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ON THE GROUPS WITH HOMOGENEOUS THEORY

By

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1. Introduction

D. MacPherson [5] proved that no infinite groups are interpretable in any finitely homogeneous structure. A countable structure M is called *finitely homogeneous* if its language is finite, its domain is countable, and every isomorphism between finite tuples in M extends to an automorphism of M.

We shall consider a similar condition which applies to general structures.

DEFINITION 1.1. Let $2 \leq m < n$. We say that a structure M is (m,n)-*homogeneous* if for any two n-tuples $\bar{a}$, $\bar{b}$ from M, $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ if and only if corresponding m-tuples from $\bar{a}$ and $\bar{b}$ have the same type. A complete theory T is (m,n)-*homogeneous* if every model of T is (m,n)-*homogeneous*.

Note that the additive group of integers $(\boldsymbol{Z},+)$ is $(2,n)$-homogeneous for any $n > 2$. But it turns out that its theory is not (n,n)-homogeneous for any m, n by the stability and Theorem 2.3 below.

In this paper, we treat the following conjecture:

CONJECTURE 1.2. *If* $(M,\cdot)$ *is a group* (*it may have other structures*) *then the theory of* $(M,\cdot)$ *is not* (m,n)-*homogeneous for any* m, n *such that* $2 \leq m < n$.

We call a theory (m,∞)-*homogeneous* if it is (m,n)-homogeneous for any $n > m$. Handa [2] studied (m,∞)-homogeneous theories and proved that no infinite Abelian p-groups are interpretable in a model of such a theory, and if the theory is ω-stable in addition then no infinite groups are interpretable.

If the above conjecture is true then no groups are interpretable in a model of (m,∞)-homogeneous theories. However, we cannot claim that no groups are interpretable in a model of an (m,n)-homogeneous theory. The following

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example suggested by Ehud Hrushovski is ω-stable, $(2,3)$-homogeneous, not $(3,4)$-homogeneous, and interprets an infinite group.

Example 1.3. Consider the projective line $\mathbf{P}^1$ over an algebraically closed field K and the action of $\mathrm{PGL}(2, K)$ on it. This group acts sharply 3-transitively on $\mathbf{P}^1$. Define a relation $R(z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4)$ on $\mathbf{P}^1$ as follows: There is a regular linear map A in $\mathrm{PGL}(2, K)$ such that $Az_i = w_i$ for each $i = 1, 2, 3$ and 4.

R is invariant under the action of $\mathrm{PGL}(2, K)$. Since this group acts sharply 3-transitively on $\mathbf{P}^1$, given two sets of three points $\{p, q, r\}$ and $\{p', q', r'\}$ for $\mathbf{P}^1$, the relation $R(z, p, q, r, w, p', q', r')$ between z and w represents an automorphism of $(\mathbf{P}^1, R)$ which belongs to $\mathrm{PGL}(2, K)$.

Now we can easily see that $Th(\mathbf{P}^1, R)$ is $(2,3)$-homogeneous but $(\mathbf{P}^1, R)$ interprets the infinite group $\mathrm{PGL}(2, K)$. As we can interpret $(\mathbf{P}^1, R)$ in the field K, $Th(\mathbf{P}^1, R)$ is ω-stable.

Moreover, the theory is not $(3,4)$-homogeneous. Choose three distinct points, a, b, c from $\mathbf{P}^1$ and a linear map A from $\mathrm{PGL}(2, K)$ sending a, b, c to b, c, a respectively. Since K is algebraically closed, A has a fixed point d in $\mathbf{P}^1$. Note that d is different from a, b and c. Choose a new point d' from $\mathbf{P}^1$ that is not fixed by A. Then $R(d, a, b, c, d, b, c, a)$ holds but $R(d', a, b, c, d', b, c, a)$ does not hold. Since there is only one 3-type realized by three distinct points, this shows that the theory is not $(3,4)$-homogeneous.

Also, we cannot claim that no groups are definable in a model of an (m,n)-homogeneous theory. The following example is due to Akito Tsuboi. This example is ω-categorical, ω-stable, $(2,3)$-homogeneous, not $(2,4)$-homogeneous, but some infinite groups are definable with three parameters.

Example 1.4. Let V_1, V_2, V_3, V_4 be four copies of $\boldsymbol{Z}_2^{(\omega)}$ where $\boldsymbol{Z}_2$ is the Ablian group of order 2. Let M be the disjoint union of these four sets, and define the relation $R(x_1, x_2, x_3, x_4)$ by $x_i \in V_i$ and $x_1 + x_2 + x_3 + x_4 = 0$. Then $Th(M, R)$ is $(2,3)$-homogeneous but $\boldsymbol{Z}_2^{(\omega)}$ is definable in it.

First, we can recover a group structure on each V_i. Fix three elements a, b, c one from each V_2, V_3 and V_4. The formula

$$\exists x_2, x_3[R(u_2, x_2, b, c) \wedge R(u_3, a, x_3, c) \wedge R(u_1, x_2, x_3, c)]$$

is equivalent to $u_1 + u_2 + u_3 + a + b + c = 0$ for u_1, u_2, u_3 in V_1 which gives a group structure on V_1. The same argument works for each V_i.

To show that there is only one 3-type realized by three distinct elements from V_1 is the most essential in the proof of $(2,3)$-homogeneity of the theory. Consider each V_i as a vector space over the prime field of characteristic 2. Let $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ be two sets of three distinct elements from V_1. Whether each set is dependent or not, we can choose c from V_1 so that $\{a_1 - c, a_2 - c, a_3 - c\}$ and $\{b_1 - c, b_2 - c, b_3 - c\}$ are both linearly independent sets. Let s be a linear automorphism on V_1 sending each $a_i - c$ to $b_i - c$. Then $\sigma(x) = s(x - c) + c$ is an automorphism of V_1 which sends a_i to b_i for $i = 1, 2, 3$. Extend σ to V_2, V_3 and V_4 by $\sigma(x) = s(x + c) - c$ on V_2, and $\sigma(x) = s(x)$ on V_3 and V_4. Then σ is an automorphism of (M, R).

We prove Conjecture 1.2 with various additional conditions such as ω-categoricity, o-minimality, stability and simplicity (in Shelah's sense), but it seems very hard to prove it in general. In the simple case, we only prove that the theory is not $(2,3)$-homogeneous. Also, we have not found a pure group with the (m,n)-homogeneous theory for some m and n.

In this paper, the language is countable and the notation follows Pillay's book [7].

2. (m,n)-Homogeneous Theory

In this section, we prove that Conjecture 1.2 holds if $Th(M)$ is ω-categorical, stable, or o-minimal.

THEOREM 2.1 *If $(M,\cdot)$ has an infinite Abelian p-subgroup then $Th(M,\cdot)$ is not (m,n)-homogeneous for any m and n.*

PROOF. The proof is much the same as Handa's proof in [2] which is a modification of Macpherson's argument [5]. We give the proof for reader's convenience.

We work in an Abelian subgroup and write the group operation additively.

It is enough to show that the theory is not $(m, m+1)$-homogeneous for any m. We find elements $a_1, \dots, a_{m+1}$ that are linearly independent over a finite prime field $\boldsymbol{F}_p$ with the p elements, and the corresponding m-tuples from $(a_1, \dots, a_m, a_1 + \cdots + a_m + a_{m+1})$ and $(a_1, \dots, a_m, a_1 + \cdots + a_m)$ have the same type. Note that we can describe this condition by a set of elementary formulas. We show that for given finite set Δ of m-formulas, we can find elements $a_1, \dots, a_{m+1}$ satisfying the above condition except that the phrase "have the same type" changed to "have the same Δ-type". Then by compactness, we get the desired tuple.

Let V be an infinite Abelian p-subgroup of $(M,\cdot)$. Consider V as a vector space over $\boldsymbol{F}_p$. We can assume that V has the countable dimension over $\boldsymbol{F}_p$. Choose a basis $(v_i : i < \omega)$ of V. Now we give a rule for coloring the m-dimensional subspaces of V.

First, we give a rule for ordering the elements of a such subspace. If U is an m-dimensional subspace of V, then the cardinality of U is p^m. Since U is a finite dimensional subspace of V, U is covered by the $\boldsymbol{F}_p$-span of $(v_i : i < n)$ for some natural number n. Every element of U can be written as a linear combination of $(v_i : i < n)$ over $\boldsymbol{F}_p$. If we list all of them, we naturally get a $|U| \times n$ matrix with entries in $\boldsymbol{F}_p$. Then we can find a unique row reduced echelon form of the matrix. It has $m(= \dim U)$ nonzero rows, and the tuple of elements of U represented by those rows is an ordered basis of U. We call it the *canonical basis* of U. Order the elements of U lexicographically according to their coordinates with respect to the canonical basis.

Now, if U and U' are m-dimensional subspaces of V, we say that U and U' have the same color if every corresponding m-tuples with respect to the above ordering have the same Δ-type. Note that the number of the colors is finite.

By the affine version of Ramsey's theorem [1], V has an $(m+1)$-dimensional subspace W all of whose m-dimensional subspaces have the same color. Let $(a_1, \dots, a_m, a_{m+1})$ be the canonical basis of W.

All we have to show is that the corresponding m-tuples from $(a_1, \dots, a_m, a_1 + \cdots + a_m + a_{m+1})$ and $(a_1, \dots, a_m, a_1 + \cdots + a_m)$ have the same Δ-type. Let U_1 be the $\boldsymbol{F}_p$-span of $\{a_1, \dots, a_m, a_1 + \cdots + a_m + a_{m+1}\} \setminus \{a_i\}$ and U_2 the $\boldsymbol{F}_p$-span of $\{a_1, \dots, a_m, a_1 + \cdots + a_m\} \setminus \{a_i\}$. Then their dimensions are both m. Since U_1 has the canonical basis $(a_1, \dots, a_i + a_{m+1}, \dots, a_m)$ and U_2 has the canonical basis $(a_1, \dots, a_i, \dots, a_m)$, $a_1 + \cdots + a_m + a_{m+1}$ in U_1 and $a_1 + \cdots + a_m$ in U_2 have the same coordinate $(1, \dots, 1)$. Thus, we get the desired result. ∎

As there exists an infinite Abelian p-subgroup in a ω-categorical group (see [5]), we have the following.

Corollary 2.2. *If $Th(M,\cdot)$ is countably categorical then it is not (m,n)-homogeneous for any m and n.*

We now turn to the stable case. In this case, Conjecture 1.2 holds by the existence of stationary generic types.

THEOREM 2.3. *If $Th(M,\cdot)$ is stable then it is not (m,n)-homogeneous for any m and n.*

PROOF. It is enough to show that the theory is not $(m,m+1)$-homogeneous for any m. Let p be a stationary generic type over a model N, and $a_1,\dots,a_m$ independent (over N) realizations of p. Let $b = a_1\cdot\dots\cdot a_m$. Since p is generic, $\mathrm{tp}(b/N)$ is also a stationary generic type, and any m elements from $a_1,\dots,a_m,b$ are independent over N.

Now choose c such that $\mathrm{tp}\,(c/a_1\dots a_mN)$ is a nonforking extension of $\mathrm{tp}(b/N)$ and consider the two $(m+1)$-tuples $(a_1,\dots,a_m,b)$ and $(a_1,\dots,a_m,c)$. They do not have the same type since b is algebraic (definable) over $\{a_1,\dots,a_m\}$ and c is independent of $\{a_1,\dots,a_m\}$. But the corresponding m-tuples from both tuples have the same type by the stationarity of types over a model. This shows that the theory is not $(m,m+1)$-homogeneous. ∎

To finish this section, we consider the o-minimal case.

THEOREM 2.4. *If $Th(M,\cdot,<)$ is o-minimal then it is not (m,n)-homogeneous for any m and n.*

PROOF. Choose algebraically independent elements $a_1,\dots,a_m$ (in the big model). If we cannot choose such elements, then by compactness, there are formulas $\psi_i(x;y_1,\dots,y_{m-1})$ $(i=1,\dots,m)$ such that any m-tuple satisfies one of ψ_1's (by permuting if necessary) and if $x,y_1,\dots,y_{m-1}$ satisfies ψ_i then x is algebraic over $y_1,\dots,y_{m-1}$. But if we choose an infinite indiscernible sequence $\langle a_i \mid i<\omega\rangle$, we get a contradiction by considering $a_k,a_{2k},\dots,a_{mk}$ for sufficiently large k.

Let $b=a_1\cdots a_m$ and consider the types

$$\mathrm{tp}(b/A_i) \quad \text{where} \quad A_i=\{a_1,\dots,a_m\}\setminus\{a_i\}.$$

Note that they are non-algebraic types. If a formula $\varphi_i(x)$ belongs to $\mathrm{tp}(b/A_i)$ then it is a finite union of intervals by o-minimality. Without loss of generality, we can assume that $\varphi_i(x)$ represents a single interval $[c_i,d_i]$ where c_i and d_i are definable elements over A_i (this may not be a closed interval, but the argument will be the same in any case). Since b is not algebraic over A_i, b belongs to the open interval (c_i,d_i). As this is true for each $i=1,\dots,m$, the type

$$\mathrm{tp}(b/A_1)\cup\cdots\cup\mathrm{tp}(b/A_m)$$

is non-algebraic by compactness. Choose $b' \neq b$ satisfying this type. Considering the tuples $(a_1, \ldots, a_m, b)$ and $(a_1, \ldots, a_m, b')$, we see that the theory is not $(m, m+1)$-homogeneous. ∎

3. (2,3)-Homogeneous Theory

If the theory is simple then we can still find a generic type, but it is not necessarily stationary. Instead, we can use the Independence Theorem due to B. Kim and A. Pillay to prove the conjecture in a special form. But we could not prove the conjecture in the general form.

We use the following definition and facts from [4] and [6].

DEFINITION 3.1. A 1-type $p(x)$ over A is called *generic* if for any a realizing p and b such that a is independent from b over A, $a \cdot b$ is independent from Ab over $\emptyset$ and so is $b \cdot a$.

FACT 3.2. *If $Th(M, \cdot)$ is simple then there is a generic type.*

FACT 3.3 (Independence Theorem). *Suppose the theory is simple. If A and B are independent over a model M and a type p_1 over A and a type q_2 over B are both nonforking extensions of a type p over M, then there is a type q over $a \cup B$ such that q extends both p_1 and p_2, and q does not fork over M.*

THEOREM 3.4. *If $Th(M, \cdot)$ is simple then it is not $(2,3)$-homogeneous.*

PROOF. Let p be a generic type over some model N, and a_1, a_2 independent realizations of p. Let $b = a_1 \cdot a_2$. Then both $\mathrm{tp}(b/a_1N)$ and $\mathrm{tp}(b/a_2N)$ do not fork over N. By the Independence Theorem, we can choose c such that $\mathrm{tp}(c/a_1a_2N)$ does not fork over N and $\mathrm{tp}(c/a_1a_2N)$ extends both $\mathrm{tp}(b/a_1N)$ and $\mathrm{tp}(b/a_2N)$. This implies that corresponding pairs from (a_1, a_2, b) and (a_1, a_2, c) have the same type. On the other hand, (a_1, a_2, b) and (a_1, a_2, c) have different types over $\emptyset$ since $b = a_1 \cdot a_2$ but c is non-algebraic over $\{a_1, a_2\} \cup N$. ∎

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