TSUKUBA J. MATH.
Vol. 15 No. 2 (1991), 389–395

COMPACTIFICATION AND FACTORIZATION THEOREMS FOR TRANSFINITE COVERING DIMENSION

By

Katsuya YOKOI

Abstract. P. Borst introduced a transfinite extension of the covering dimension. In this paper we obtain compactification and factorization theorems for this dimension function.

1. Introduction.

In this paper we assume that all spaces are normal.

A space X is called *weakly infinite-dimensional in the sense of Smirnov*, abbreviated S-w.i.d., if for every sequence $\{(A_i, B_i): i \in \mathbf{N}\}$ of pairs of disjoint closed sets in X there is a partition L_i in X between A_i and B_i for each $i \in \mathbf{N}$ such that $\bigcap_{i=1}^{n} L_i = \phi$ for some $n \in \mathbf{N}$.

P. Borst [2] defined a new transfinite dimension function, trdim, by generalizing a necessary and sufficient condition of n-dimensionality (in the sense of covering dimension) to transfinite ordinals and he classified S-w.i.d. spaces by use of the dimension function.

This paper is concerned with this dimension function. In section 3 we prove factorization theorem for the above transfinite covering dimension. Recently T. Kimura [6] showed that every space X has a compactification αX of X such that $\operatorname{trdim} \alpha X \leqq \operatorname{trdim} X$ and $w(\alpha X) \leqq w(X)$. He constructed this space by Wallman-type compactification. In section 4 we give another proof for the theorem by the standard way in dimension theory. We extend an earlier result of A.B. Forge [4].

2. Definitions and preliminaries.

We need some preparations for the definition of Borst's paper.

2.1. DEFINITION. Let L be a set. By Fin L we denote the collection of all non-empty finite subsets of L. For a subset M of Fin L and an element

Received December 6, 1990.

$\sigma\in\{\phi\}\cup\mathrm{Fin}\,L$ we put

$$M^{\sigma}=\{\tau\in\mathrm{Fin}\,L:\sigma\cup\tau\in M \text{ and } \sigma\cap\tau=\phi\}.$$

We abbreviate $M^{\{a\}}$ to M^{a} for each $a\in L$.

2.2. DEFINITION. Let L and M be as in Definition 2.1. We define the *ordinal number* $\mathrm{Ord}\,M$ inductively as follows,

$\mathrm{Ord}\,M=0$ iff $M=\phi$,

$\mathrm{Ord}\,M\leqq\alpha$ iff for every $a\in L$, $\mathrm{Ord}\,M^{a}<\alpha$.

$\mathrm{Ord}\,M=\alpha$ iff $\mathrm{Ord}\,M\leqq\alpha$ and $\mathrm{Ord}<\alpha$ is not true, and

$\mathrm{Ord}\,M=\infty$ iff $\mathrm{Ord}\,M>\alpha$ for every ordinal number α.

2.3. DEFINITION. Let X be a space. We put

$$L(X)=\{(A, B): A \text{ and } B \text{ are disjoint closed sets in } X\}.$$

A collection $\sigma=\{(A_i, B_i): i=1, \cdots, n\}\in\mathrm{Fin}\,L(X)$ is called *inessential* if there is a partition L_i in X between A_i and B_i for each $i=1, \cdots, n$ such that $\bigcap_{i=1}^{n} L_i=\phi$. Otherwise σ is called *essential*. For arbitrary $L\subset L(X)$ we set

$$M_L=\{\sigma\in\mathrm{Fin}\,L: \sigma \text{ is essential in } X\}.$$

2.4. DEFINITION. For a space X we put

$$\mathrm{trdim}\,X=\mathrm{Ord}\,M_{L(X)}.$$

2.5. REMARK. P. Borst [2] showed that the above dimension function, trdim, coincides with the covering dimension if the covering dimension is *finite*.

For more detailed information about transfinite covering dimension, the reader is refered to Borst's paper [2].

3. Factorization theorem.

The following Mardešić's factorization theorem [7] is well-known. For every continuous mapping $f: X\to Z$ of a compact space X to a compact space Z there exist a compact space Y and continuous mappings $g: X\to Y$ and $h: Y\to Z$ such that $\dim Y\leqq\dim X$, $w(Y)\leqq w(Z)$, $g(X)=Y$ and $f=hg$.

In this section we extend this result to trdim. The idea of the proof is similar to B. A. Pasynkov's paper [9, 1].

3.1. LEMMA [2]. *Let L and L' be sets, $M\subset \mathrm{Fin}\, L$, $M'\subset \mathrm{Fin}\, L'$, and $\varphi: L\to L'$ be a function satisfying the following condition* (*):

$$(*) \quad \textit{for every } \sigma\in M, \textit{ we have } \varphi(\sigma)\in M' \textit{ and } |\sigma|=|\varphi(\sigma)|.$$

Then we have $\mathrm{Ord}\, M\leqq \mathrm{Ord}\, M'$.

3.2. LEMMA [2]. *Let X be a space and $L\subset L(X)$. Further assume that for every $(A, B)\in L(X)$ there exists $(G, H)\in L$ such that $A\subset G$ and $B\subset H$. Then we have* $\mathrm{Ord}\, M_L=\mathrm{Ord}\, M_{L(X)}$.

We shall give a notation.

Let $f: X\to Z$ be a continuous mapping from a space X to a space Z and $\mathcal{F}_Z=\{(A_\alpha, B_\alpha): \alpha\in\mathcal{A}\}$ be a collection of pairs of disjoint closed sets in Z. Then we denote by $\mathcal{B}(f: X\to Z, \mathcal{F}_Z)$ the following set $\{\beta=(\alpha_1, \cdots, \alpha_n): \{(f^{-1}(A_{\alpha_i}), f^{-1}(B_{\alpha_i})): i=1, \cdots, n\}$ is inessential in X, $\alpha_i\in\mathcal{A}$ and $n\in\mathbf{N}.\}$

3.3. LEMMA [1]. *Let $f: X\to Z$ be a continuous mapping from a compact space X to a compact space Z with $w(Z)=\tau$. For every collection $\mathcal{F}_Z$ of pairs of disjoint closed sets in Z with the cardinality $\leqq\tau$, there exist a compact space $Y=Y(f: X\to Z, \mathcal{F}_Z)$ and continuous mappings $g=g(f: X\to Z, \mathcal{F}_Z): X\to Y$ and $h=h(f: X\to Z, \mathcal{F}_Z): Y\to Z$ such that $w(Y)\leqq w(Z)$, $g(X)=Y$, $f=hg$ and $\{(h^{-1}(A_{\alpha_i}), h^{-1}(B_{\alpha_i})): i=1, \cdots, n\}$ is inessential in Y for every $\beta=(\alpha_1, \cdots, \alpha_n)\in\mathcal{B}=\mathcal{B}(f: X\to Z, \mathcal{F}_Z)$.*

PROOF. We shall give an outline of the proof. For each $\beta=(\alpha_1, \cdots, \alpha_n)\in\mathcal{B}$ there exists a partition L_i in X between $f^{-1}(A_{\alpha_i})$ and $f^{-1}(B_{\alpha_i})$ for each $i=1, \cdots, n$ such that $\bigcap_{i=1}^{n} L_i=\phi$. We construct continuous mappings $\varphi_{\alpha_i}: X\to I_i=[0, 1]$ for each $i=1, \cdots, n$ such that $\varphi_{\alpha_i}(f^{-1}(A_{\alpha_i}))=0$, $\varphi_{\alpha_i}(f^{-1}(B_{\alpha_i}))=1$ and $\varphi_{\alpha_i}(L_i)=1/2$. Then we may assume $L_i=\varphi_{\alpha_i}{}^{-1}(1/2)$. Now we put

$$g_\beta=\mathop{\Delta}_{\alpha_i\in\beta}\varphi_{\alpha_i}: X\longrightarrow I_\beta=\prod_{i=1}^{n} I_i \qquad \text{and}$$

$$g=f\Delta\mathop{\Delta}_{\beta\in\mathcal{B}} g_\beta: X\longrightarrow Z\times\prod_{\beta\in\mathcal{B}} I_\beta\,.$$

Moreover we put $Y=g(X)$ and $h=pr_Z|_Y: Y\to Z$; where $pr_Z: Z\times\prod_{\beta\in\mathcal{B}} I_\beta\to Z$ is the natural projection. Then the conditions are satisfied. □

Let X be a compact space with $w(X)=\tau$. Then there is a large base $\mathcal{U}_X$ for X such that $|\mathcal{U}_X|=\tau$. We put

$$\mathcal{V}_X=\{(Cl_XU, Cl_XV): U, V\in\mathcal{U}_X \text{ and } Cl_XU\cap Cl_XV=\phi\}.$$

3.4. THEOREM. *For every continuous mapping $f: X\to Z$ from a compact space X to a compact space Z there exist a compact space Y and continuous mappings $g: X\to Y$ and $h: Y\to Z$ such that* $\operatorname{trdim} Y\leqq \operatorname{trdim} X$, $w(Y)\leqq w(Z)$, $g(X)=Y$ *and* $f=hg$.

PROOF. Let be $Y_0=Z$, $g_0=f$ and $\mathcal{F}_{Y_0}=\mathcal{V}_{Y_0}$. By Lemma 3.3 we can construct compact spaces Y_i, continuous mappings $g_i: X\to Y_i$ and $h_{i,i-1}: Y_i\to Y_{i-1}$ and collections $\mathcal{F}_{Y_i}$ of pairs of disjoint closed sets in Y_i inductively satisfying the following conditions:

(1) $Y_i=Y(g_{i-1}: X\to Y_{i-1}, \mathcal{F}_{Y_{i-1}})$,

(2) $g_i=g(g_{i-1}: X\to Y_{i-1}, \mathcal{F}_{Y_{i-1}}): X\to Y_i$,

(3) $h_{i,i-1}=h(g_{i-1}: X\to Y_{i-1}, \mathcal{F}_{Y_{i-1}}): Y_i\to Y_{i-1}$,

(4) $\mathcal{F}_{Y_i}=\mathcal{V}_{Y_i}\cup h_{i,i-1}{}^{-1}(\mathcal{F}_{Y_{i-1}})$,

(5) $w(Y_i)\leqq w(Y_{i-1})$,

(6) $g_i(X)=Y_i$,

(7) $g_{i-1}=h_{i,i-1}g_i$ and

(8) $\{(h_{i,i-1}{}^{-1}(A_{\alpha_k}), h_{i,i-1}{}^{-1}(B_{\alpha_k})): k=1, \cdots, n\}$ is inessential in Y_i for every $\beta=(\alpha_1, \cdots, \alpha_n)\in\mathcal{B}_i=\mathcal{B}(g_{i-1}: X\to Y_{i-1}, \mathcal{F}_{Y_{i-1}})$.

Now we put $Y=\varprojlim\{Y_i, h_{i,i-1}\}$, $g=\varprojlim g_i: X\to Y$ and $h=pr_0: Y\to Y_0$, where $pr_i: Y\to Y_i$ is the natural projection. Then the conditions are satisfied. We can easily prove that $w(Y)\leqq w(Z)$, $g(X)=Y$ and $f=hg$. So, we shall show $\operatorname{trdim} Y\leqq \operatorname{trdim} X$.

By Lemma 3.2 it is sufficient to prove that $\operatorname{Ord} M_{\mathcal{V}_Y}\leqq \operatorname{Ord} M_{L(X)}$. For each $(A_\alpha, B_\alpha)\in\mathcal{V}_Y$ there exists $k(\alpha)\in\mathbf{N}\cup\{0\}$ such that $pr_{k(\alpha)}(A_\alpha)\cap pr_{k(\alpha)}(B_\alpha)=\phi$. So we can select an element $(C_{\gamma(\alpha)}, D_{\gamma(\alpha)})\in\mathcal{V}_{Y_{k(\alpha)}}$ such that $pr_{k(\alpha)}(A_\alpha)\subset C_{\gamma(\alpha)}$ and $pr_{k(\alpha)}(B_\alpha)\subset D_{\gamma(\alpha)}$. Then we find an element $(E_{\eta(\alpha)}, F_{\eta(\alpha)})\in\mathcal{V}_X$ such that $g^{-1}(pr_{k(\alpha)}{}^{-1}(C_{\gamma(\alpha)}))\subset E_{\eta(\alpha)}$ and $g^{-1}(pr_{k(\alpha)}{}^{-1}(D_{\gamma(\alpha)}))\subset F_{\eta(\alpha)}$.

Let $\varphi: \mathcal{V}_Y\to L(X)$ be the function defined by

$$\varphi((A_\alpha, B_\alpha))=(E_{\eta(\alpha)}, F_{\eta(\alpha)})$$

for every $(A_\alpha, B_\alpha)\in\mathcal{V}_Y$. Then we can show that the function φ has the property (*) of Lemma 3.1.

Suppose $\{(E_{\eta(\alpha_i)}, F_{\eta(\alpha_i)}): i=1, \cdots, n\}$ is inessential for some $\varphi((A_{\alpha_i}, B_{\alpha_i}))=(E_{\eta(\alpha_i)}, F_{\eta(\alpha_i)})$ and $i=1, \cdots, n$. We put

$$m=\max\{k(\alpha_i): i=1, \cdots, n\}+1.$$

Then we have

$$g^{-1}(pr_{k(\alpha_i)}{}^{-1}(C_{\gamma(\alpha_i)}))=g^{-1}(pr_m{}^{-1}(h_{m,k(\alpha_i)}{}^{-1}(C_{\gamma(\alpha_i)}))$$
$$=g_m{}^{-1}(h_{m,k(\alpha_i)}{}^{-1}(C_{\gamma(\alpha_i)})),$$

and similarly

$$g^{-1}(pr_{k(\alpha_i)}{}^{-1}(D_{\gamma(\alpha_i)}))=g_m{}^{-1}(h_{m,k(\alpha_i)}{}^{-1}(D_{\gamma(\alpha_i)})).$$

Therefore by the above hypothesis and the constructions,

$$\{(h_{m+1,m}{}^{-1}(h_{m,k(\alpha_i)}{}^{-1}(C_{\gamma(\alpha_i)})),\ h_{m+1,m}{}^{-1}(h_{m,k(\alpha_i)}{}^{-1}(D_{\gamma(\alpha_i)}))):\ i=1,\cdots,n\}$$

is inessential in Y_{m+1}.

On the other hand, we have

$$A_{\alpha_i}\subset pr_{k(\alpha_i)}{}^{-1}(C_{\gamma(\alpha_i)})=pr_{m+1}{}^{-1}(h_{m+1,m}{}^{-1}(h_{m,k(\alpha_i)}{}^{-1}(C_{\gamma(\alpha_i)})))$$

and similarly

$$B_{\alpha_i}\subset pr_{m+1}{}^{-1}(h_{m+1,m}{}^{-1}(h_{m,k(\alpha_i)}{}^{-1}(D_{\gamma(\alpha_i)}))).$$

Thus $\{(A_{\alpha_i}, B_{\alpha_i}):\ i=1,\cdots,n\}$ is inessential in Y. This completes the proof of Theorem 3.4. □

Now we can prove the following corollary by the similar way [7].

3.5. COROLLARY. *For every compact space X such that* trdim $X\leqq\alpha$ *there exists an inverse system $\boldsymbol{S}=\{X_\sigma, \pi_{\sigma,\rho}, \Sigma\}$, where $|\Sigma|\leqq w(X)$, consisting of metrizable compact spaces of dimensions $\leqq\alpha$ whose limit is homeomorphic to X.*

4. Compactification theorems.

Let X be a space such that the covering dimension $\dim X$ has a finite. Then the following facts are well-known [3].

(a) The covering dimension of the Stone-Čech compactification βX of X coincides with the covering dimension of X.

(b) There exists a compactification αX of X such that $\dim\alpha X\leqq\dim X$ and $w(\alpha X)\leqq w(X)$.

In this section we extend these results (c. f. [6]).

4.1. THEOREM. *For every space X we have*

$$\operatorname{trdim}\beta X=\operatorname{trdim} X.$$

PROOF. Let $\varphi: L(X)\to L(\beta X)$ be the function defined by

$$\varphi((A, B))=(Cl_{\beta X}A,\ Cl_{\beta X}B)$$

for every $(A, B)\in L(X)$. Then it is obvious that $\varphi(\sigma)\in M_{L(\beta X)}$ for every $\sigma\in M_{L(X)}$. By Lemma 3.1, we have $\operatorname{trdim} X\leqq\operatorname{trdim}\beta X$.

Conversely, we shall show that $\operatorname{trdim}\beta X\leqq\operatorname{trdim} X$. For every $(A, B)\in L(\beta X)$ we can select open sets U_A, U_B in βX such that $A\subset U_A$, $B\subset U_B$ and $Cl_{\beta X}U_A\cap Cl_{\beta X}U_B=\phi$. Let $\varphi: L(\beta X)\rightarrow L(X)$ be the function defined by

$$\varphi((A, B))=(Cl_{\beta X}U_A\cap X,\ Cl_{\beta X}U_B\cap X)$$

for every $(A, B)\in L(\beta X)$. Then we can easily see that $\varphi(\sigma)\in M_{L(X)}$ for every $\sigma\in M_{L(\beta X)}$. By Lemma 3.1 we have $\operatorname{trdim}\beta X\leqq\operatorname{trdim} X$. □

4.2. THEOREM. *Every space X has a compactification αX of X such that* $\operatorname{trdim}\alpha X\leqq\operatorname{trdim} X$ *and $w(\alpha X)\leqq\omega(X)$. Further assume that $f_a: X\rightarrow I_a=[0, 1]$ be continuous mappings for $a\in\mathcal{A}$, where $|\mathcal{A}|\leqq w(X)$. Then each f_a is extendable to a continuous mapping $\tilde{f}_a: \alpha X\rightarrow I_a$.*

PROOF. We can suppose that $\operatorname{trdim} X$ exists and $w(X)=\tau$. There exists a homeomorphic embedding $i: X\rightarrow I^\tau$ of the space X into the Tychonoff cube I^τ of weight τ. We put

$$F=\mathop{\Delta}_{a\in\mathcal{A}} f_a\Delta i: X\longrightarrow \prod_{a\in\mathcal{A}} I_a\times I^\tau .$$

Then we note that F is the homeomorphic embedding and $w(\prod_{a\in\mathcal{A}} I_a\times I^\tau)=\tau$. Let $\beta F: \beta X\rightarrow \prod_{a\in\mathcal{A}} I_a\times I^\tau$ be the extension of F over βX. By virtue of Theorem 3.4 there exist a compact space Y and continuous mappings $g: \beta X\rightarrow Y$ and $h: Y\rightarrow \prod_{a\in\mathcal{A}} I_a\times I^\tau$ such that $\operatorname{trdim} Y\leqq\operatorname{trdim}\beta X$, $w(Y)\leqq\tau$, $g(\beta X)=Y$ and $\beta F=hg$. Now we put $\alpha X=Y$ and $\tilde{f}_a=pr_a h: \alpha X\rightarrow I_a$, where $pr_a: \prod_{a\in\mathcal{A}} I_a\times I^\tau\rightarrow I_a$ is the natural projection. Then by Theorem 4.1 and the construction, the conditions are satisfied. □

In particular, we have the following.

4.3. THEOREM. *For every metrizable separable space X and every sequence $\{f_i: X\rightarrow I: i\in\mathbf{N}\}$ of continuous mappings, there exists a metrizable comactification αX of X such that* $\operatorname{trdim}\alpha X\leqq\operatorname{trdim} X$ *and each f_i is extendable to a continuous mapping $\tilde{f}_i: \alpha X\rightarrow I$.*

In the same way of the proof of Theorem 4.2 we can show the following Corollary.

4.4. COROLLARY. *Let $f_a: X \to X_a$ be continuous mappings from a space X to compact spaces X_a such that $w(X_a) \leqq w(X)$, $a \in \mathcal{A}$ and $|\mathcal{A}| \leqq w(X)$. Then there exists a compactification αX of X such that* $\mathrm{trdim}\, \alpha X \leqq \mathrm{trdim}\, X$, $\omega(\alpha X) \leqq w(X)$ *and each f_a is extendable to a continuous mapping $\tilde{f}_a: \alpha X \to X_a$.*

5. Comments.

The following fact is well-known [3].

For every non-negative integer n and every infinite cardinal number τ there exists a compact universal space Pn, τ for the class of all normal spaces whose covering dimension is not larger than n whose weight is not larger than τ.

We can see this fact by use of the factorization theorem. But we can not apply this theorem to the transfinite covering dimension with infinite value. We should consider the space $\bigoplus_{n \in \mathbf{N}} I^n$. Thus the next question is natural.

QUESTION. For every infinite ordinal number α and every infinite cardinal number τ does there exist a universal space $P\alpha, \tau$ for the class of all normal spaces whose transfinite covering dimension is not larger than α and whose weight is not larger than τ?

References

[1] Alexandroff, P. S. and Pasynkov, B. A., Introduction to Dimension theory (Russian), Moskva, 1973.

[2] Borst, P., Classification of weakly-infinite dimensional spaces. Part I; A transfinite extension of the covering dimension, Fund. Math. 130 (1988), 1-25.

[3] Engelking, R., Dimension theory, PWN, Warszawa, 1978.

[4] Forge, A. B., Dimension preserving compactifications allowing extensions of continuous functions, Duke Math. J. 88 (1961), 625-627.

[5] Hurewicz, W. and Wallman, H., Dimension theory, Princeton, 1941.

[6] Kimura, T., Compactification and product theorems for Borst's transfinite dimension, preprint.

[7] Mardešić, S., On covering dimension and inverse limits of compact spaces, Illinois J. Math. 4 (1960), 278-291.

[8] Nagata, J., Modern Dimension Theory, Heldermann Verlag, Berlin 1983.

[9] Pasynkov, B. A., On the dimension of normal spaces, Soviet Math. Dokl. 12 (1971), 1784-1787.

[10] Pol, R., On classification of weakly infinite-dimensional compacta, Fund. Math. 116 (1983), 169-188.

Institute of Mathematics
University of Tsukuba
Tsukuba-shi, Ibaraki, 305
Japan