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IDEMPOTENT RINGS WHICH ARE EQUIVALENT TO RINGS WITH IDENTITY

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Let A be a ring such that $A=A^2$, but which does not necessarily have an identity element. In studying properties of the ring A through properties of its modules, it is pointless to consider the category A-MOD of all the left A-modules: for instance, every abelian group -with trivial multiplication- is in A-MOD. The natural choice for an interesting category of left A-modules seems to be the following: if a left A-module ${}_AM$ is *unital* when $AM=M$, and is *A-torsionfree* when the annihilator $\iota_M(A)$ is zero, then A-mod will be the full subcategory of A-MOD whose objects are the unital and A-torsionfree left A-modules. The category A-mod appears in a number of papers (for instance, [7-9]) and when A has local units [1, 2] or is a left s-unital ring [6, 12], then the objects of A-mod are the unital left A-modules. A-mod is a Grothendieck category and we study here the question of finding necessary and sufficient conditions on the ring A for A-mod to be equivalent to a category R-mod of modules over a ring with 1. This was already considered for rings with local units in [1], [2] or [3], and for left s-unital rings in [6]. Our situation is therefore more general.

In this paper, all rings will be associative rings, but we do not assume that they have an identity. A ring A has local units [2] when for every finite family $a_1, \cdots, a_n$ of elements of A there is an idempotent $e\in A$ such that $ea_j=a_j=a_je$ for all $j=1, \cdots, n$. A left A-module M is said to be unital if M has a spanning set (that is, if $AM=M$); and M has a finite spanning set when $M=\sum Ax_i$ for a finite family of elements $x_1, \cdots, x_n$ of M. The module ${}_AM$ will be called A-torsionfree when $\iota_M(A)=0$. A ring A is said to be left nondegenerate if the left module ${}_AA$ is A-torsionfree, and A is nondegenerate when it is both left and right nondegenerate (see [10, p. 88]). Clearly, a ring with local units is nondegenerate. The ring A will be called (left) s-unital [12] in case for each $a\in A$ (equivalently, for every finite family $a_1, \cdots, a_n$ of elements

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of A) there is some $u \in A$ such that $ua = a$ (respectively, $ua_i = a_i$, for all i): see [12, Theorem 1]. Any left s-unital ring is idempotent and left nondegenerate.

We will say that a ring A is generated by the element $a \in A$ in case $A = AaA$. The above mentioned results of Abrams and Ánh-Márki [1], [2], and Komatsu [6] may be stated as follows: if A has local units, then A-mod is equivalent to a category of modules over a ring with 1 if and only if A is generated by an idempotent e [2, Proposition 3.5]; if A is left e-unital and A-mod is equivalent to the category of left modules over a ring with 1, then A is generated by some element a [6, Proposition 4.7].

In the sequel, we will be dealing with left modules, and so we follow the convention of denoting the composition $g \circ f$ of two module homomorphisms as the product fg. On the other hand, if R is a ring with 1, ${}_RM$ is a left R-module and $E = \mathrm{End}({}_RM)$ is its endomorphism ring, then we will denote by $E_0 = f\,\mathrm{End}({}_RM)$ the following subring -in general, without identity- of E: $E_0 = \{f \in E \mid f: M \to M$ factors through a finitely generated free module$\}$.

We now state and prove the following result.

THEOREM. *Let A be an idempotent ring. Then the category A-mod is eqivalent to the category R-mod of left modules over a ring R with* 1 *if and only if there is some integer $n \geqq 1$ such that the matrix ring $M_n(A)$ is generated by an idempotent.*

PROOF. We divide the proof in several steps.

Step 1. For any idempotent ring A, let us put $\mathrm{ann}(A) = \{x \in A \mid AxA = 0\}$ and $A' := A/\mathrm{ann}(A)$. Then A' is a nondegenerate idempotent ring and A-mod and A'-mod are equivalent categories.

The fact that A' is nondegenerate is easy to verify. On the other hand, if $\varepsilon: A \to A'$ is the canonical projection, then one may see that the restriction of scalars functor ε_* gives indeed a functor from A'-mod to A-mod. Now, if ${}_AM$ belongs to A-mod and $a \in \mathrm{ann}(A)$, then $AaM = AaAM = 0$, so that $aM \subseteq t_M(A)$, and $aM = 0$, because M is A-torsionfree. As a consequence, there is a functor F: A-mod$\to A'$-mod which views each ${}_AM$ of A-mod as a left A'-module. Then F and ε_* are inverse equivalences and hence A-mod and A'-mod are equivalent categories.

Step 2. For each $n \geqq 1$, let Δ be the matrix ring $M_n(A)$. Then A-mod and Δ-mod are also equivalent categories.

To see this, consider the bimodules ${}_A(A^n)_\Delta$ and ${}_\Delta(A^n)_A$, and the natural mappings Φ: $A^n \otimes_A A^n \to \Delta$, Ψ: $A^n \otimes_\Delta A^n \to A$. It is clear that they are bimodule homomorphisms which give a Morita context between A and Δ (if we represent elements in ${}_A(A^n)_\Delta$ in row form, and elements of ${}_\Delta(A^n)_A$ in column form, then Φ and Ψ are induced by products of matrices). Also, the fact that A is idempotent allows us to deduce that Φ and Ψ are surjective. Then, by [7, Theorem], A-mod and Δ-mod are equivalent categories.

Step 3. We prove now the sufficiency of the condition of the Theorem. Assume that $\Delta=M_n(A)$ is generated by an idempotent. By step 1, Δ is equivalent to $\Delta'=\Delta/\mathrm{ann}(\Delta)$. But $\Delta=\Delta e\Delta$ for the idempotent e implies that $\Delta'=\Delta'e'\Delta'$ for the idempotent $e'=e+\mathrm{ann}(\Delta)$; so, we can assume that Δ is a nondegenerate ring. Then Δ belongs to the category Δ-mod and is a generator of this category. But ${}_\Delta(\Delta e)$ generates Δ, so that it is also a generator of Δ-mod. Δe, being finitely spanned, is clearly a finitely generated object of Δ-mod [11, p. 121]. Finally, let $p: Y\to X$ be an epimorphism in Δ-mod, and put $U=\mathrm{Im}\ p$, $V=X/U$, $W=V/t_V(A)$. Then W belongs to Δ-mod and hence the canonical projection from X to W must be 0; thus, $\Delta V=0$ and $X=U$, so that p is a surjective homomorphism. If $f:\Delta e\to X$ is now a homomorphism, then $f(e)=ea$ for some $a\in X$, and $\alpha(e):=ey$, with y such that $p(y)=ea$, gives a morphism α with $f=\alpha\cdot p$. This shows that Δe is projective. It follows that Δ-mod is equivalent to the category of left modules over the ring $\mathrm{End}_\Delta(\Delta e)\cong e\Delta e$. By step 2, A is equivalent to a ring with 1.

Step 4. Let us now suppose that A is an idempotent and left nondegenerate ring and that there is an equivalence F: A-mod$\to R$-mod, R being a ring with 1. We are to show that $M_n(A)$ is generated by an idempotent, for some $n\geqq 1$.

By [4, Theorem 2.4], there exists a generator ${}_RM$ of R-mod with the property that, if $E=\mathrm{End}({}_RM)$, and $E_0=f\,\mathrm{End}({}_RM)$, then A is isomorphic to some right ideal T of E_0 such that $E_0T=E_0$.

We now point out that we can further assume that there is an epimorphism of left R-modules $\pi: M\to R$. Indeed, this is true for some M^k, and we put $S:=\mathrm{End}({}_RM^k)$, $S_0:=f\,\mathrm{End}({}_RM^k)$, so that there is an isomorphism $S\cong M_k(E)$. We assert that, in this isomorphism, $S_0\cong M_k(E_0)$; in fact, the inclusion $S_0\subseteqq M_k(E_0)$ is obvious, and the inclusion $M_k(E_0)\subseteqq S_0$ depends on the easily verified fact that morphisms $M^r\to M$ or $M\to M^s$ factor through free modules of finite type whenever they are induced by endomorphisms of ${}_RM$ belonging to E_0. By substituting M^k, S and S_0 for M, E and E_0, we have that the matrix ring $M_k(A)$ is still (isomorphic to) a right ideal of S_0 in such a way that -assuming the obvious identification- $S_0\cdot M_k(A)=S_0$. So, by replacing A by $M_k(A)$ if necessary (note that $M_k(A)$ is again idempotent and left nondegenerate), we may indeed assume that $\pi: M\to R$ is an epimorphism.

Let $x\in M$ be such that $\pi(x)=1$. Since $E_0A=E_0$ and $\sum_{\sigma\in E_0}\mathrm{Im}\ \sigma=M$ we deduce that $\sum_{\sigma\in A}\operatorname{Im}\sigma=M$. Therefore there exists a homomorphism $\alpha: M^n\to M$ such that $x\in\operatorname{Im}\alpha$; and each component $\alpha_j:=\mu_j\cdot\alpha$, with $\mu_j: M\to M^n$ being the canonical inclusion, satisfies $\alpha_j\in A$. So we have that $\alpha\cdot\pi: M^n\to R$ is an epimorphism and hence there is $g: R\to M^n$ with $g\alpha\pi=1_R$ and $\alpha\pi g=e$ an idempotent in the ring $\operatorname{End}({}_RM^n)\cong M_n(E)$. Moreover, each of the components of e, when considered as a matrix, consists of $\mu_j\alpha\pi g p_k=\alpha_j(\pi g p_k)\in\alpha_j E\subseteq A$ (where the p_k are the canonical projections $M^n\to M$). This means that $e\in M_n(A)$.

As before, we may put $S:=\operatorname{End}({}_RM^n)\cong M_n(E)$, $S_0:=f\operatorname{End}({}_RM^n)\cong M_n(E_0)$ so that $M_n(A)$ is an idempotent right ideal in S_0 which satisfies $S_0M_n(A)=S_0$. Thus, e is an idempotent element in $M_n(A)\subseteq S_0$ and is an endomorphism of M^n such that $\operatorname{Im} e$ is a direct aummand of M^n isomorphic to R. Consequently, $\operatorname{Im} e$ generates M^n and hence, if we let t range over all the elements in eS_0, we have $\sum_t \operatorname{Im} t=M^n$. This shows that eS_0 is a right ideal of S which satisfies $M^n\cdot(eS_0)=M^n$. If we apply now [5, Proposition 2.5], we see that this implies $S_0eS_0=S_0$.

Since $A=A^2$, $M_n(A)\cdot S_0=M_n(A)$ and so we have: $M_n(A)\cdot e\cdot M_n(A)=M_n(A)\cdot S_0e\cdot S_0=M_n(A)\cdot S_0=M_n(A)$. This proves that $M_n(A)$ is generated by an idempotent element.

Step 5. Now we complete the proof of the Theorem. Let A be an idempotent ring (but not necessarily left nondegenerate), and assume that there is an equivalence of categories between A-mod and R-mod for R a ring with 1. Put $\iota_A(A)=\{a\in A\mid Aa=0\}$, and $A^*=A/\iota_A(A)$. In a way analogous to that of Step 1, we may show that A and A^* are equivalent rings, so that we can deduce from stea 4, that for some $n\geqq 1$, the matrix ring $M_n(A^*)$ is generated by an idempotent. Thus, all that is left to show is that this property can be lifted from $M_n(A^*)$ to $M_n(A)$. But we have that $M_n(A^*)=M_n(A/\iota_A(A))\cong(M_n(A))/(M_n(\iota_A(A)))$, and this last quotient is nothing else than $M_n(A)/\iota_{M_n(A)}(M_n(A))$, that is, $(M_n(A))^*$. Therefore, it will suffice to prove that if a ring of the form $A^*=A/\iota_A(A)$ is generated by an idempotent, then so is the ring A.

So, let us assume that $A^*=A^*\cdot e\cdot A^*$ for some idempotent e. There is $u\in A$ with $u+\iota_A(A)=e$, and then $u^2-u\in\iota_A(A)$, from which we see that $u^3=u^2=u^4$. Therefore, $w=u^2$ is an idempotent of A such that $w+\iota_A(A)=e$. Now, let a, $b\in A$; by hypothesis, $b+\iota_A(A)=\sum\alpha_j\cdot e\cdot\beta_j$ in the ring A^*, so that $b-\sum a_j\cdot w\cdot b_j\in\iota_A(A)$, for some a_j and b_j in A. Then $ab=\sum aa_jwb_j$ and $ab\in AwA$. But since A is idempotent, we have finally that $A=AwA$ and A is generated by an idempotent.

REMARKS. 1) It follows from the Theorem that an idempotent ring A

which is equivalent to a ring with 1 must be finitely generated as a bimodule over A: the coordinates of the idempotent matrix e in the adequate $M_n(A)$ give the family of generators. When A is left s-unital this gives as a consequence the already mentioned result of Komatsu [6, Proposition 4.7]. If A has local units, we get [2, Proposition 3.5].

2) However, the condition that A be finitely generated as a bimodule over itself is not sufficient for A to be equivalent to a ring with 1. To see this, take a ring A such that $A=A^2$, A is finitely generated as an $A-A$-bimodule, is nondegenerate and coincides with its Jacobson radical (Sasiada's example [10, p. 314] of a simple radical ring fulfills these requirements). It is not difficult to show that the Jacobson radical of such a ring is the intersection of all the subobjects of A in A-mod which give a simple quotient of A in A-mod, so that A has no simple quotients in A-mod. Suppose that the category A-mod were equivalent to R-mod for R a ring with 1. Then, if ${}_RM$ corresponds to A in this equivalence, we would have that ${}_RM$ is a generator of R-mod without simple quotients. But this is absurd, since R is isomorphic to a summand of some M^k.

3) It may happen that A be an idempotent ring such that A-mod is equivalent to a category R-mod for a ring R with 1 but, nevertheless, A is not generated by an idempotent. For instance, let R be a simple domain which is not a division ring and let I be a right ideal of R such that $I\neq 0$, $I\neq R$. Then $RI=R$, $I=IR=I^2$ and I is a faithful right ideal of R, so that we can view I as a left nondegenerate and idempotent ring contained in $R=f\,\mathrm{End}({}_RR)$. By [4, Theorem 2.4], we see that I-mod is equivalent to the category R-mod. But I contains no idempotent other than 0, so that I is not generated by an idempotent.

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