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TOTALLY GEODESIC SUBMANIFOLDS OF NATURALLY REDUCTIVE HOMOGENEOUS SPACES

By

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1. Introduction.

B. Y. Chen and T. Nagano [2] investigated the totally geodesic submanifolds in Riemannian symmetric spaces, and as one of their results, the following holds.

FACT 1.1. *Spheres and hyperbolic spaces are the only simply connected, irreducible symmetric spaces admitting a totally geodesic hypersurface.*

In this paper we shall study totally geodesic submanifolds in the naturally reductive homogeneous spaces which are known as a natural generalization of Riemannian symmetric spaces.

At first, in a naturally reductive space (M, g), we express a necessary and sufficient condition of the existence of a totally geodesic submanifolds in the language of the Lie algebra of a Lie group of isometries of M (Theorem 3.2), which generalizes the notion of the Lie triple system due to E. Cartan.

Next, as an application of that, by making use of the results in Kowalski and Vanhecke [5, 6], we shall prove that simply connected, irreducible naturally reductive spaces of dimension n $(n = 3, 4, 5)$ admitting a totally geodesic hypersurface are spheres and hyperbolic spaces (Theorem 4.1).

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2. Totally geodesic submanifolds of Riemannian spaces.

Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection of (M, g). Let p be a point of M and u a vector in the tangent space T_pM to M at p. P_u denotes the parallel transport with respect to ∇ along the geodesic $\gamma_u(t) = \mathrm{Exp}_p(tu)$ from p to $\gamma_u(1)$, where Exp denotes the Riemannian exponential map. Let R be the curvature tensor defined by

$$R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y],$$

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where X and Y are vector fields on M.

Then the following formulas are well-known.

$$R(X,Y,Z,W)=-R(X,Y,W,Z)=R(Z,W,X,Y) \tag{2.1}$$

$$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0, \tag{2.2}$$

for vector fields X, Y, Z and W. Here $R(X,Y,Z,W)=g(R(X,Y)Z,W)$.

Define a (1,3)-tensor $R_u(t)$ on T_pM as follows:

$$R_u(t)(x,y)z=P_{tu}\circ R_{\gamma u(t)}(P_{tu}(x),P_{tu}(y))P_{tu}(z)\quad (x,y,z\in T_pM).$$

Then the following theorem is known with respect to the existence of a totally geodesic submanifold. (see Cartan [1] and Hermann [3].)

THEOREM 2.1. *Let V be a subspace of T_pM. Then the following conditions are equivalent.*

(*a*) *There exists a totally geodesic submanifold tangent to V at $p\in M$.*

(b) *There is a positive number ε such that for each $t\in(-\varepsilon,\varepsilon)$ and each $u\in V(|u|=1)$, the following is satisfied:*

$$R_u(t)(V,V)V\subset V \tag{2.3}$$

(c) *There is a positive number ε such that for each $t\in(-\varepsilon,\varepsilon)$ and each $u\in V(|u|=1)$, the following is satisfied;*

$$r_u(t)(V,V)\subset V, \tag{2.4}$$

where $r_u(t)(x,y)=R_u(t)(u,x)y$.

In particular, if condition (*a*) is satisfied, the following holds.

$$R(V,V)V\subset V. \tag{2.5}$$

3. Naturally reductive spaces.

Let $(M, g) = G/H$ be a naturally reductive homogeneous space. Then there exists an Ad(H)-invariant decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of Lie group$G$ ($\mathfrak{h}$ is the Lie algebra of H.) such that

$$\langle [x,y]_{\mathfrak{p}},z\rangle+\langle y,[x,z]_{\mathfrak{p}}\rangle=0\quad (x,y,z\in\mathfrak{p}) \tag{3.1}$$

where $\langle\ ,\ \rangle$denotes the induced scalar product on $\mathfrak{p}$ from the metric g by using the canonical identification $\mathfrak{p}\equiv T_{p_o}M(p_o=\{H\})$. Since g is G-invariant, $\langle\ ,\ \rangle$is Ad(H)-invariant.

As is well-known, the geodesic $\gamma_x (x \in \mathfrak{p})$ with $\gamma_x(0) = p_o, \gamma_x'(0) = x$ and the curvature tensor R_o at p_o are given by

$$\gamma_x(t) = \tau(\exp tx)(p_o) \tag{3.2}$$

$$\begin{aligned} R_o(y,z)\omega &= [[y,z]_{\mathfrak{h}}, \omega] + \frac{1}{2}[[y,z]_{\mathfrak{p}}, \omega]_{\mathfrak{p}} \\ &\quad - \frac{1}{4}[y,[z,\omega]_{\mathfrak{p}}]_{\mathfrak{p}} + \frac{1}{4}[z,[y,\omega]_{\mathfrak{p}}]_{\mathfrak{p}} \quad (y,z,\omega \in \mathfrak{p}) \end{aligned} \tag{3.3}$$

where exp and $\tau(h)$ denote the Lie exponential map of G and the left transformation of G/H induced by $h \in G$, respectively.

According to Nomizu [7], the connection function $\Lambda : \mathfrak{p} \times \mathfrak{p} \to \mathfrak{p}$ which is associated to ∇ is given by

$$\Lambda(x)(y) = \frac{1}{2}[x,y]_{\mathfrak{p}} \quad (x,y \in \mathfrak{p}). \tag{3.4}$$

From now on, we put $\varphi_x = \Lambda(x)$ for simplicity. Moreover, we put

$$e^{-\varphi_x} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \varphi_x{}^l.$$

By (3.1), a linear map φ_x is skew-symmetric. Therefore a mapping $e^{-\varphi_x} : (\mathfrak{p}, \langle\,,\rangle) \to (\mathfrak{p}, \langle\,,\rangle)$ is an isometry.

By using (3.4), we shall prove the following lemma.

LEMMA 3.1. *The parallel vector field $Y(t)$ along γ_x such that $Y(0) = y (\in \mathfrak{p})$ is given by*

$$Y(t) = \tau(\exp tx)_*(e^{-\varphi_{tx}}(y)).$$

PROOF. From (3.4), we have

$$\nabla_{\gamma_x'(t)} \tau(\exp tx)_*(z) = \tau(\exp tx)_*(\varphi_x(z)) \quad (z \in \mathfrak{p})$$

Then we have

$$\begin{aligned} \nabla_{\gamma_x'(t)} Y(t) &= \tau(\exp tx)_* \left(\frac{d}{dt} e^{-\varphi_{tx}}(y) \right) + \tau(\exp tx)_*(\varphi_x \circ e^{-\varphi_{tx}}(y)) \\ &= \tau(\exp tx)_* \left(\sum_{l=1}^{\infty} \frac{(-1)^l}{(l-1)!} t^{l-1} \varphi_x{}^l(y) + \varphi_x \circ \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} t^l \varphi_x{}^l(y) \right) \\ &= 0 \end{aligned}$$

This prove the lemma. □

Now, from (3.3) and lemma 3.1, theorem 2.1 can be written in terms of the bracket operation.

THEOREM 3.2. *Let $(M, g) = G/H$ be a naturally reductive homogeneous space and V a subspace of $\mathfrak{p}$. Then there exists a totally geodesic submanifold tangent to V at p_o if and only if for any $x \in V$, the following is satisfied.*

$$R_o(x, e^{-\varphi_x}(V))e^{-\varphi_x}(V) \subset e^{-\varphi_x}(V) \tag{3.5}$$

PROOF. For $x, y, z \in V(|x| = 1), \xi \in V^{\perp}$ and $t \in (-\varepsilon, \varepsilon)$, we have from lemma 3.1,

$$\begin{aligned} \langle r_x(t)(y,z),\xi\rangle &= \langle P_{tx}{}^{-1} \circ R(P_{tx}(x), P_{tx}(y))P_{tx}(z), \xi\rangle \\ &= g\,(\tau(\exp tx)_* R_o(x, e^{-\varphi_{tx}}(y))e^{-\varphi_{tx}}(z), \tau(\exp tx)_*(e^{-\varphi_{tx}}(\xi))) \\ &= \langle R_o(x, e^{-\varphi_{tx}}(y))e^{-\varphi_{tx}}(z), e^{-\varphi_{tx}}(\xi)\rangle \end{aligned} \tag{3.6}$$

Put $f(t) = \langle R_o(x, e^{-\varphi_{tx}}(y))e^{-\varphi_{tx}}(z), e^{-\varphi_{tx}}(\xi)\rangle$.

If condition (c) in theorem 2.1 is satisfied, then $f(t) = 0$ on $(-\varepsilon, \varepsilon)$ from (3.6), and obviously $t\,f(t) \equiv 0$. Since $e^{-\varphi_{tx}}$ is isometry, (3.5) is derived from the real analyticity of f.

Conversely, we suppose that (3.5) is satisfied. Then we have $t\,f(t) = 0$ for all $t \in \boldsymbol{R}$.

Therefore if $t \neq 0$, then we get $f(t) = 0$. By continuity of f at $t = 0$, we obtain $f(t) = 0$ for all $t \in \boldsymbol{R}$. Hence from (3.6), condition (c) holds.
We have thus proved the theorem. □

REMARK 3.3. *Using the same method as in the above proof, from (2.3) we have*

$$R_o(e^{-\varphi_x}(V), e^{-\varphi_x}(V))e^{-\varphi_x}(V) \subset e^{-\varphi_x}(V) \tag{3.7}$$

for any $x \in V$. Then (3.7) is equivalent to (3.5) since condition (b) is equivalent to condition (c).

REMARK 3.4. If $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$ (then G/H is a locally symmetric space), then (3.5) (or (3.7)) turns into the relation

$$[V, [V, V]] \subset V,$$

i.e. V is a Lie triple system. In this sense it can be said that (3.5) is a generalization of the notion of the Lie triple system in Riemannian symmetric spaces due to E. Cartan.

We say that $V^{\perp}$ satisfies condition (T − G) if there exists a totally geodesic submanifold tangent to V at p_o.

PROPOSITION 3.5. *Suppose* $V^{\perp}$ *satisfies condition* (T − G). *Then* $\mathrm{Ad}(H)(V^{\perp})$ *satisfy condition* (T − G). *Moreover, for each* $x \in V$, *a subspace* $e^{-\varphi_{tx}}(V^{\perp})$ *satisfies condition* (T − G).

PROOF. Considering the isotropy representation, the first part of the proposition is obvious.

Now, we shall prove the second part. Let S be a totally geodesic submanifold tangent to V at p_o. According to Hermann [3], any totally geodesic submanifold of a real analytic complete Riemannian manifold can be extended to complete one. Then we assume S is complete.

For each $x \in V$, a vector space $\tau(\exp x)_*(e^{-\varphi_x}(V))$ coincides with the tangent space to S at $\tau(\exp x)(p_o)$. Hence $\tau(\exp -x)(S)$ is a totally geodesic submanifold tangent to $e^{-\varphi_{tx}}(V)$. This proves the second part and completes the proof of the proposition. □

4. Totally geodesic hypersurfaces.

In this section we shall prove the following theorem.

THEOREM 4.1. *Let* $(M, g) = G/H$ *be a simply connected, irreducible naturally reductive homogeneous space of dimension* n $(n = 3, 4, 5)$. *If* M *admits a totally geodesic hypersurface, then* (M, g) *is either sphere or hyperbolic space.*

From fact 1.1 and the following, we only consider the case that (M, g) is not symmetric and $n = 3$ or 5.

FACT 4.2. (O. Kowalski and L. Vanhecke [5]) *Let* (M, g) *be a simply connected, irreducible naturally reductive space of dimension four. Then* (M, g) is symmetric.

Let G/H be a m-dimensional $(m \geq 2)$ naturally reductive space. Let V be a hyperplane of $\mathfrak{p}$ and $v \in \mathfrak{p}$ a unit vector normal to V.

Throughout this section we assume that v satisfies condition (T - G).
From (2.1) and (2.5), for each $x \in V$ there is a number λ such that

$$R_o(x, v)x = \lambda v$$

LEMMA 4.3.

$$R_o(x, \varphi_x{}^l(v))x = \lambda \varphi_x{}^l(v) \quad (l = 0, 1, 2, \cdots)$$

PROOF. By theorem 3.2, for all $t \in \boldsymbol{R}$ we get

$$\langle R_0(tx, e^{-\varphi_{tx}}(V))e^{-\varphi_{tx}}(tx), e^{-\varphi_{tx}}(v)\rangle = 0$$

Therefore we obtain

$$\langle R_0(tx, e^{-\varphi_{tx}}(V))tx, e^{-\varphi_{tx}}(v)\rangle = 0$$

By continuity, we get

$$\langle R_0(x, e^{-\varphi_{tx}}(v))x, e^{-\varphi_{tx}}(V)\rangle = 0$$

for all $t \in \boldsymbol{R}$. Hence we have

$$R_0(x, e^{-\varphi_{tx}}(v))x = \lambda(t)e^{-\varphi_{tx}}(v)$$
$$\lambda(t) = \langle R_0(x, e^{-\varphi_{tx}}(v))x, e^{-\varphi_{tx}}(v)\rangle.$$

On the other hand, the eigenvalues of $R_0(x, \cdot)x$ do not depend on t. Then we obtain

$$\lambda(t) \equiv \lambda(= \lambda(0)).$$

From this, the lemma is easily derived. □

Now we shall prove theorem 4.1.

The case dimM = 3.

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of $\mathfrak{p}$ such that $V = \{e_1, e_2\}$. From (3.1), there is a number c such that the following relation is satisfied.

$$[e_1, e_2]_{\mathfrak{p}} = ce_3, [e_2, e_3]_{\mathfrak{p}} = ce_1, [e_3, e_1]_{\mathfrak{p}} = ce_2. \tag{4.1}$$

By (2.5), we have

$$R_0(e_1, e_3)e_1 = \lambda e_3, R_0(e_2, e_3)e_2 = \mu e_3$$

for some λ and μ. By fact 1.1, we suppose $c \neq 0$. From lemma 4.3 and (4.1), we get

$$R_0(e_1, e_2)e_1 = \lambda e_2, R_0(e_2, e_1)e_2 = \mu e_1.$$

Hence we have $\lambda = \mu$, and then for any $a, b \in \boldsymbol{R}(a^2 + b^2 = 1)$, we obtain

$$R_0(ae_1 + be_2, be_1 - ae_2)(ae_1 + be_2) = \lambda(be_1 - ae_2).$$

Therefore, also by lemma 4.3, we have

$$R_0(ae_1 + be_2, e_3)(ae_1 + be_2) = \lambda e_3. \tag{4.2}$$

From (2.1) and (2.5), we get

$$R_0(e_1, e_2)e_3 = 0.$$

Therefore by (2.2), we obtain

$$R_0(e_1,e_3)e_2 = R_0(e_2,e_3)e_1. \tag{4.3}$$

By (4.3), we have

$$R_0(ae_1+be_2,e_3)(ae_1+be_2) = \lambda e_3 + 2abR_0(e_1,e_3)e_2.$$

Therefore it follows from (4.2) and (4.3) that

$$R_0(e_1,e_3)e_2 = R_0(e_2,e_3)e_1 = 0.$$

In this case (M, g) has constant sectional curvature λ.

The case dimM = 5.

Let $\{e_1,\cdots,e_5\}$ be an orthonormal basis of $\mathfrak{p}$.

Define linear mappings A_{ij} $(i \neq j, i,j = 1,\cdots,4)$ associated to $\{e_1,\cdots,e_5\}$ as follows:

$$A_{ij}(e_i) = e_j,\ A_{ij}(e_j) = -e_i,\ A_{ij}(e_k) = 0 (k \neq i,j).$$

Let $\mathfrak{k}$ be the holonomy algebra, i.e. $\mathfrak{k}$ is a Lie algebra generated by $ad([x,y]_{\mathfrak{h}})(x,y \in \mathfrak{p})$.

The following two lemmas are due to Kowalski and Vanhecke (see [6]).

LEMMA 4.4. *Let (M, g) be a simply connected, irreducible naturally reductive space of dimension five which is not symmetric. Then there exists an orthonormal basis $\{e_1,\cdots,e_5\}$ of $\mathfrak{p}$ such that the following are satisfied:*

$$\left\{\begin{array}{l} [e_1,e_2]_{\mathfrak{p}} = -\rho e_5, [e_1,e_5]_{\mathfrak{p}} = \rho e_2, [e_2,e_5]_{\mathfrak{p}} = -\rho e_1, \\ [e_3,e_4]_{\mathfrak{p}} = -\lambda e_5, [e_3,e_5]_{\mathfrak{p}} = \lambda e_4, [e_4,e_5]_{\mathfrak{p}} = -\lambda e_3, \\ [e_1,e_3]_{\mathfrak{p}} = [e_1,e_4]_{\mathfrak{p}} = [e_2,e_3]_{\mathfrak{p}} = [e_2,e_4]_{\mathfrak{p}} = 0 \end{array}\right\}, \tag{4.4}$$

where ρ and λ are non-zero constants.

LEMMA 4.5. *Under the same assumption as in the above lemma, there is an orthonormal basis $\{e_1,\cdots,e_5\}$ satisfying (4.4) such that the algebra $\mathfrak{k}$ has one of the following four cases:*

(1) $\mathfrak{k} = (\alpha A_{12} + \beta A_{34})$ *with* $\alpha \bullet \beta \neq 0$. *Moreover*

$ad([e_1,e_2]_{\mathfrak{h}}) = uP$

$ad([e_3,e_4]_{\mathfrak{h}}) = vP, (P = \alpha A_{12} + \beta A_{34}, u \cdot v \neq 0)$

and the others are zero.

(2) $\mathfrak{k} = (A_{12}, A_{34})$ *Moreover*

$ad([e_1,e_2]_{\mathfrak{h}}) = \gamma A_{12} - \rho\lambda A_{34}$

$ad([e_3,e_4]_{\mathfrak{h}}) = -\rho\lambda A_{12} + \delta A_{34} (\gamma,\delta \in \boldsymbol{R}, \gamma\delta \neq \rho^2\lambda^2)$
and the others are zero.

(3) $\mathfrak{f} = (A_{34} - A_{12}, B, C)$ *(in this case, $\rho = \lambda$.) Moreover*

$$ad([e_1,e_2]_{\mathfrak{h}}) = -ad([e_3,e_4]_{\mathfrak{h}}) = \frac{\lambda^2}{3}(A_{12} - A_{34}),$$

$$ad([e_1,e_3]_{\mathfrak{h}}) = ad([e_2,e_4]_{\mathfrak{h}}) = \frac{\lambda^2}{3} B,$$

$$ad([e_1,e_4]_{\mathfrak{h}}) = -ad([e_2,e_3]_{\mathfrak{h}}) = \frac{\lambda^2}{3} C$$

and the others are zero.

(4) $\mathfrak{f} = (A_{12}, A_{34}, B, C)$ *(in this case, $\rho = \lambda$.) Moreover*

$$ad([e_1,e_2]_{\mathfrak{h}}) = (4_\epsilon - \lambda^2)A_{12} + (2_\epsilon - \lambda^2)A_{34},$$
$$ad([e_3,e_4]_{\mathfrak{h}}) = (2_\epsilon - \lambda^2)A_{12} + (4_\epsilon - \lambda^2)A_{34},$$
$$ad([e_1,e_3]_{\mathfrak{h}}) = ad([e_2,e_4]_{\mathfrak{h}}) = {}_\epsilon B,$$
$$ad([e_1,e_4]_{\mathfrak{h}}) = -ad([e_2,e_3]_{\mathfrak{h}}) = {}_\epsilon C (\epsilon \neq 0),$$

and the others are zero.

Here $B = A_{13} + A_{24}$, $C = A_{14} + A_{32}$.

In particular, we have $ad([e_i,e_5]_{\mathfrak{h}}) = 0$ $(i = 1,2,3,4)$ *in all cases.*

Considering the action of $\mathfrak{f}$, we can assume that $e_1 \in V$ and put

$$v = ae_2 + be_3 + ce_4 + de_5 (a^2 + b^2 + c^2 + d^2 = 1).$$

We suppose that $a \cdot d \neq 0$ (If $a \cdot d = 0$, then take $e^{-\varphi_{te1}}(v)$ or $e^{-\varphi_{tx}}(v)$ $(x = be_4 - ce_3)$ instead of v.)

From (4.4), we have

$$[e_1, v]_{\mathfrak{p}} = \rho(-ae_5 + de_2)$$
$$[e_1, [e_1, v]_{\mathfrak{p}}]_{\mathfrak{p}} = \rho^2 (ae_2 + de_5)$$

Then we get $R_0(e_1,e_2)e_1 = \mu e_2$, $R_0(e_1,e_5)e_1 = \mu e_5$ for some μ.

On the other hand, from lemma 4.5, we get

$$R_0(e_1,e_5)e_1 = \frac{\rho^2}{4} e_5$$

$$R_0(e_1,e_2)e_1 = \frac{\rho^2}{4} e_2 + [[e_1,e_2]_{\mathfrak{h}}, e_1]$$

Hence $[[e_1,e_2]_{\mathfrak{h}}, e_1] = 0$.

By lemma 4.5, the only possible cases are (2) and (4). In particular, in the case (4) we can see by a straightforward computation that (M, g) has constant sectional curvature $\rho^2/4$.

Finally we check the case (2).

By lemma 4.5, we have

$$R_0(e_1, e_3)e_1 = R_0(e_1, e_4)e_1 = 0.$$

This implies $b = c = 0$.(i.e. $e_3, e_4 \in V$.).

But by lemma 4.5, we have

$$R_0(e_3, v)e_3 = \frac{\lambda^2 d}{4} e_5 \quad (v = ae_2 + de_5, a \cdot d \neq 0)$$

This contradicts (2.5).

Therefore there is no tangent vector satisfying condition (T – G) in the case (2). We have thus proved the theorem.

REMARK 4.7. It is known that $SL(2, \boldsymbol{R})$ with a special left invariant metric admits a totally geodesic hypersurface, (cf. Tsukada [9]) but it is not naturally reductive.

Also it is known that $SL(2, \boldsymbol{R})$ admits a naturally reductive (left invariant) metric. (see Tricerri and Vanhecke [8], for details.)

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