2k-2)^2z^2)(-(2k-4)^2z^2)\cdots(-2^2z^2)\cdot 2z}{(-(2k-1)^2z^2)(-(2k-3)^2z^2)\cdots(-3^2z^2)(-1^2z^2)}(4k-1)$$

$$= -\frac{2(4k-1)}{z}\left(\frac{(2k-2)!!}{(2k-1)!!}\right)^2.$$

From $2[a_0;a_1,a_2,a_3,\ldots] = \left[2a_0;\frac{a_1}{2},2a_2,\frac{a_3}{2},\ldots\right]$ and Lemma 2 we obtain

$$\frac{1}{2}\log\frac{1+z}{1-z} = zF\left(\frac{1}{2},1,\frac{3}{2};z^2\right)$$

$$= \left[0;\overline{\frac{4k-3}{z}\left(\frac{(2k-3)!!}{(2k-2)!!}\right)^2,-\frac{4k-1}{z}\left(\frac{(2k-2)!!}{(2k-1)!!}\right)^2}\right]_{k=1}^{\infty}$$

$$\overset{-}{=} \left[0;\overline{\frac{2k-1}{z}\left(\frac{(k-2)!!}{(k-1)!!}\right)^2}\right]_{k=1}^{\infty}.$$

But, the partial quotients are inadmissible if they are not integers. Then this expression means nonsense. Therefore, we must take $z^{-1} = (N!)^2$ so that

$$\frac{2k-1}{z}\left(\frac{(k-2)!!}{(k-1)!!}\right)^2$$

can become integral for $k = 1,2,\ldots,N+1$.

PROOF OF THEOREM 2. From [2], (2.1.19) (p. 24)

$$\arctan z = \frac{z}{1+}\,\frac{z^2}{3+}\,\frac{4z^2}{5+}\,\frac{9z^2}{7+}\,\frac{16z^2}{9+}\cdots.$$

Since $a_{2k-1}^* = 4k-3$, $a_{2k}^* = 4k-1$ $(k \ge 1)$, $b_k^* = (k-1)^2 z^2$ $(k \ge 2)$ with $b_1^* = z$, we have for $k \ge 1$

$$a_{2k-1} = \frac{(2k-3)^2(2k-5)^2\cdots 3^2 1^2}{(2k-2)^2(2k-4)^2\cdots 4^2 2^2\cdot z}(4k-3) = \left(\frac{(2k-3)!!}{(2k-2)!!}\right)^2\cdot\frac{4k-3}{z}$$

and

$$a_{2k} = \frac{(2k-2)^2(2k-4)^2\cdots 2^2\cdot z}{(2k-1)^2(2k-3)^2\cdots 3^2 1^2 z^2}(4k-1) = \left(\frac{(2k-2)!!}{(2k-1)!!}\right)^2\cdot\frac{4k-1}{z}$$

Hence, when $z^{-1} = (N!)^2$, a_i is integral for $i = 1, 2, \ldots, N+1$.

PROOF OF THEOREM 3. From [2], (6.1.20) (p. 203)

$$\frac{\arcsin z}{\sqrt{1-z^2}} = \frac{zF\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};z^2\right)}{F\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2};z^2\right)}$$

$$= \frac{z}{1-}\,\frac{1\cdot 2z^2}{3-}\,\frac{1\cdot 2z^2}{5-}\,\frac{3\cdot 4z^2}{7-}\,\frac{3\cdot 4z^2}{9-}\,\frac{5\cdot 6z^2}{11-}\,\frac{5\cdot 6z^2}{13-}\cdots.$$

Since $a_{2k-1}^* = 4k-3$, $a_{2k}^* = 4k-1$, $b_{2k}^* = b_{2k+1}^* = -(2k-1)(2k)z^2$ $(k \ge 1)$ with $b_1^* = z$, we have for $k \ge 1$

$$a_{2k-1} = \frac{(2k-2)(2k-3)\cdot(2k-4)(2k-5)\cdots 2\cdot 1}{(2k-2)(2k-3)\cdot(2k-4)(2k-5)\cdots 2\cdot 1\cdot z}(4k-3) = \frac{4k-3}{z}$$

and

$$a_{2k} = -\frac{(2k-2)(2k-3)\cdot(2k-4)(2k-5)\cdots 2\cdot 1\cdot z}{(2k)(2k-1)\cdot(2k-2)(2k-3)\cdots 2\cdot 1\cdot z^2}(4k-1) = -\frac{4k-1}{(2k)(2k-1)z}.$$

Hence, when $z^{-1} = N!$, we have $a_{2k-1} \in \boldsymbol{N}$ $(k = 1, 2, \ldots)$ and $-a_{2k} \in \boldsymbol{N}$ $(k = 1, 2, \ldots, \lfloor N/2 \rfloor)$. Together with Lemma 2 we have the desired result.

PROOF OF THEOREM 4. From [2], (6.1.19) (p. 203)

$$\int_0^z \frac{dt}{1+t^n} = zF\left(\frac{1}{n}, 1, 1+\frac{1}{n}; -z^n\right)$$

$$= \frac{z}{1+}\,\frac{1^2\cdot z^n}{n+1+}\,\frac{(1\cdot n)^2 z^n}{2n+1+}\,\frac{(n+1)^2 z^n}{3n+1+}\,\frac{(2n)^2\cdot z^n}{4n+1+}\,\frac{(2n+1)^2 z^n}{5n+1+}\cdots.$$

Since $a_k^* = ((k-1)n+1)$, $b_1^* = z$, $b_{2k}^* = ((k-1)n+1)^2 z^n$ and $b_{2k+1}^* = (kn)^2 z^n$ $(k \geq 1)$, we have for $k \geq 1$

$$a_{2k-1} = \frac{((k-2)n+1)^2((k-3)n+1)^2 \cdots (n+1)^2}{((k-1)n)^2((k-2)n)^2 \cdots n^2 z}((2k-2)n+1)$$

and

$$a_{2k} = \frac{((k-1)n)^2((k-2)n)^2 \cdots n^2 z}{((k-1)n+1)^2((k-2)n+1)^2 \cdots (n+1)^2 z^n}((2k-1)n+1).$$

Hence, when $z^{-1} = \prod_{i=1}^N (in)^2(in+1)^2$, a_{2k-1} and a_{2k} become integral for $k = 1, 2, \ldots, N+1$.

5. General Hypergeometric Functions

The continued fraction expansion of Gauss $F(a,b+1,c+1;z)/F(a,b,c;z)$ or $\Phi(1,c;z)$ is well-known, but it is not easy to transform it into the simple one. We consider the modified expression about $zF(a,b+1,c+1;z^2)/F(a,b,c;z^2)$ or $z\Phi(1,c;z^2)$.

THEOREM 5.

$$\frac{\Phi(1,c;B_N^{-2})}{B_N} = \left[0; \left\{ \frac{(c+k-3)!}{(k-1)!(c-1)!}(c+2k-3)B_N - 1, 1, \right.\right.$$
$$\left.\left. \frac{(k-1)!(c-1)!}{(c+k-2)!}(c+2k-2)B_N - 1 \right\}_{k=1}^{N-c+2}, \ldots \right],$$

where c is a positive integer and $B_N = N!$.

PROOF. Since from [2], (7.1.52) and (7.1.54)

$$\Phi(1,c;z) = 1 + \frac{1}{c}z + \frac{1}{c(c+1)}z^2 + \frac{1}{c(c+1)(c+2)}z^3 + \cdots$$
$$= \frac{1}{1-}\,\frac{z}{c+}\,\frac{1 \cdot z}{c+1-}\,\frac{cz}{c+2+}\,\frac{2z}{c+3-}\,\frac{(c+1)z}{c+4}\,\frac{3z}{+c+5-}\cdots,$$

we have

$$z\Phi(1,c;z^2) = z + \frac{1}{c}z^3 + \frac{1}{c(c+1)}z^5 + \frac{1}{c(c+1)(c+2)}z^7 + \cdots$$
$$= \frac{z}{1-}\,\frac{z^2}{c+}\,\frac{1 \cdot z^2}{c+1-}\,\frac{cz^2}{c+2+}\,\frac{2z^2}{c+3-}\,\frac{(c+1)z^2}{c+4}\,\frac{3z^2}{+c+5-}\cdots.$$

Since $a_1^* = 1$, $a_k^* = c + k - 2$ $(k \geq 2)$, $b_1^* = z$, $b_2^* = -z^2$, $b_{2k+1}^* = kz^2$ and $b_{2k+2}^* = -(c + k - 1)z^2$ $(k \geq 1)$, we have for $k \geq 1$

$$a_{2k-1} = \frac{(-(c+k-3)z^2)(-(c+k-4)z^2)\cdots(-cz^2)(-z^2)}{(k-1)z^2 \cdot (k-2)z^2 \cdots 1z^2 \cdot z}(c+2k-3)$$

$$= (-1)^{k-1}\frac{(c+k-3)!}{(k-1)!(c-1)!}\frac{c+2k-3}{z}$$

and

$$a_{2k} = \frac{(k-1)z^2 \cdot (k-2)z^2 \cdots 1z^2 \cdot z}{(-(c+k-2)z^2)(-(c+k-3)z^2)\cdots(-cz^2)(-z^2)}(c+2k-2)$$

$$= (-1)^k \frac{(k-1)!(c-1)!}{(c+k-2)!}\frac{c+2k-2}{z}.$$

Since $[\dots, a, -b, \gamma] = [\dots, a-1, 1, b-1, -\gamma]$ as seen in [6], Section 6, one has

$$[0; a_1', -a_2', -a_3', a_4', a_5', -a_6', -a_7', \dots]$$
$$= [0; a_1' - 1, 1, a_2' - 1, a_3' - 1, 1, a_4' - 1, a_5' - 1, 1, a_6' - 1, a_7' - 1, 1, \dots].$$

Hence,

$$z\Phi(1, c; z^2) = [0; \overline{|a_{2k-1}| - 1, 1, |a_{2k}| - 1}]_{k=1}^{\infty}.$$

When $z^{-1} = N!$, $a_{2k-1} \in \boldsymbol{N}$ $(k = 1, 2, \dots, N+1)$ and $a_{2k} \in \boldsymbol{N}$ $(k = 1, 2, \dots, N - c + 2)$.

THEOREM 6.

$$\frac{F(a, b+1, c+1; D_N^{-2})}{D_N \cdot F(a, b, c; D_N^{-2})} = [0; D_N - 1, \{1, |a_{2k}| - 2, 1, a_{2k+1} - 2\}_{k=1}^{N'}, \dots]$$

with $N' = N - \max(b, c-a, a-1, c-b-1)$, *where for* $k = 1, 2, \dots, N$

$$a_{2k} = -\frac{(b+k-1)!(c-a+k-1)!(a-1)!(c-b-1)!}{(a+k-1)!(c-b+k-1)!b!(c-a)!}(c+2k-1)cD_N$$

and

$$a_{2k+1} = \frac{(a+k-1)!(c-b+k-1)!b!(c-a)!}{(b+k)!(c-a+k)!(a-1)!(c-b-1)!}\frac{c+2k}{c}D_N$$

with $D_N = c(N!)^2$.

PROOF. Since

$$\frac{zF(a,b+1,c+1;z^2)}{F(a,b,c;z^2)}=\frac{z}{1}\,-\,\frac{u_1z^2}{1}\,-\,\frac{u_2z^2}{1}\,-\,\frac{u_3z^2}{1}\,-\,\frac{u_4z^2}{1}\,-\,\cdots,$$

where

$$u_{2k+1}=\frac{(a+k)(c-b+k)}{(c+2k)(c+2k+1)}\quad(k=0,1,2,\dots)$$

and

$$u_{2k}=\frac{(b+k)(c-a+k)}{(c+2k-1)(c+2k)}\quad(k=1,2,3,\dots)$$

([2], Theorem 6.1), we have for $k=0,1,2,\dots$

$$\begin{aligned}a_{2k+1}&=\frac{(-u_{2k-1}z^2)(-u_{2k-3}z^2)\cdots(-u_1z^2)}{(-u_{2k}z^2)(-u_{2k-2}z^2)\cdots(-u_2z^2)z}=\frac{u_{2k-1}u_{2k-3}\cdots u_1}{u_{2k}u_{2k-2}\cdots u_2}\frac{1}{z}\\&=\frac{(a+k-1)!(c-b+k-1)!b!(c-a)!}{(b+k)!(c-a+k)!(a-1)!(c-b-1)!}\frac{c+2k}{cz}\end{aligned}$$

and for $k=1,2,\dots$

$$\begin{aligned}a_{2k}&=\frac{(-u_{2k-2}z^2)(-u_{2k-4}z^2)\cdots(-u_2z^2)z}{(-u_{2k-1}z^2)(-u_{2k-3}z^2)\cdots(-u_1z^2)}=-\frac{u_{2k-2}u_{2k-4}\cdots u_2}{u_{2k-1}u_{2k-3}\cdots u_1}\frac{1}{z}\\&=-\frac{(b+k-1)!(c-a+k-1)!(a-1)!(c-b-1)!}{(a+k-1)!(c-b+k-1)!b!(c-a)!}\frac{(c+2k-1)c}{z}.\end{aligned}$$

When $z^{-1}=c(N!)^2$, $a_{2k+1}\in N$ $(k\le N-\max(b,c-a))$ and $a_{2k}\in N$ $(k\le N+1-\max(a,c-b))$. Together with Lemma 2 we have the desired result.

Denote

$$\Omega(a,b;z)=1+abz+a(a+1)b(b+1)\frac{z^2}{2!}+a(a+1)(a+2)b(b+1)(b+2)\frac{z^3}{3!}+\cdots$$

([9], (89.5)). One has

$$\Omega(a,b;-z)=\frac{1}{\Gamma(a)}\int_0^\infty\frac{e^{-u}u^{a-1}\,du}{(1+zu)^b}.$$

Since

$$\frac{\Omega(a,b;-z)}{\Omega(a,b-1;-z)}=\frac{1}{1}\,+\,\frac{az}{1}\,+\,\frac{bz}{1}\,+\,\frac{(a+1)z}{1}\,+\,\frac{(b+1)z}{1}\,+\,\frac{(a+2)z}{1}\,+\,\cdots$$

([9], (92.1)), we have

$$z\frac{\Omega(a,b;-z^2)}{\Omega(a,b-1;-z^2)}=\frac{z}{1+}\,\frac{az^2}{1+}\,\frac{bz^2}{1+}\,\frac{(a+1)z^2}{1+}\,\frac{(b+1)z^2}{1+}\,\frac{(a+2)z^2}{1+}\cdots,$$

yielding the following.

Theorem 7. *For positive integers a and b we have*

$$\frac{\Omega(a,b;-B_N^{-2})}{B_N\cdot\Omega(a,b-1;-B_N^{-2})}=\left[0;\left\{\frac{(a+k-2)!(b-1)!}{(b+k-2)!(a-1)!}B_N,\frac{(b+k-2)!(a-1)!}{(a+k-1)!(b-1)!}B_N\right\}_{k=1}^{N'},\dots\right]$$

with $N'=N+2-\max(a+1,b)$, where $B_N=N!$.

If we take $\prod_{n=\min(a,b)}^{\max(a,b)+N-1}n$ instead of B_N, this continued fraction expansion is valid from $k=1$ up to $k=N$.

Since

$$\int_0^\infty\frac{e^{-u}\,du}{(z+u)^a}=\frac{z^{1-a}}{z+}\,\frac{a}{1+}\,\frac{1}{z+}\,\frac{a+1}{1+}\,\frac{2}{z+}\,\frac{a+2}{1+}\,\frac{3}{z+}\cdots,$$

([9], (92.8)), we have the following.

Corollary 2.

$$E_N^{2a-1}\int_0^\infty\frac{e^{-u}\,du}{(u+E_N^2)^a}=\left[0;\left\{\binom{a+k-2}{a-1}E_N,\binom{a+k-1}{a-1}^{-1}\frac{E_N}{k}\right\}_{k=1}^{N},\dots\right],$$

where $E_N=\prod_{n=0}^N(n+a)$.

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