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# ON EXPONENTIAL SUMS OVER PRIMES IN ARITHMETIC PROGRESSIONS

By

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## 1. Introduction

I. M. Vinogradov's proof of the ternary Goldbach problem is based upon bounds for the exponential sum

$$\sum_{n\le x}\Lambda(n)e(\alpha n) \tag{1}$$

with a wide uniformity in real $\alpha$, where $\Lambda$ is the von Mangoldt function and, for real $\theta$, $e(\theta)=\exp(2\pi i\theta)$. By using a combinatrial identity, R. C. Vaughan presented an elegant simple argument on it, see [2], for instance.

J.-r. Chen's theorem on the binary Goldbach problem is built upon the linear sieve and the mean prime number theorem, vide [5]. According to H. Iwaniec [6], the Rosser's weight of the linear sieve has the well-factorable property. An arithmetic function $\lambda$ is called "well-factorable of level $D$", if for any $D_1, D_2\ge 1$, $D=D_1D_2$, there exist two functions $\lambda_1$ and $\lambda_2$ supporting in $(0,D_1]$ and $(0,D_2]$ respectively such that $|\lambda_1|\le 1$, $|\lambda_2|\le 1$ and $\lambda=\lambda_1*\lambda_2$. Also the mean prime number theorem has been surprisingly developed by E. Fouvry and H. Iwaniec [4], E. Fouvry [3] and E. Bombieri, J.-B. Friedlander and H. Iwaniec [1]. In [1] they established a non-trivial bound of the averaging sum

$$\sum_{(d,c)=1}\lambda(d)\left(\sum_{\substack{n\le x\\ n\equiv c\ (\mathrm{mod}\, d)}}\Lambda(n)-\frac{x}{\varphi(d)}\right) \tag{2}$$

for any fixed integer $c\ne 0$ and for any well-factorable function $\lambda$ of level $D=x^{4/7-\varepsilon}$, $\varepsilon>0$.

Recently D. I. Tolev mixed the ternary problem with the binary problem, and was led to a blend of (1) and (2):

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$$\sum_{\substack{d \le D \\ (d,c)=1}} \gamma(d) \sum_{\substack{n \le x \\ n \equiv c \pmod{d}}} \Lambda(n) e(\alpha n). \tag{3}$$

In [8] he successfully estimated (3) with a wide uniformity in $\alpha$, providing that $\gamma \ll 1$ and $D = x^{1/3}(\log x)^{-B}$ where $B > 0$ is some constant. As the sequence $\gamma$ is regarded as sieving weights, it is of some interest to extend the level of distribution $D$ in (3). Thus the purpose of this paper is to show that, if $\gamma$ is well-factorable, then the above exponent 1/3 may be replaced by 4/9.

THEOREM. *Suppose that* $|\alpha - a/q| \le q^{-2}$ *with* $(a, q) = 1$. *Let* $c \ne 0$ *be an integer. Let* $B > 0$ *be given. Then, for any well-factorable function* $\lambda$ *of level* $D = x^{4/9}(\log x)^{-B}$, *we have that*

$$\sum_{(d,c)=1} \lambda(d) \sum_{\substack{n \le x \\ n \equiv c \pmod{d}}} \Lambda(n) e(\alpha n) \ll x^{7/8}(xq^{-1} + x(\log x)^{-4B} + q)^{1/8}(\log x)^{13}$$

*where the implied constant depends only on* $B$.

This assertion would be applicable to the problems of [7, 8, 9] and capable to make a modest improvement upon these results. As well as [8], our argument is elementary.

The notation of this paper is standard in Number Theory. Although the symbol $\|\cdot\|$ is used in two different meanings, there would be no confusion. For real $\theta$, $\|\theta\|$ is the distance from $\theta$ to the nearest integer. For sequence $a = (a(n))$, $\|a\|$ stands for the $l^2$-norm. $n \sim N$ means that $N < n \le cN$ with some constant $0 < c \le 2$. We use the abbreviation $L = \log x$.

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## 2. Proof of Theorem

We may assume that $q \le x$, for otherwise our assertion is trivial. We choose the parameters of the well-factorable property as $D_1 = x^{1/3}L^{-B}$ and $D_2 = x^{1/9}$, so that $D = D_1 D_2 = x^{4/9}L^{-B}$. By a dyadic decomposition of summation ranges, it is sufficient to show that

$$P := \sum_{\substack{m \sim M \\ (m,c)=1}} \sum_{\substack{n \sim N \\ (n,c)=1}} f(m)g(n) \sum_{\substack{k \le x \\ k \equiv c \pmod{mn}}} \Lambda(k)e(\alpha k) \ll x^{7/8}(xq^{-1} + xL^{-4B} + q)^{1/8}L^{11} \tag{4}$$

uniformly for

$$1 \ll M \ll x^{1/3}L^{-B}, \quad 1 \ll N \ll x^{1/9}; \quad f \ll 1, \ g \ll 1. \tag{5}$$

We next decompose $\Lambda$ by means of the combinatrial identity of R. C. Vaughan. We take the parameters in [2, § 24] as $U = V = x^{1/3}$. Then $\Lambda$ is written as the sum of $\Lambda_0$ and $\Lambda_{ij}$'s, $1 \le i, j \le 2$, where

$$\Lambda_0(k) = \begin{cases} \Lambda(k) & k \le x^{1/3} \\ 0 & \text{otherwise;} \end{cases}$$

$$\Lambda_{1j}(k) = \sum_{\substack{th=k \\ t \le x^{1/3}}} a_j(t) l_j(h); \quad \Lambda_{2j}(k) = \sum_{\substack{th=k \\ x^{1/3} < t, h \le x^{2/3}}} b_j(t) d_j(h)$$

with $a_1(n) = b_1(n) \ll \log n$, $a_2(n) \ll 1$, $l_1(n) = d_1(n) = 1$, $l_2(n) = \log n$, $b_2(n) \ll \Lambda(n)$ and $d_2(n) \ll \tau(n)$.

The contribution of $\Lambda_0$ to $P$ is at most

$$\sum_{m \sim M} \sum_{n \sim N} L\left(\frac{x^{1/3}}{mn} + 1\right) \ll (x^{1/3} + MN)L \ll x^{1/2}.$$

Let $Q_{ij}$ be the partial sum of $P$ corresponding to $\Lambda_{ij}$, $1 \le i, j \le 2$. Then

$$P \ll x^{1/2} + \sum_{i=1,2} \sum_{j=1,2} |Q_{ij}|. \tag{6}$$

We first consider the "type I" sum $Q_{1j}$, $j = 1, 2$. Since $l_1(h) = 1$, we see that

$$Q_{11} \ll \sum_{\substack{m \sim M \\ (m,c)=1}} \sum_{\substack{n \sim N \\ (n,c)=1}} \sum_{t \le x^{1/3}} |a_1(t)| \left| \sum_{\substack{th \le x \\ th \equiv c \ (\mathrm{mod}\ mn)}} e(\alpha th) \right|.$$

The above congruence is soluble if and only if $(t, mn) = 1$, and equivalent to $h \equiv r \pmod{mn}$ with some $r$. Writing $h = r + mnk$, we change the variable $h$ for $k$. Then $k$ runs through some interval of length $\le x(tmn)^{-1}$. Here we note that $tmn \ll x^{1/3}MN \ll x$ or $x(tmn)^{-1} \gg 1$. Hence we have that

$$\begin{aligned} Q_{11} &\ll L \sum_m \sum_n \sum_t \left| \sum_k e(\alpha tmnk) \right| \\ &\ll L \sum_{\substack{m \sim M \\ (m,ct)=1}} \sum_{\substack{n \sim N \\ (n,ct)=1}} \sum_{t \le x^{1/3}} \min\left(\frac{x}{tmn}, \frac{1}{\|\alpha tmn\|}\right) \\ &\ll L \sum_{k \ll MNx^{1/3}} \tau_3(k) \min\left(\frac{x}{k}, \frac{1}{\|\alpha k\|}\right) \end{aligned} \tag{7}$$

$$\ll L\left(\sum_k \tau_3(k)^2 \frac{x}{k}\right)^{1/2} \left(\sum_k \min\left(\frac{x}{k}, \frac{1}{\|\alpha k\|}\right)\right)^{1/2}$$

$$\ll L(xL^9)^{1/2}((xq^{-1} + MNx^{1/3} + q)L)^{1/2}$$

$$\ll x^{1/2}(xq^{-1} + MNx^{1/3} + q)^{1/2}L^6$$

by Cauchy's inequality and [2, §25, (3)]. The estimation of $Q_{12}$ is similar.

We proceed to the "type II" sum $Q_{2j}$, $j = 1, 2$. Put

$$R = R(M, N, U, V; f, g, r, s)$$

$$= \sum_{\substack{m \sim M \\ (m,c)=1}} \sum_{\substack{n \sim N \\ (n,c)=1}} f(m)g(n) \sum_{u \sim U} \sum_{\substack{v \sim V \\ uv \le x \\ uv \equiv c \ (\mathrm{mod}\ mn)}} r(u)s(v)e(\alpha uv).$$

By a dyadic decomposition of summation ranges, we find that

$$|Q_{21}| + |Q_{22}| \ll L^2 \sup |R| \tag{8}$$

where the supremum is taken over all parameters $M, N, U, V$ and all sequences $f, g, r, s$ satisfying (5) and

$$x^{1/3} \ll U, \quad V \ll x^{2/3}; \quad r(k) \ll \log k, \quad s(k) \ll \tau(k). \tag{9}$$

In the next section we shall show that

$$|R|^2 \ll \|r\|^2 \|s\|^2 x^{3/4} (xq^{-1} + xL^{-4B} + q)^{1/4} L^{13} \tag{10}$$

uniformly. We here note that, by symmetry, we may assume

$$V \ll U. \tag{11}$$

Therefore, since $\|r\|^2 \|s\|^2 \ll xL^5$, (4) follows from (6), (7), (8) and (10). Our proof of Theorem is thus reduced to the estimation (10) for $R$ under the conditions (5), (9) and (11).

### 3. Type II Sum

In order to show (10), we first arrange $R$ in the following three ways:

$$\sum_u \left|\sum_m \sum_n \sum_v\right|; \quad \sum_u \sum_m \left|\sum_n \sum_v\right|; \quad \sum_u \sum_m \sum_n \left|\sum_v\right|.$$

We then examine each of these, and compare the three resulting bounds for $R$.

We begin by taking the second way. It follows from Cauchy's inequality that

$$(12)\qquad |R|^2 \le \|r\|^2 M \sum_{u\sim U} \sum_{\substack{m\sim M\\(m,c)=1}} \left| \sum_{\substack{n\sim N\\(n,c)=1}} g(n) \sum_{\substack{v\sim V\\ uv\le x\\ uv\equiv c \pmod{mn}}} s(v)e(\alpha uv) \right|^2$$

$$= \|r\|^2 MS, \text{ say.}$$

We expand the square is $S$ and bring the sum over $u$ inside to obtain

$$S = \sum_{\substack{m\sim M\\(m,c)=1}} \sum_{\substack{n_1\sim N\\(n_1,c)=1}} \sum_{\substack{n_2\sim N\\(n_2,c)=1}} g(n_1)\overline{g(n_2)} \sum_{v_1\sim V} \sum_{v_2\sim V} s(v_1)\overline{s(v_2)} \sum_{\substack{u\sim U\\ uv_1,uv_2\le x\\ uv_1\equiv c \pmod{mn_1}\\ uv_2\equiv c \pmod{mn_2}}} e(\alpha u(v_1-v_2)).$$

The above simultaneous congruences are soluble if and only if $(v_1, mn_1) = (v_2, mn_2) = 1$ and $v_1 \equiv v_2 \pmod{m(n_1,n_2)}$, and reduce to the single equation $u \equiv b \pmod{m[n_1,n_2]}$ with some $b$. Writing $v_1 = v_2 + m(n_1,n_2)k$ and $u = b + m[n_1,n_2]l$, we change the variables $(v_1, v_2, u)$ for $(k, v, l)$. Then we see that

$$(13)\qquad |m(n_1,n_2)k| = |v_1 - v_2| \le V,$$

and that $l$ runs through some interval of length $\le U(m[n_1,n_2])^{-1}$. Also

$$\begin{aligned} u(v_1 - v_2) &= (b + m[n_1,n_2]l)m(n_1,n_2)k \\ &= bm(n_1,n_2)k + m^2 n_1 n_2 kl. \end{aligned}$$

Hence we have that

$$S \ll \sum_m \sum_{n_1} \sum_{n_2} \sum_k \sum_v |s(v + m(n_1,n_2)k)||s(v)| \left| \sum_l e(\alpha m^2 n_1 n_2 kl) \right|.$$

The terms with $k = 0$ contribute

$$(14)\qquad \sum_m \sum_{n_1} \sum_{n_2} \|s\|^2 \sum_l 1 \ll \|s\|^2 \sum_{m\sim M} \sum_{n_1\sim N} \sum_{n_2\sim N} \left( \frac{U}{m[n_1,n_2]} + 1 \right)$$

$$\ll \|s\|^2 (UL^3 + MN^2).$$

As for the terms with $k \ne 0$, we may assume $k > 0$. Put $n_1 n_2 k = j$. Then, by (13), the condition on $j$ becomes

$$0 < mj = mn_1 n_2 k = [n_1,n_2]m(n_1,n_2)k \ll N^2 V.$$

Also the trivial bound for the sum over $l$ is

$$\ll \frac{U}{m[n_1, n_2]} + 1 = \frac{Um(n_1, n_2)k}{m^2 n_1 n_2 k} + 1 \ll \frac{UV}{m^2 j} + 1 \ll \frac{x}{m^2 j} + 1.$$

Moreover the sum over $v$ is $O(\|s\|^2)$ because of $ab \ll a^2 + b^2$. Hence the sum under consideration is bounded by

$$(15) \qquad \sum_m \sum_j \tau_3(j) \|s\|^2 \left| \sum_l e(\alpha m^2 jl) \right|$$
$$\ll \|s\|^2 \sum_{m \sim M} \sum_{mj \ll N^2 V} \tau_3(j) \min\left(\frac{x}{m^2 j} + 1, \frac{1}{\|\alpha m^2 j\|}\right).$$

Here we note that $\min(a+1, b) \le \min(a, b) + 1$. Thus, substituting (14) and (15) into (12), we have that

$$(16) \quad |R|^2 \ll \|r\|^2 \|s\|^2$$
$$\cdot \left\{ M \sum_{m \sim M} \sum_{mj \ll N^2 V} \tau_3(j) \min\left(\frac{x}{m^2 j}, \frac{1}{\|\alpha m^2 j\|}\right) + MUL^3 + M^2 N^2 + MN^2 VL^3 \right\}.$$

Now, in the above double sum, we split up the summation range for $j$. We then appeal to

LEMMA. *For $0 < M,\ J \le x$, we have that*

$$G := M \sum_{m \sim M} \sum_{j \sim J} \tau_3(j) \min\left(\frac{x}{m^2 j}, \frac{1}{\|\alpha m^2 j\|}\right)$$
$$\ll M^2 JL^3 + x^{3/4}(xq^{-1} + xM^{-1} + q)^{1/4} L^8.$$

We put our proof of this lemma off until the next section. Therefore, through the second way, we reach the following estimation.

$$(17) \quad |R|^2 \ll \|r\|^2 \|s\|^2 L^9 \{x^{3/4}(xq^{-1} + xM^{-1} + q)^{1/4} + MU + M^2 N^2 + MN^2 V\}$$
$$= \|r\|^2 \|s\|^2 L^9 Y, \text{ say.}$$

Next, turning back to the begining, we take the third way. In place of the form $\sum_u \sum_m |\sum_n \sum_v|$, our starting point is now $\sum_u \sum_m \sum_n |\sum_v|$. Then, by the similar argument as above, we get the similar bound to (16), in which the pair of parameters $(M, N)$ is replaced by $(MN, 1)$. We thus have that

(18)

$$|R|^2 \ll \|r\|^2\|s\|^2L^3\left\{MN\sum_{d\sim MN}\sum_{dj\ll V}\min\left(\frac{x}{d^2j},\frac{1}{\|\alpha d^2j\|}\right)+MNU+M^2N^2+MNV\right\}$$

$$\ll \|r\|^2\|s\|^2L^{12}\{x^{3/4}(xq^{-1}+x(MN)^{-1}+q)^{1/4}+MNU+M^2N^2+MNV\}$$

$$= \|r\|^2\|s\|^2L^{12}Z, \text{ say,}$$

by Lemma again.

Finally we take the first way. Restarting from $\sum_u|\sum_m\sum_n\sum_v|$, we argue as before. We then have the similar estimation to (16), replacing $(M,N)$ by $(1,MN)$. Hence we see that

$$|R|^2 \ll \|r\|^2\|s\|^2\left\{\sum_{h\ll M^2N^2V}\tau_5(h)\min\left(\frac{x}{h},\frac{1}{\|\alpha h\|}\right)+UL^8+M^2N^2+M^2N^2V\right\}.$$

The square of the above sum over $h$ is at most

$$\sum_{k\ll M^2N^2V}\tau_5(k)^2\frac{x}{k}\sum_{h\ll M^2N^2V}\min\left(\frac{x}{h},\frac{1}{\|\alpha h\|}\right)\ll x(xq^{-1}+M^2N^2V+q)L^{26},$$

by Cauchy's inequality and [2, §25, (3)]. Hence, going through the first way, we get that

$$|R|^2 \ll \|r\|^2\|s\|^2L^{13}\{x^{1/2}(xq^{-1}+M^2N^2V+q)^{1/2}+U+M^2N^2+M^2N^2V\} \tag{19}$$

$$= \|r\|^2\|s\|^2L^{13}X, \text{ say.}$$

In conjunction with (17), (18) and (19), we conclude that

$$|R|^2 \ll \|r\|^2\|s\|^2L^{13}\min(X,Y,Z). \tag{20}$$

Now we recall the conditions (5), (9) and (11). It follows from (17) and (18) that

$$\min(Y,Z) \ll x^{3/4}(xq^{-1}+xM^{-1}+q)^{1/4}+M^2N^2+\min(MU+MN^2V,MNU)$$

since $\min(a+b,a+c)=a+\min(b,c)$, $N\gg 1$ and $V\ll U$. The above last term is

$$\le MU+\min(MN^2V,MNU)$$

$$\le MU+(MN^2V)^{1/2}(MNU)^{1/2}$$

$$\ll Mx^{2/3}+MN^{3/2}x^{1/2}$$

$$\ll xL^{-B}.$$

Here we used the inequality that $\min(a,b) \le a^s b^t$, $s+t=1$, $s,t \ge 0$. Hence

$$\begin{aligned}\min(Y,Z) &\ll x^{3/4}(xq^{-1}+xL^{-4B}+q)^{1/4}+xM^{-1/4}\\ &= W+xM^{-1/4}, \text{ say.}\end{aligned}$$

Also, from (19), we see that

$$X \ll W + x^{1/2}(M^2N^2V)^{1/2} + M^2N^2V$$

because of $1 \le q \le x$. In consequence, it turns out that

$$\begin{aligned}\min(X,Y,Z) &= \min(X,\min(Y,Z))\\ &\ll W+\min(x^{1/2}(M^2N^2V)^{1/2}+M^2N^2V, xM^{-1/4})\\ &\ll W+(x^{1/2}(M^2N^2V)^{1/2})^{1/5}(xM^{-1/4})^{4/5}+(M^2N^2V)^{1/9}(xM^{-1/4})^{8/9}\\ &\ll W+x^{9/10}(N^2V)^{1/10}+x^{8/9}(N^2V)^{1/9}\\ &\ll W\end{aligned}$$

since $N^2V \ll x^{8/9}$.

Substituting this into (20), we get the required bound (5) for $R$. Therefore we have Theorem, except for the verification of Lemma.

## 4. Proof of Lemma

It remains to estimate $G$. To this end, we employ a well-known Fourier series: *For* $H > 2$,

$$\min(H, \|\theta\|^{-1}) = \sum_{h\in Z} w_h e(\theta h)$$

*where*

$$w_h = w_h(H) \ll \min\left(\log H, \frac{H}{|h|}, \frac{H^2}{h^2}\right).$$

Put $H = x(M^2J)^{-1}$. Unless $H > 2$, we trivially have that

$$G \ll M \sum_{m\sim M}\sum_{j\sim J}\tau_3(j) \ll M^2JL^2.$$

So we may use the above expansion to obtain

$$\min\left(H, \frac{1}{\|\alpha m^2 j\|}\right) = O(L) + \sum_{0<|h|\le H^2} w_h e(\alpha m^2 jh).$$

Substituting this into $G$, we see that

$$\begin{aligned}
G &\ll M^2JL^3 + M \sum_{0<|h|\le H^2} |w_h| \sum_{j\sim J} \tau_3(j) \left| \sum_{m\sim M} e(\alpha m^2 jh) \right| \qquad (21)\\
&= M^2JL^3 + F, \text{ say.}
\end{aligned}$$

Here we consider

$$\left| \sum_{m\sim M} e(\alpha m^2 jh) \right|^2 = \sum_{m_1\sim M} \sum_{m_2\sim M} e(\alpha(m_1^2 - m_2^2) jh).$$

We write $m_1 - m_2 = g$, so that $|g| \le M$ and $m_1^2 - m_2^2 = 2m_2 g + g^2$. The above sum is then bounded by

$$\begin{aligned}
&\ll \sum_{m\sim M} 1 + \sum_{g\le M} \left| \sum_{m\sim M} e(\alpha 2mgjh) \right| \qquad (22)\\
&\ll M + \sum_{g\le 2M} \min\left(M, \frac{1}{\|\alpha gjh\|}\right).
\end{aligned}$$

Hence it follows from (21), Cauchy's inequality and (22) that

$$\begin{aligned}
F^2 &\ll M^2 \sum_{k\le H^2} |w_k| \sum_{l\sim J} \tau_3(l)^2 \sum_{h\le H^2} |w_h| \sum_{j\sim J} \left| \sum_{m\sim M} e(\alpha m^2 jh) \right|^2 \qquad (23)\\
&\ll M^2HJL^9 \left\{ HJML + \sum_{h\le H^2} \min\left(\log H, \frac{H}{h}\right) \sum_{j\sim J} \sum_{g\le 2M} \min\left(M, \frac{1}{\|\alpha gjh\|}\right) \right\}\\
&= xL^9 \left\{ \frac{x}{M} L + E \right\}, \text{ say.}
\end{aligned}$$

We proceed to $E$. Dividing the interval $(0, H^2]$ into the subintervals $(0, H]$ and $(H2^{k-1}, H2^k]$, $1 \le k \ll L$, we find that

$$E \ll L \max_{1\le T\ll H} \frac{1}{T} \sum_{h\le 2HT} \sum_{j\sim J} \sum_{g\le 2M} \min\left(M, \frac{1}{\|\alpha gjh\|}\right). \qquad (24)$$

Put $l = gjh$. Then $l \ll MJHT \ll (x/M)T$ or $M \ll xT/l$. Hence, by Cauchy's inequality and [2, §25, (3)], the triple sum in (24) is at most

$$\sum_{l \ll (x/M)T} \tau_3(l) \min\left(M, \frac{1}{\|\alpha l\|}\right)$$

$$\ll \left(\sum_{l \ll (x/M)T} \tau_3(l)^2 M\right)^{1/2} \left(\sum_{l \ll (x/M)T} \min\left(\frac{xT}{l}, \frac{1}{\|\alpha l\|}\right)\right)^{1/2}$$

$$\ll Tx^{1/2}\left(\frac{x}{q} + \frac{x}{M} + \frac{q}{T}\right)^{1/2} L^5.$$

We therefore have that

$$E \ll x^{1/2}\left(\frac{x}{q} + \frac{x}{M} + q\right)^{1/2} L^6.$$

Combining this with (23) and (21), we get the required bound for $G$.

This completes our proof of Theorem.

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